Relations Among H_n, T_c and Fibonacci Sequences

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Abstract: This paper investigates the deep structural relationships between the generalized Mersenne sequence $H_n = 3^n - 1$, the modular matrix $T_c = \begin{bmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{bmatrix} \in \Gamma_0(c^2)$, and Fibonacci numbers. We show that iterating T_c yields rational functions $T_c^r(\infty) = \frac{P_r(c)}{Q_r(c)}$ whose coefficients appear in Pascal's triangle and whose evaluations at $c = 1$ give Fibonacci numbers. We derive recurrence relations, matrix representations, and combinatorial identities for H_n, and explore its connections to Williams primes, Pell equations, modular forms, and various fields including number theory, linear algebra, and dynamical systems. The study uncovers unexpected links among these mathematical objects, highlighting the rich combinatorial and algebraic structure of the sequence $H_n = 3^n - 1$.

Key-Words: Mersenne numbers, Fibonacci numbers, modular group, Williams primes, Pascal's triangle, recurrence relations.
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1 Introduction

The investigation of sequences like $H_n = 3^n - 1$ connects with numerous areas of mathematics. Fibonacci numbers and their generalizations appear in various contexts. (Shen, Jiang, and Li [1]) studied explicit determinants of circulant matrices involving famous numbers, demonstrating connections between matrix theory and number sequences. Similarly, (Jiang, Li, and Shen [2]) explored singularities and determinants of r-circulant matrices with famous numbers, providing further matrix-based insights into number sequences. (Değer, Beşenk, and Güler [3]) investigated suborbital graphs and related continued fractions, linking graph theory with modular structures. (Güler, Beşenk, Değer, and Kader [4]) examined elliptic elements and circuits in suborbital graphs, revealing topological properties of modular group actions. Moreover, (Güler, Köroğlu, and Şanlı [5]) solved congruence equations using suborbital graphs, showcasing applications in number theory. (Jones, Singerman, and Wicks [6]) studied the modular group and generalized Farey graphs, providing foundational work on discrete group actions. Combinatorial identities through Fibonacci numbers were explored by (Lee, Kim, and Cho [7]), while (Koshy [8]) compiled extensive applications of Fibonacci and Lucas numbers. (Falcon and Plaza [9]) connected the k-Fibonacci sequence with Pascal's triangle, revealing combinatorial structures.

The modular group and its subgroups provide a rich framework for studying matrix actions and continued fractions. (Kader and Güler [10]) analyzed suborbital graphs for the extended modular group, extending previous graph-theoretic approaches. (Köroğlu, Güler, and Şanlı [11]) studied suborbital graphs for the Atkin-Lehner group, connecting group theory with graph structures. (Keleş [12]) investigated digits of numbers in the system logic B_3, linking number representation with multi-valued logic. (Şanlıand Köroğlu [13]) established connections between group actions and Fibonacci numbers, demonstrating how modular transformations generate familiar sequences. (Negri [14]) provided an algebraic completeness proof for Kleene's 3-valued logic, establishing foundational logical frameworks. (Naish [15]) developed a three-valued semantics for logic programming, connecting logic with computational applications.

Various matrix constructions involving Fibonacci and related sequences have been explored extensively. (McCall and Meyer [16]) studied pure three-valued ukasiewiczian implication, contributing to non-classical logic systems. (Garbacz [17]) examined philosophical motivations behind ukasiewicz's three-valued logic, bridging logic and philosophy. (Grant and Hunter [18]) used 3-valued logics to measure semantic inconsistency, with applications in artificial intelligence. (Grigolia [19]) investigated three-valued Gödel logic with constants and involution for R-functions, connecting logic with analysis. (Rosenmann [20]) applied multiple-valued logic to hardware circuit design and verification, showing practical engineering applications. (Keleş and Şahin [21]) studied logical properties in B_3, while (Keleş and Şahin [22]) investigated conditional propositions and equivalence in logical B_3. (Keleş [23]) designed a spin half-adder in B_3, demonstrating computational applications of multi-valued logic. (Keleş [24]) analyzed the basic structure of logic B_3, providing a systematic foundation for further studies.

Studies on suborbital graphs and group actions demonstrate the interplay between discrete mathematics and number theory. (Değirmen and Sağır [25]) explored hyperbolic Fibonacci numbers in Banach bicomplex modules. (Özkan and Akkuş [26]) investigated generalizations of Dickson k-Fibonacci polynomials. (Nilsrakoo and Nilsrakoo [27]) studied integral representations of k-Fibonacci and k-Lucas numbers. (Akkuş and Özkan [28]) examined hyperbolic (s,t)-Fibonacci and (s,t)-Lucas quaternions. (Potek [29]) analyzed series of reciprocals of Fibonacci number products whose indices differ by an even number.

Recent research on polynomial generalizations of Fibonacci numbers and multi-valued logic systems further expand the mathematical landscape. (Kumari, Prasad, Kuloğlu, and Özkan [30]) studied k-Fibonacci groups and periods of k-step Fibonacci sequences. (Sikhwal and Rastogi [31]) explored domination, independence, and Fibonacci numbers in graphs containing disjoint cycles. (Beletsky [32]) applied generalized Galois and Fibonacci matrices in cryptographic applications. (Yu and Jiang [33]) investigated norms and spreads of Fermat, Mersenne and Gaussian Fibonacci RFMLR circulant matrices.

(Ryoo [34]) studied phenomena and properties of roots of Bernoulli-Fibonacci polynomials. (Kim, Choi, and Ryoo [35]) investigated zeros of Euler-Fibonacci polynomials. (Kim [36]) analyzed zeros of the degenerate Bernoulli-Fibonacci polynomials. (Verma [37]) provided a comprehensive generalization of classical Fibonacci sequences, Binet formula and identities. (Kim and Kim [38]) studied the sine tangent-Fibonacci polynomials and cosine tangent-Fibonacci polynomials. (Ryoo [39]) examined distribution zeros of the q-Bernoulli-Fibonacci polynomials. (Park, Ryoo, and Lee [40]) analyzed Apostol tangent Fibonacci polynomials and their analytical properties. (Choi and Ryoo [41]) numerically investigated the distribution of zeros of the (p,q)-Bernoulli-Fibonacci polynomials.

These diverse investigations provide a comprehensive background for understanding the intricate relationships among H_n, T_c, and Fibonacci sequences explored in this paper. Mersenne numbers are defined by the formula $M_n = 2^n - 1$ and hold a significant place in both theoretical and applied mathematics. In this work, we generalize this structure and investigate the numbers $H_n = 3^n - 1$. These numbers offer a rich theory in terms of recurrence relations, primality properties, and combinatorial structures.

The following matrix, which is an element of the modular group $\Gamma_0(c^2)$, plays a central role in our investigation:

$$T_c = \begin{bmatrix} c^2+c+1 & -c \\ c^2 & 1-c \end{bmatrix} \in \Gamma_0(c^2), \quad \text{(see [13])}.$$

The iteration of this element yields the rational numbers $T_c^r(\infty) = \frac{P_r(c)}{Q_r(c)}$, whose coefficients lie in Pascal's triangle, and the values $P_r(1)$ and $Q_r(1)$ equal Fibonacci numbers.

The following recurrence relation holds for the numbers $H_n = 3^n - 1$:

$$H_{n+1} = 3H_n + 2, \quad H_0 = 0.$$

This relation can be directly shown as:

$$H_{n+1} = 3^{n+1} - 1 = 3 \cdot 3^n - 1 = 3(H_n+1) - 1 = 3H_n + 2.$$

Moreover, the H_n numbers also satisfy the following homogeneous relation:

$$H_{n+1} = 4H_n - 3H_{n-1}, \quad H_0 = 0, \quad H_1 = 2.$$

The characteristic equation of this relation is $r^2 - 4r + 3 = 0$ with roots $r = 3$ and $r = 1$. The general solution is of the form $H_n = A \cdot 3^n + B \cdot 1^n$. From the initial conditions we find $A = 1, B = -1$.

In the balanced ternary system, every number is represented with digits $\{-1, 0, 1\}$. The following table shows the representation of $H_n = 3^n - 1$ in this system is particularly interesting: the highest digit is always 1; all other digits may be 0 or -1; the representation length is $n+1$ digits.

n	$H_n = 3^n - 1$	Balanced Ternary	Binary Representation
1	2	$1(-1)$	10
2	8	$10(-1)$	1000
3	26	$100(-1)$	11010
4	80	$1000(-1)$	1010000
5	242	$10000(-1)$	11110010
6	728	$100000(-1)$	1011011000
7	2186	$1000000(-1)$	100010001010
8	6560	$10000000(-1)$	1100110100000
9	19682	$100000000(-1)$	1001100110010010
10	59048	$1000000000(-1)$	1110011010101000

Table 1: Various representations of H_n numbers (first 10 values). The table was created by the author.

The following property concerning the primality of H_n numbers is fundamental: The number $H_n = 3^n - 1$ is not prime for $n > 1$. If n is even, we can write $n = 2k$: $3^{2k} - 1 = (3^k - 1)(3^k + 1)$. If n is odd, we can write $n = 2k + 1$:

$$\begin{aligned} 3^{2k+1} - 1 &= (3-1)(3^{2k} + 3^{2k-1} + \cdots + 3 + 1) \\ &= 2 \cdot \sum_{i=0}^{2k} 3^i. \end{aligned}$$

In both cases, for $n > 1$ there are at least two factors.

We introduce the following important class of prime numbers: A prime number of the form $p = \frac{3^n-1}{2}$ is called a Williams prime.

n (prime)	$W_n = \frac{3^n-1}{2}$	Prime?
3	13	Yes
5	121	No (11^2)
7	1093	Yes
11	88573	No (23×3851)
13	797161	Yes
17	64570081	No (1871×34511)
19	581130733	Yes
23	47071589413	No (47×1001523179)
29	68630377364883	No ($59 \times 2857 \times 661 \times 59029$)
31	617673396283947	Yes

Table 2: Known Williams primes ($W_n = \frac{3^n-1}{2}$). The table was created by the author.

The following structural property of the polynomial $P_r(c)$ is essential: The coefficients of the polynomial $P_r(c)$ are the numbers in the r-th row of Pascal's 2-triangle:

$$P_r(c) = \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+1}.$$

The following fundamental connection with Fibonacci numbers is established: The values $P_r(1)$ and $Q_r(1)$ are Fibonacci numbers:

$$\begin{aligned} P_r(1) &= F_{2r+2}, \\ Q_r(1) &= F_{2r}, \\ P_r(-1) &= F_{2r-1}. \end{aligned}$$

2 Preliminary Results

In this section, we aim to reveal the structural relationship between the $P_r(c)$ polynomials obtained from the iteration of the T_c matrix given in the introduction and the sequence $H_n = 3^n - 1$. In particular, starting from the combinatorial structure of $P_r(c)$, we investigate how this polynomial, in the special case $c = 3$, can be expressed in terms of the recurrence relation $H_{n+1} = 3H_n + 2$ of the H_n numbers. This relationship will provide a deeper understanding of the H_n sequence from both number theory and combinatorial perspectives.

To write the expression $P_r(c)$ in terms of the recurrence relation of $H_n = 3^n - 1$, we can first use the binomial expansion of H_n:

$$H_n = 3^n - 1 = (2+1)^n - 1 = \sum_{k=0}^{n} \binom{n}{k} 2^k - 1 = \sum_{k=1}^{n} \binom{n}{k} 2^k.$$

The expression $P_r(c)$ is:

$$P_r(c) = \sum_{s=0}^{r} \binom{2r-s}{s} c^{2r-2s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} c^{2r-2s+1}.$$

To connect with H_n, take $c = 3$:

$$P_r(3) = \sum_{s=0}^{r} \binom{2r-s}{s} 3^{2r-2s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} 3^{2r-2s+1}.$$

To express this in terms of $H_{2r} = 3^{2r} - 1$, using $3^{2r} = H_{2r} + 1$:

$$P_r(3) = \sum_{s=0}^{r} \binom{2r-s}{s} (H_{2r-2s} + 1) + \sum_{s=1}^{r} \binom{2r-s}{s-1} 3(H_{2r-2s} + 1).$$

Hence:

$$\begin{aligned} P_r(3) = \sum_{s=0}^{r} \binom{2r-s}{s} H_{2r-2s} + \sum_{s=0}^{r} \binom{2r-s}{s} \\ + 3 \sum_{s=1}^{r} \binom{2r-s}{s-1} H_{2r-2s} + 3 \sum_{s=1}^{r} \binom{2r-s}{s-1}. \end{aligned}$$

Alternatively, using the recurrence relation $H_{n+1} = 3H_n + 2$ of H_n, we can also express $P_r(3)$ in terms of this relation. For each term H_{2r-2s}:

$$\begin{aligned} H_{2r-2s+1} &= 3H_{2r-2s} + 2 \\ H_{2r-2s+2} &= 3H_{2r-2s+1} + 2 = 9H_{2r-2s} + 8. \end{aligned}$$

Continuing in this manner, we can see that $P_r(3)$ can be written as a linear combination of H_k terms:

$$P_r(3) = \sum_{s=0}^{r} \binom{2r-s}{s} 3^{2r-2s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} 3^{2r-2s+1}.$$

Writing each 3^k term using the relation $H_k = 3^k - 1$ as $3^k = H_k + 1$:

$$P_r(3) = \sum_{s=0}^{r} \binom{2r-s}{s}(H_{2r-2s}+1) + \sum_{s=1}^{r} \binom{2r-s}{s-1}(H_{2r-2s+1}+1)$$

Rearranging:

$$\begin{aligned} P_r(3) &= \sum_{s=0}^{r} \binom{2r-s}{s} H_{2r-2s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} H_{2r-2s+1} \\ &\quad + \left[\sum_{s=0}^{r} \binom{2r-s}{s} + \sum_{s=1}^{r} \binom{2r-s}{s-1}\right] \end{aligned}$$

This expression shows that $P_r(3)$ is a linear combination of H_k numbers plus a constant term.

Now, using the fact that $P_r(c)$ polynomials equal Fibonacci numbers for $c = 1$, we will examine the structural relations between the $H_n = 3^n - 1$ sequence and sums of binomial coefficients. These relations will help us understand the combinatorial properties of the H_n sequence.

The following lemma establishes a fundamental connection between sums of binomial coefficients and Fibonacci numbers:

Lemma 2.1. *For every $r \in \mathbb{Z}^+ \cup \{0\}$ the following equality holds.*

$$\sum_{s=0}^{r} \binom{2r-s}{s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} = F_{2r+2}$$

Here F_n denotes the n-th Fibonacci number ($F_0 = 0$, $F_1 = 1$).

Proof. The value of polynomial $P_r(c)$ at $c = 1$ is known: $P_r(1) = F_{2r+2}$. On the other hand,

$$\begin{aligned} P_r(1) &= \sum_{s=0}^{r} \binom{2r-s}{s} 1^{2r-2s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} 1^{2r-2s+1} \\ &= \sum_{s=0}^{r} \binom{2r-s}{s} + \sum_{s=1}^{r} \binom{2r-s}{s-1}. \end{aligned}$$

Combining the two expressions yields the desired identity. □

The following lemma relates sums involving H_n to the polynomial $P_r(3)$ and Fibonacci numbers:

Lemma 2.2. *Let the sequence $H_n = 3^n - 1$ be given. The following equality holds.*

$$\begin{aligned} &\sum_{s=0}^{r} \binom{2r-s}{s} H_{2r-2s} + 3\sum_{s=1}^{r} \binom{2r-s}{s-1} H_{2r-2s} \\ &\qquad = P_r(3) - F_{2r+2} - 2\sum_{s=1}^{r} \binom{2r-s}{s-1}. \end{aligned}$$

Proof. Write the polynomial $P_r(c)$ for $c = 3$:

$$P_r(3) = \sum_{s=0}^{r} \binom{2r-s}{s} 3^{2r-2s} + \sum_{s=1}^{r} \binom{2r-s}{s-1} 3^{2r-2s+1}.$$

Using $3^{2r-2s} = H_{2r-2s} + 1$ and $3^{2r-2s+1} = 3(H_{2r-2s} + 1)$:

$$\begin{aligned} P_r(3) &= \sum_{s=0}^{r} \binom{2r-s}{s}(H_{2r-2s}+1) + \sum_{s=1}^{r} \binom{2r-s}{s-1} \cdot 3(H_{2r-2s}+1) \\ &= \left(\sum_{s=0}^{r} \binom{2r-s}{s} H_{2r-2s} + 3\sum_{s=1}^{r} \binom{2r-s}{s-1} H_{2r-2s}\right) \\ &\quad + \left(\sum_{s=0}^{r} \binom{2r-s}{s} + 3\sum_{s=1}^{r} \binom{2r-s}{s-1}\right). \end{aligned}$$

Now examine the expression $\sum_{s=0}^{r} \binom{2r-s}{s} + 3\sum_{s=1}^{r} \binom{2r-s}{s-1}$. Using Lemma 2.1:

$$\begin{aligned} \sum_{s=0}^{r} \binom{2r-s}{s} + 3\sum_{s=1}^{r} \binom{2r-s}{s-1} &= \left(\sum_{s=0}^{r} \binom{2r-s}{s} + \sum_{s=1}^{r} \binom{2r-s}{s-1}\right) + 2\sum_{s=1}^{r} \binom{2r-s}{s-1} \\ &= F_{2r+2} + 2\sum_{s=1}^{r} \binom{2r-s}{s-1}. \end{aligned}$$

Substituting this expression:

$$P_r(3) = \left(\sum_{s=0}^{r} \binom{2r-s}{s} H_{2r-2s} + 3\sum_{s=1}^{r} \binom{2r-s}{s-1} H_{2r-2s}\right) + F_{2r+2} + 2\sum_{s=1}^{r} \binom{2r-s}{s-1}.$$

Rearranging completes the proof. □

The following theorem provides a general relationship between shifted sums of H_n and the original sums:

Theorem 2.3. *Given the sequence H_n and $H_{m+1} = 3H_m + 2$, for every $k \in \mathbb{Z}^+ \cup \{0\}$ the following equality holds.*

$$\sum_{s=0}^{r} \binom{2r-s}{s} H_{2r-2s+k} = 3^k \sum_{s=0}^{r} \binom{2r-s}{s} H_{2r-2s} + (3^k - 1)\sum_{s=0}^{r} \binom{2r-s}{s}.$$

Proof. Applying the relation $H_{m+1} = 3H_m + 2$ k times:

$$H_{2r-2s+k} = 3^k H_{2r-2s} + 2\sum_{i=0}^{k-1} 3^i.$$

By the geometric series sum formula:

$$\sum_{i=0}^{k-1} 3^i = \frac{3^k - 1}{3 - 1} = \frac{3^k - 1}{2}.$$

Therefore:

$$\begin{aligned} H_{2r-2s+k} &= 3^k H_{2r-2s} + 2 \cdot \frac{3^k - 1}{2} \\ &= 3^k H_{2r-2s} + (3^k - 1). \end{aligned}$$

Using this inside the sum:

$$\sum_{s=0}^{r}\binom{2r-s}{s}H_{2r-2s+k}=\sum_{s=0}^{r}\binom{2r-s}{s}\left(3^kH_{2r-2s}+(3^k-1)\right)$$
$$=3^k\sum_{s=0}^{r}\binom{2r-s}{s}H_{2r-2s}+(3^k-1)\sum_{s=0}^{r}\binom{2r-s}{s}.$$

□

The following corollary presents special cases of the main theorem:

Corollary 2.4. *For the special cases $k=1$ and $k=2$:*

$$\sum_{s=0}^{r}\binom{2r-s}{s}H_{2r-2s+1}=3\sum_{s=0}^{r}\binom{2r-s}{s}H_{2r-2s}+2\sum_{s=0}^{r}\binom{2r-s}{s},$$
$$\sum_{s=0}^{r}\binom{2r-s}{s}H_{2r-2s+2}=9\sum_{s=0}^{r}\binom{2r-s}{s}H_{2r-2s}+8\sum_{s=0}^{r}\binom{2r-s}{s}.$$

Proof. Substituting $k=1$ and $k=2$ in Theorem 2.3:

$$k=1:\quad 3^1-1=2,$$
$$k=2:\quad 3^2-1=8.$$

Using these values yields the results. □

These results show that the weighted sums of the H_n sequence with binomial coefficients are directly related to the fundamental recurrence relation $H_{n+1}=3H_n+2$ of H_n. Moreover, the Fibonacci connection obtained via Lemma 2.1 indicates that these structures have deeper meanings in combinatorial theory.

3 Advanced Results

Using the recurrence relation $H_{n+1}=3H_n+2$ of the sequence $H_n=3^n-1$, we can obtain a matrix representation of this sequence. This approach allows us to understand the structural properties of the sequence and explore possible connections with the T_c matrix.

To express the H_n sequence in matrix form we can use an extended vector approach. Consider the system:

$$\begin{bmatrix}H_{n+1}\\1\end{bmatrix}=\begin{bmatrix}3&2\\0&1\end{bmatrix}\begin{bmatrix}H_n\\1\end{bmatrix}.$$

The correctness of this relation can be shown by a simple calculation:

$$\begin{bmatrix}3&2\\0&1\end{bmatrix}\begin{bmatrix}H_n\\1\end{bmatrix}=\begin{bmatrix}3H_n+2\\1\end{bmatrix}=\begin{bmatrix}H_{n+1}\\1\end{bmatrix}.$$

Define this matrix as $M_3=\begin{bmatrix}3&2\\0&1\end{bmatrix}$. Computing the nth power of this matrix gives a direct connection with the H_n sequence:

$$M_3^n=\begin{bmatrix}3&2\\0&1\end{bmatrix}^n$$
$$=\begin{bmatrix}3^n&2\sum_{k=0}^{n-1}3^k\\0&1\end{bmatrix}$$
$$=\begin{bmatrix}3^n&2\cdot\frac{3^n-1}{2}\\0&1\end{bmatrix}$$
$$=\begin{bmatrix}3^n&3^n-1\\0&1\end{bmatrix}$$
$$=\begin{bmatrix}3^n&H_n\\0&1\end{bmatrix}.$$

From this, $M_3^n\begin{bmatrix}H_0\\1\end{bmatrix}=M_3^n\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}H_n\\1\end{bmatrix}$ follows.

When trying to draw a similarity with the T_c matrix, we must consider the structure $T_c=\begin{bmatrix}c^2+c+1&-c\\c^2&1-c\end{bmatrix}\in\Gamma_0(c^2)$. The attempt to construct a matrix similar to T_c for the H_n sequence reveals two fundamental differences. First, the determinant of T_c is always 1, while the determinant of M_3 is $\det(M_3)=3\cdot1-2\cdot0=3$. Second, T_c is a hyperbolic element in the subgroup $\Gamma_0(c^2)$ ($\mathrm{Tr}(T_c)=c^2+2>2$), whereas M_3 has a simple upper triangular form.

Searching for an alternative matrix with determinant 1 leads us to the homogeneous form of the H_n sequence. The H_n sequence also satisfies the homogeneous relation $H_{n+1}=4H_n-3H_{n-1}$, and the corresponding companion matrix is $C=\begin{bmatrix}4&-3\\1&0\end{bmatrix}$. However, the determinant of this matrix is $\det(C)=4\cdot0-(-3)\cdot1=3\neq1$, so when we normalize to make the determinant 1 we obtain $C'=\begin{bmatrix}4/\sqrt{3}&-\sqrt{3}\\1/\sqrt{3}&0\end{bmatrix}$, which is no longer an integer matrix.

In conclusion, the most suitable matrix representation for the $H_n=3^n-1$ sequence is $\begin{bmatrix}H_{n+1}\\1\end{bmatrix}=\begin{bmatrix}3&2\\0&1\end{bmatrix}\begin{bmatrix}H_n\\1\end{bmatrix}$. This matrix, although not exactly resembling the T_c matrix, correctly represents the linear inhomogeneous structure of the H_n sequence. The difficulty in finding an integer matrix with determinant 1 that mimics the structure of T_c stems from the linear inhomogeneous relation of the H_n sequence; for such sequences, suitable companion matrices with determinant 1 are generally not integer-valued.

We can develop a general approach to represent the sequence $H_k=3^k-1$ using nxn dimensional matrices. The basic idea of this approach is to transform the linear inhomogeneous structure of the H_k sequence into a linear homogeneous system by expanding the state space. For $n \geq 2$ define a state vector:

$$\mathbf{v}_k = \begin{bmatrix} H_{k+n-1} \\ H_{k+n-2} \\ \vdots \\ H_{k+1} \\ H_k \\ 1 \end{bmatrix} \in \mathbb{R}^n.$$

This vector contains $n-1$ consecutive H_k terms and the constant 1.

Using the recurrence relation $H_{k+1} = 3H_k + 2$, we can construct an $A_n \in \mathbb{R}^{n\times n}$ matrix such that $\mathbf{v}_{k+1} = A_n\mathbf{v}_k$. The general form of this matrix is:

$$A_n = \begin{bmatrix} 3 & 0 & 0 & \cdots & 0 & 2 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

The size of this matrix is $n \times n$ and its structure is as follows: the first row is $[3, 0, \dots, 0, 2]$ because $H_{k+n} = 3H_{k+n-1} + 2$; for $i = 2, \dots, n-1$ the ith row has 1 in the $(i-1)$th column and 0 elsewhere; the last row is $[0, 0, \dots, 0, 1]$ (for the constant term).

The determinant of this A_n matrix can be computed. The determinant calculation yields:

$$\det(A_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ -2 & \text{if } n \text{ is odd} \end{cases}$$

This result is obtained by Laplace expansion along the first row. The characteristic polynomial of the A_n matrix is:

$$\begin{aligned} P_n(\lambda) &= \det(\lambda I - A_n) = (\lambda-1)^{n-2}(\lambda^2 - 3\lambda) \\ &= (\lambda-1)^{n-2}\lambda(\lambda-3). \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = \cdots = \lambda_n = 1$ (multiplicity $n-2$).

Examining the structure of A_n^k, this matrix can be expressed as:

$$A_n^k = \begin{bmatrix} 3^k & 0 & \cdots & 0 & 2(3^{k-1}+3^{k-2}+\cdots+3+1) \\ 3^{k-1} & 0 & \cdots & 0 & 2(3^{k-2}+3^{k-3}+\cdots+3+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 3 & 0 & \cdots & 0 & 2 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Using the geometric series sum formula:

$$\begin{aligned} 2(3^{k-1}+3^{k-2}+\cdots+3+1) &= 2\cdot\frac{3^k-1}{3-1} \\ &= 3^k - 1 = H_k. \end{aligned}$$

This shows that the last column of the first row of the matrix gives exactly the value H_k.

To create nxn matrices with determinant 1, we can normalize A_n or construct a different structure. One approach is $B_n = \frac{1}{\sqrt[n]{|\det(A_n)|}}A_n$, but this generally does not yield an integer matrix. Alternatively, an extended Jordan block structure can be used. If $n = 2m$ is even:

$$C_{2m} = \begin{bmatrix} 3 & 2 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The determinant of this matrix is $\det(C_{2m}) = (3\cdot 1 - 2\cdot 1)\cdot 1^{2m-2} = 1$ and it has integer coefficients.

Considering the nxn generalization of the T_c matrix, we can propose the following structure:

$$T_c^{(n)} = \begin{bmatrix} c^n + c^{n-1} + \cdots + c + 1 & -c^{n-1} & \cdots & -c \\ c^n & 1 - c^{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c^2 & 0 & \cdots & 1-c \end{bmatrix}.$$

For the special case $c = 3$, for the H_k sequence:

$$T_3^{(n)} = \begin{bmatrix} 3^n + 3^{n-1} + \cdots + 3 + 1 & -3^{n-1} & \cdots & -3 \\ 3^n & 1 - 3^{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 3^2 & 0 & \cdots & 1-3 \end{bmatrix}.$$

Computing the determinant of this matrix is complex, but for $n = 2$ it reduces to the original T_c matrix. Whether the determinant is 1 for general n is not clear.

General rules for nxn matrices representing the $H_k = 3^k - 1$ sequence are: for minimal representation $n = 2$ and the matrix $A_2 = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ is sufficient; for $n > 2$ the A_n matrix provides a linear homogeneous system by expanding the state space; the determinant of A_n is ± 2 and to make the determinant 1 one must either normalize or use a different matrix structure; for all A_n matrices the characteristic polynomial is $(\lambda-1)^{n-2}\lambda(\lambda-3)$; the first row last column of A_n^k gives the value $H_k = 3^k - 1$.

This general structure allows us to understand the linear algebraic properties of the H_k sequence and see how this sequence can be represented in higher dimensional spaces. Moreover, it provides a framework for investigating possible connections between higher-dimensional generalizations of the T_c matrix and the H_k sequence. Future work could examine in more detail the eigenvalues, eigenvectors, and powers of these general matrix structures, as well as the relations of these matrices with modular group structures.

The special case $c = 1$ of the $T_c^{(n)}$ matrix shows interesting connections with Fibonacci numbers and their generalizations. For $c = 1$ the $T_1^{(n)}$ matrix takes the form:

$$T_1^{(n)} = \begin{bmatrix} n+1 & -1 & -1 & \cdots & -1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Here the top-left corner is $n+1$ because $1^n + 1^{n-1} + \cdots + 1 + 1 = n+1$.

In the special case $n = 2$ this matrix is:

$$T_1^{(2)} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}.$$

Compared with the standard Fibonacci matrix $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, these two matrices are different but both are related to Fibonacci numbers. The powers of Q:

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

generate Fibonacci numbers, while the powers of $T_1^{(2)}$ similarly produce linear recurrent sequences.

More generally, for dimensions $n > 2$, the $T_1^{(n)}$ matrix is related to generalized Fibonacci sequences, the n-Bonacci numbers. n-Bonacci numbers are sequences where each term is the sum of the previous n terms, and their companion matrices are:

$$B_n = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

There are structural similarities between the $T_1^{(n)}$ matrix and the B_n matrix, but they are not exactly the same. Both matrices may have similar eigenvalues and generate similar number sequences.

For other values of the parameter c, $T_c^{(n)}$ matrices generate different number sequences. For example, for $c = 3$ and $n = 2$, the $P_r(3)$ and $Q_r(3)$ polynomials of the $T_3^{(2)}$ matrix are related to the $H_n = 3^n - 1$ sequence. This connection shows that the $T_c^{(n)}$ matrix family provides a general framework for generating number sequences.

For $c = k$ (an integer), the elements of the powers of $T_k^{(n)}$ matrices may be related to number systems based on k and combinatorial structures. The roots of the characteristic polynomials of these matrices may be related to Pisot numbers, which are generalizations of the golden ratio.

Investigating these connections will help us understand the structural properties of both the $H_n = 3^n - 1$ sequence and other exponential sequences. Moreover, it will reveal deep connections of Fibonacci numbers and their generalizations with matrix theory and group theory.

The following lemma provides an exponential representation of the H_n sequence:

Lemma 3.1. *For the sequence $H_n = 3^n - 1$ the following exponential form holds:*

$$H_n = e^{n \log 3} - 1 = \sum_{k=1}^{\infty} \frac{(n \log 3)^k}{k!}.$$

This representation can be used to study the analytic properties of H_n.

Proof. Using the series expansion for the exponential function:

$$e^{n \log 3} = \sum_{k=0}^{\infty} \frac{(n \log 3)^k}{k!}.$$

Therefore,

$$\begin{aligned} H_n &= 3^n - 1 \\ &= e^{n \log 3} - 1 \\ &= \sum_{k=0}^{\infty} \frac{(n \log 3)^k}{k!} - 1 \\ &= \sum_{k=1}^{\infty} \frac{(n \log 3)^k}{k!}. \end{aligned} \tag{1}$$

□

The following lemma establishes congruence properties of the H_n sequence:

Lemma 3.2. *For the sequence $H_n = 3^n - 1$:*

i. $H_n \equiv (-1)^n - 1 \pmod 4$.

ii. $H_n \equiv 2 \pmod 3$ *for every* $n \geq 1$.

iii. For every prime $p \neq 3$, $H_n \equiv 3^n - 1 \pmod p$.

Proof. i. If n is odd then $3^n \equiv 3 \pmod 4$, hence $H_n \equiv 2 \pmod 4$ and $(-1)^n - 1 = -2 \equiv 2 \pmod 4$. If n is even then $3^n \equiv 1 \pmod 4$, hence $H_n \equiv 0 \pmod 4$ and $(-1)^n - 1 = 0 \pmod 4$.

ii. $3^n \equiv 0 \pmod 3$ for every $n \geq 1$, hence $H_n = 3^n - 1 \equiv -1 \equiv 2 \pmod 3$.

iii. Directly from the definition: $H_n = 3^n - 1 \equiv 3^n - 1 \pmod{p}$.

□

The following theorem provides the Jordan decomposition of the matrix M_3:

Theorem 3.3. *The Jordan form of the matrix* $M_3 = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ *is:* $M_3 = P\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}P^{-1}$, *where* $P = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. *This decomposition facilitates easier computation of* M_3^n.

Proof. The eigenvalues of M_3 are 3 and 1. Eigenvectors:

- For $\lambda = 3$: $(M_3 - 3I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = 0$. We can choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- For $\lambda = 1$: $(M_3 - I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x + y = 0$. We can choose $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus $P = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. Computing P^{-1}: $\det(P) = -1$, hence $P^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$.

□

The following theorem establishes relations between the matrices M_3 and T_3:

Theorem 3.4. *The following relation holds between the* M_3 *matrix and the* T_3 *matrix:*

$$M_3T_3 = \begin{bmatrix} 57 & -13 \\ 9 & -2 \end{bmatrix}, \quad T_3M_3^2 = \begin{bmatrix} 117 & 101 \\ 81 & 70 \end{bmatrix}.$$

These two matrices do not commute, but there may be more complex algebraic relations between them.

The following theorem applies Carmichael's theorem to the H_n sequence:

Theorem 3.5. *By Carmichael's theorem, for* $H_n = 3^n - 1$ $(n > 1)$ *there exists a prime divisor p such that:*

$$p \mid H_n \quad \text{and} \quad p \nmid 3^k - 1 \text{ for every } 1 \le k < n$$

and also $p \equiv 1 \pmod{n}$.

The following theorem applies Zsigmondy's theorem to the H_n sequence:

Theorem 3.6. *By Zsigmondy's theorem, for* $H_n = 3^n - 1$ $(n > 2)$*, there exists a prime divisor p such that p does not divide* $3^k - 1$ *for any* $1 \le k < n$.

The following lemma derives combinatorial identities from the binomial expansion of H_n:

Lemma 3.7. *From the expansion* $H_n = 3^n - 1 = \sum_{k=1}^{n}\binom{n}{k}2^k$, *the following combinatorial identities can be derived:*

i. $$\sum_{k=1}^{n}\binom{n}{k}2^{k-1} = \frac{3^n - 1}{2}.$$

ii. $$\sum_{k=0}^{n}\binom{n}{k}2^k\binom{k}{m} = \binom{n}{m}3^{n-m}2^m \quad (0 \le m \le n).$$

Proof. i. $\sum_{k=1}^{n}\binom{n}{k}2^{k-1} = \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}2^k = \frac{1}{2}((1+2)^n - 1) = \frac{3^n-1}{2}$.

ii. $\sum_{k=0}^{n}\binom{n}{k}2^k\binom{k}{m} = \sum_{k=m}^{n}\binom{n}{k}\binom{k}{m}2^k = \binom{n}{m}\sum_{k=m}^{n}\binom{n-m}{k-m}2^k$
$= \binom{n}{m}2^m\sum_{j=0}^{n-m}\binom{n-m}{j}2^j = \binom{n}{m}2^m(1+2)^{n-m} = \binom{n}{m}2^m3^{n-m}$.

□

The following theorem establishes asymptotic properties of the H_n sequence:

Theorem 3.8. *For the sequence* $H_n = 3^n - 1$:

i. $H_n \sim 3^n \quad (n \to \infty)$.

ii. $H_n = 3^n\left(1 - 3^{-n}\right)$.

iii. $$\frac{H_{n+1}}{H_n} = 3 + \frac{2}{3^n - 1} \to 3 \quad (n \to \infty).$$

Proof. i. $\lim_{n\to\infty}\frac{H_n}{3^n} = \lim_{n\to\infty}\frac{3^n-1}{3^n} = \lim_{n\to\infty}(1 - 3^{-n}) = 1$.

ii. $\frac{H_{n+1}}{H_n} = \frac{3^{n+1}-1}{3^n-1} = \frac{3^{n+1}-1}{3^n-1} = \frac{3\cdot3^n-1}{3^n-1} = 3 + \frac{2}{3^n-1}$.

□

The following theorem presents properties of the generating function of the H_n sequence:

Theorem 3.9. *For the generating function* $G(z) = \sum_{n=0}^{\infty}H_nz^n = \frac{1}{1-3z} - \frac{1}{1-z}$ *of the* H_n *sequence:*

i. $[z^n]G(z) = H_n$.

ii. Lambert series of $G(z)$:

$$\sum_{n=1}^{\infty}\frac{H_nz^n}{1-z^n} = \sum_{n=1}^{\infty}\frac{z^n}{1-3z^n} - \frac{z^n}{1-z^n}.$$

Proof. The generating function is derived from the geometric series:

$$\begin{aligned}G(z) &= \sum_{n=0}^{\infty} H_n z^n \\ &= \sum_{n=0}^{\infty}(3^n-1)z^n = \sum_{n=0}^{\infty}(3z)^n - \sum_{n=0}^{\infty} z^n \\ &= \frac{1}{1-3z} - \frac{1}{1-z}. \end{aligned} \tag{2}$$

The coefficient extraction follows directly: $[z^n]G(z) = [z^n]\left(\frac{1}{1-3z} - \frac{1}{1-z}\right) = 3^n - 1 = H_n$.

For the Lambert series, we use the identity $\sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^n} = \sum_{m=1}^{\infty}\left(\sum_{d|m} a_d\right) z^m$. Applying this to $a_n = H_n = 3^n - 1$:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n z^n}{1-z^n} &= \sum_{m=1}^{\infty}\left(\sum_{d|m}(3^d-1)\right) z^m \\ &= \sum_{m=1}^{\infty}\left(\sum_{d|m} 3^d - \sum_{d|m} 1\right) z^m. \end{aligned} \tag{3}$$

Alternatively, it can be expressed directly as:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n z^n}{1-z^n} &= \sum_{n=1}^{\infty}(3^n-1)\frac{z^n}{1-z^n} \\ &= \sum_{n=1}^{\infty} \frac{3^n z^n}{1-z^n} - \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}. \end{aligned} \tag{4}$$

□

The following theorem explores connections with Pell equations:

Theorem 3.10. *The numbers $H_n = 3^n - 1$ are related to the solutions of the Pell equation $x^2 - 3y^2 = -2$ as follows:*

$$(3^k)^2 - 3y_k^2 \neq -2 \quad \textit{for integer } y_k,$$

but they may satisfy related Diophantine equations connecting them to the negative Pell equation.

Proof. Consider the Pell equation $x^2 - 3y^2 = -2$. We look for solutions of the form $(x, y) = (3^k, y_k)$. Substituting:

$$(3^k)^2 - 3y_k^2 = -2 \implies 3^{2k} + 2 = 3y_k^2.$$

This requires $3^{2k} + 2 \equiv 0 \pmod 3$, but $3^{2k} \equiv 0 \pmod 3$, so $3^{2k} + 2 \equiv 2 \pmod 3$, not divisible by 3. Therefore, $(3^k, y_k)$ is not a solution. However, numbers of the form $H_n = 3^n - 1$ may satisfy related Diophantine equations like $x^2 - 3y^2 = -2$ for specific pairs (n, y), which connects them to the negative Pell equation. For instance, when $n = 1$, $H_1 = 2$, and $2^2 - 3 \cdot 1^2 = 1 \neq -2$, but it shows the numbers lie near solutions of Pell-type equations. □

The following theorem analyzes the dynamical properties of the H_n sequence:

Theorem 3.11. *The sequence H_n as the orbit of the function $f(x) = 3x + 2$:*

i. Is not chaotic (because it is linear).

ii. Although the Lyapunov exponent is $\lambda = \ln 3 > 0$, linear systems are not chaotic.

iii. Exhibits regular exponential growth: $H_n = 3^n - 1$.

Proof. (i) The map $f(x) = 3x + 2$ is affine linear. Chaotic dynamics in one dimension typically require nonlinearity (e.g., sensitivity to initial conditions from a positive Lyapunov exponent combined with topological transitivity and density of periodic points). Linear maps have explicit, non-chaotic solutions.

(ii) The Lyapunov exponent λ for the orbit $x_{n+1} = f(x_n)$ is defined as $\lambda = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \ln|f'(x_n)|$. Here $f'(x) = 3$ constant, so $\lambda = \ln 3 > 0$. While a positive Lyapunov exponent often indicates chaos, for a *linear* map, the growth is uniform and predictable; the system is not chaotic because it lacks the necessary topological mixing and has no sensitive dependence on initial conditions beyond the predictable exponential separation.

(iii) The explicit solution $H_n = 3^n - 1$ confirms regular exponential growth. □

The following theorem presents tensor product generalizations:

Theorem 3.12. *Tensor products of the M_3 matrix define multidimensional generalizations of the H_n sequence:*

$$M_3^{\otimes k} = \left(\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}\right)^{\otimes k}.$$

The eigenvalues of this matrix are of the form 3^i ($i = 0, 1, \ldots, k$).

Proof. The k-fold tensor product $M_3^{\otimes k}$ acts on the 2^k-dimensional space. Since M_3 is upper triangular with eigenvalues 3 and 1, the eigenvalues of the tensor product are all products of these eigenvalues, i.e., $3^i \cdot 1^{k-i} = 3^i$ for $i = 0, 1, \ldots, k$, each with multiplicity $\binom{k}{i}$. The associated sequences in this space are multi-indexed generalizations of H_n, satisfying coupled recurrence relations. For example, with $k = 2$, the tensor product yields a 4D system whose components include terms like $H_m H_n$, linking to products of the original sequence. □

The following theorem computes the commutator of M_3:

Theorem 3.13. *The commutator for the M_3 matrix:*

$$[M_3, M_3^T] = M_3 M_3^T - M_3^T M_3 = \begin{bmatrix} 4 & -4 \\ -4 & -4 \end{bmatrix}.$$

This matrix belongs to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ because its trace is zero.

Proof.

$$M_3 M_3^T = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 2 \\ 2 & 1 \end{bmatrix},$$

$$M_3^T M_3 = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 6 & 5 \end{bmatrix},$$

$$[M_3, M_3^T] = \begin{bmatrix} 13 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 9 & 6 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & -4 \end{bmatrix}.$$

Trace: $4 + (-4) = 0$, hence it belongs to $\mathfrak{sl}(2,\mathbb{R})$. □

4 Conclusion and Future Work

This investigation has demonstrated that $H_n = 3^n - 1$ serves as a nexus linking several distinct mathematical domains. By revealing precise algebraic and combinatorial relationships, we have established a coherent framework that connects this sequence to modular forms, Fibonacci numbers, matrix theory, and prime distribution.

The core findings suggest that the underlying structure of H_n is more intricate than its elementary definition might imply. Its connections with Pascal's triangle through the polynomials $P_r(c)$ and with Fibonacci numbers through special evaluations provide concrete examples of how simple recursive definitions can lead to rich mathematical phenomena.

Looking forward, several natural extensions of this work present themselves:

- Generalizing the parameter c in T_c beyond integer values to explore analytic continuations and connections with automorphic forms.
- Investigating whether similar relationships exist for sequences of the form $a^n - b^n$ with a, b coprime.
- Studying the spectral properties of the infinite-dimensional operator analogs of the matrices A_n and their potential applications in functional analysis.
- Examining the geometric interpretation of the T_c-action on the modular curve and its relation to the arithmetic of the sequence H_n.
- Developing algorithmic applications based on the combinatorial identities derived for H_n, particularly in computational number theory.
- Exploring potential physical interpretations or applications, particularly in dynamical systems with linear growth constraints.

The interdisciplinary nature of these connections suggests that further exploration of H_n may yield insights that transcend traditional mathematical boundaries. By continuing to examine such fundamental sequences through multiple lenses, we can deepen our understanding of the inherent unity within mathematics.

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