Variable exponential $L^{p(\cdot)}$ operator algebra dynamical system

MYKOLA YAREMENKO
Department of Partial Differential Equations,
The National Technical University of Ukraine,
"Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, Ukraine,
UKRAINE

Abstract: - Let (G, A, ω) be a variable exponential $L^{p(\cdot)}$ operator algebra dynamical system, let A be $p(\cdot)$-incompressible. We establish that when $p(\cdot) \in P(\Omega)$, $1 < p_m \le p(g) \le p_S < \infty$, $g \in G$ is not identical to 2 then mapping $\Phi^{p(\cdot)}$ from $F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right)$ to $LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}$ is an isometric isomorphism if and only if the group G is finite. When G is finite then the isometric isomorphism $\Phi^{p(\cdot)}$ is equivariant for actions double dual actions: $\hat{\hat{\omega}} : G \to F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right)$ and $\omega \otimes Ad(\rho) : G \to LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}$.

Key-Words: - variable exponent Lebesgue space, Takai duality, spectral theory, variable exponential operator crossed product, variable exponential operator algebra.
Subject classification codes: 46H05, 46H35, 22D35, 47L65, 43A25.

Received: June 11, 2024. Revised: January 6, 2025. Accepted: April 9, 2025. Published: June 6, 2025.

1 Introduction

The preeminent inducement of Pontryagin and Takai duality is the definition of the dual group $\hat{G}$ of locally compact group G as a topological group homomorphism from G to the circle group S^1. The Takai duality theorem [14, 15] states that the iterated crossed product $\hat{G} \times_{\hat{\omega}} (G \times_{\omega} \mathrm{A})$ is isomorphic to $KL\left(L^2(G, \Sigma, \mu)\right) \otimes_{\max} \mathrm{A}$. The proof of this statement depends on the Fourier transform and so on the Hilbertian properties of $L^2(G, \Sigma, \mu)$.

In the present article, we establish the analog of the Takai duality for the variable exponential $L^{p(\cdot)}$ operator algebra dynamical systems [14, 15]. Our main goal is to answer the question: what are conditions under which there exists an isomorphism $\Phi^{p(\cdot)}$ between the iterated variable exponent $L^{p(\cdot)}$ operator crossed product $F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right)$ and the tensor product $LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}$? Here, we introduced a variable exponential $L^{p(\cdot)}$ operator algebra as isometrically isomorphic of a norm closed subalgebra of $LB\left(L^{p(\cdot)}(\Omega, \Sigma, \mu)\right)$ for some appropriate measure space (G, Σ, μ); and the dual action $\hat{\omega}: \ \hat{G} \to Aut\left(F^{p(\cdot)}(G, \mathrm{A}, \omega)\right)$ is defined as an extension of $*$-isomorphism $\hat{\omega}_{\chi}: \ C_C(G, \mathrm{A}, \omega) \to C_C(G, \mathrm{A}, \omega)$ given by $\hat{\omega}_{\chi}(f)(g) = \overline{\chi(g)} f(g)$ for $g \in G, \chi \in \hat{G}, \quad f \in C_C(G, \mathrm{A}, \omega)$. By continuity, isometrically isomorphism $\hat{\omega}_{\chi}: \ C_C(G, \mathrm{A}, \omega) \to C_C(G, \mathrm{A}, \omega)$ extends to $\hat{\omega}: \ \hat{G} \to Aut\left(F^{p(\cdot)}(G, \mathrm{A}, \omega)\right)$. The double dual action $\hat{\hat{\omega}}: \ G \to F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right)$ is given $\hat{\hat{\omega}}_h(F)(\chi, g) = \overline{\chi(h)} F(\chi, g)$ for all $F \in C_C\left(\hat{G} \times G, \mathrm{A}\right)$, $g, h \in G, \quad \chi \in \hat{G}$.

Classical results on the duality theory for crossed products of Banach algebras were obtained by H. Takai (1975) and M. Takesaki (1973) [14, 15]. Due to the variety of applications to the K-theory of operator algebra, the duality theory for operator crossed products is a rapidly developing branch of mathematics [2, 11, 13, 16, 17]. We mention the work of Y. C. Chung, K. Li studied an isometric isomorphism between the l^p uniform Roe algebras they obtained a bijective coarse equivalence between the underlying metric spaces [4]; N. C. Phillips investigated the crossed products of L^p operator algebras [11]; Z. Wang and Y. Zeng, and S. Zhu studied the Takai type duality for L^p operator crossed products [16, 17]; for general reference see [1, 3, 5 – 10, 12, 19, 20].

In this article, we establish that homomorphism

$$\Phi^{p(\cdot)}: \quad F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right) \to$$
$$\to LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}$$

is an isometric isomorphism if and only if the group G is finite. We denote right representation by $\rho: \quad G \to LB\left(l^{p(\cdot)}(G)\right)$ and $Ad\rho: \quad G \to LK\left(l^{p(\cdot)}(G)\right)$ so that $(Ad\rho)(g)(T) = \rho(g)T\rho\left(g^{-1}\right)$. If the group G is finite, the isometric isomorphism $\Phi^{p(\cdot)}: \quad F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right) \to LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}$ is equivariant for actions $\hat{\hat{\omega}}: G \to F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right)$ and $\omega \otimes Ad(\rho): G \to LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}$.

2. Variable exponential $L^{p(\cdot)}$ operator matrix norms

Although we will use matrix normed spaces, we are going to define a more general concept of operator tensor algebra.

Definition 1. ***Let*** $M_n, n \in N$ ***be the algebra of*** $n \times n$ ***complex matrices. Let*** A ***be a complex algebra. The algebra*** $M_n(\mathrm{A})$ ***is defined as*** $M_n \otimes \mathrm{A}$ ***given by***

$$\left(a_{j,k}\right)_{j,k=1,\dots,n} \mapsto \sum_{j,k=1,\dots,n} e_{j,k} \otimes a_{j,k}. \tag{1}$$

We can generalize this definition to the tensor algebra T_n as follows.

Definition 2. ***Let*** T_n ***be the standard algebra of*** $\underbrace{n \times \ldots \times n}_{i}$ ***complex tensors of rank i. Let*** A ***be a complex algebra. The algebra*** $T_n(\mathrm{A})$ ***is defined as*** $T_n \otimes \mathrm{A}$ ***given by***

$$\left(a_{k_1,\dots,k_i}\right)_{k_1,\dots,k_i=1,\dots,n} \mapsto \sum_{k_1,\dots,k_i=1,\dots,n} e_{k_1,\dots,k_i} \otimes a_{k_1,\dots,k_i}. \tag{2}$$

The operations xa and ax for $x \in T_n$ and $a \in T_n(\mathrm{A})$ is defined accordingly to standard tensor law.

Definition 3. ***A tensor normed algebra is a complex algebra*** A ***equipped with a function*** $\|\cdot\|_n : T_n(\mathrm{A}) \to R$ ***that satisfies the following condition:***

1) for any $a = \left(a_{k_1,\dots,k_i}\right)_{k_1,\dots,k_i=1,\dots,n} \in T_n(\mathrm{A})$ ***, for any injective functions***

$$\tau_k: \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}, \quad m \le n, \quad k \le i, \tag{3}$$

we request

$$\left\|\left(a_{k_1,\dots,k_i}\right)_{k_1,\dots,k_i=1,\dots,m}\right\|_m \le \|a\|_n; \tag{4}$$

2) for any $a \in T_n(\mathrm{A})$ ***and for any set*** $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \in C^n$ ***so that*** $z = diag(\lambda_1, \lambda_2, \dots, \lambda_n) \in T_n$***, we request***

$$\max\left\{\|az\|_n, \ \|za\|_n\right\} \le \le \max\left\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|\right\}\|a\|_n; \tag{5}$$

3) for any $a \in T_n(\mathrm{A})$ and any $b \in T_m(\mathrm{A})$ we request

$$\|diag(a,b)\|_{n+m} = \max\{\|a\|_n, \ \|b\|_m\}. \qquad (6)$$

Proposition 1. ***Let A be a tensor normed algebra, then estimate***

$$\max_{k_1,\dots,k_i=1,\dots,n} \|a_{k_1,\dots,k_i}\|_n \le \|a\|_n \le \sum_{k_1,\dots,k_i=1,\dots,n} \|a_{k_1,\dots,k_i}\|_n \qquad (7)$$

hold for all $a = (a_{k_1,\dots,k_i})_{k_1,\dots,k_i=1,\dots,n} \in T_n(\mathrm{A})$.

The proof is straightforward.

Proposition 2. ***Let A be a tensor normed algebra and let J be a closed ideal. Then, the norm $\|\cdot\|_n : T_n(\mathrm{A}/J) \to R$ is identified with $T_n(\mathrm{A})/T_n(J)$, and the algebra A/J is a tensor normed algebra such that the quotient map is completely contractive.***

Proof. We assume $\tau_k : \{1,2,\dots,m\} \to \{1,2,\dots,n\}, \quad m \le n, \quad k \le i$. We denote the quotient mapping $q : \mathrm{A} \to \mathrm{A}/J$, and

$$a = (a_{k_1,\dots,k_i})_{k_1,\dots,k_i=1,\dots,n} \in T_n(\mathrm{A})$$

and

$$y = (y_{k_1,\dots,k_i})_{k_1,\dots,k_i=1,\dots,n} \in T_n(\mathrm{A}/J)$$

such that $q_n(a) = y$ and $\|a\|_n < \|y\|_n + \varepsilon$ for arbitrary $\varepsilon > 0$. We obtain

$$\left\| \left(y_{q(k_1),\dots,q(k_i)} \right)_{k_1,\dots,k_i=1,\dots,m} \right\|_m \le$$

$$\le \left\| \left(a_{q(k_1),\dots,q(k_i)} \right)_{k_1,\dots,k_i=1,\dots,m} \right\|_m \le$$
$$\le \|a\|_n < \|y\|_n + \varepsilon,$$

which proves proposition 2.

Definition 4. ***Let $p(\cdot) \in P(\Omega)$, $1 < p_m \le p(x) \le p_S < \infty, \quad x \in \Omega$, and let (Ω, Σ, μ) be a measure space. The Banach algebra $\mathrm{A} \subset LB\left(L^{p(\cdot)}(\Omega, \Sigma, \mu)\right)$ can be endowed with the matrix norms by prescription $M_n(\mathrm{A})$ to closed subalgebra of $LB\left(L^{p(\cdot)}(\{1,2,\dots,n\} \times \Omega, \kappa \times \mu)\right)$, where the set function κ is countable measure on $\{1,2,\dots,n\}$.***

The natural Luxembourg-Nakano norm is given by

$$\|\zeta\|_{l^{p(\cdot)}} = \inf\left\{ \lambda > 0 : \ \sum_j \left(\frac{|\zeta_j|}{\lambda} \right)^{p(j)} \le 1 \right\} \qquad (8)$$

for all $\zeta \in l^{p(\cdot)}$.

Definition 5. ***Let $p(\cdot) \in P^{\log}(E)$, $1 < p_m \le p(x) \le p_S < \infty$. Let E be a measurable set with countable measures on E. Then, variable exponential spaces $l^{p(\cdot)}$ and $l_d^{p(\cdot)}$ are defined as $l^{p(\cdot)}(\{z \in Z : \ z > 0\})$ and $l_d^{p(\cdot)} = l^{p(\cdot)}(\{1,2,\dots,d\})$ with the operator norm.***

Let $p(\cdot) \in P^{\log}(E)$, $1 < p_m \le p(x) \le p_S < \infty$, and let E_1 be a finite subset of E. The set $M_{E_1}^{p(\cdot)}$ consists of all elements $a \in L\left(l^{p(\cdot)}(E)\right)$ such that $a\zeta = 0$, $\zeta|_{E_1} = 0$, and $a\zeta \in l^{p(\cdot)}(E_1) \subset l^{p(\cdot)}(E)$ for all $\zeta \in l^{p(\cdot)}(E)$, then the set $M_E^{p(\cdot)}$ is given $M_E^{p(\cdot)} = \bigcup_{E_1 \subset E} M_{E_1}^{p(\cdot)}$ for all finite E_1; and the $L^{p(\cdot)}$ operator algebra $clos_{LB\left(l^{p(\cdot)}(E)\right)}\left(M_E^{p(\cdot)}\right)$ contains the operator algebra $LK\left(l^{p(\cdot)}(E)\right)$ or more precisely $clos_{LB\left(l^{p(\cdot)}(E)\right)}\left(M_E^{p(\cdot)}\right) = LK\left(l^{p(\cdot)}(E)\right)$ since all elements of $L\left(l^{p(\cdot)}(E)\right)$ have finite rank. Thus, the

$L^{p(\cdot)}$ operator algebra $clos_{LB\left(l^{p(\cdot)}(E)\right)}\left(M_E^{p(\cdot)}\right)$ is a closed subalgebra of $L\left(l^{p(\cdot)}(E)\right)$ such that $LK\left(l^{p(\cdot)}(E)\right)$.

3. Variable exponential $L^{p(\cdot)}$ operator algebras

Let (G, A, ω) be a variable exponential dynamical system, where G is a commutative locally compact, separable group, let A be a Banach algebra, and $\omega : G \to Aut(\mathrm{A})$ be a continuous homomorphism. Let $\hat{G}$ be a dual of G.

Definition 6. ***Let*** $p(\cdot) \in P(\Omega)$, $1 < p_m \le p(x) \le p_S < \infty$, $x \in \Omega$, ***and let*** (Ω, Σ, μ) ***be a measure space. The Banach algebra*** A***, which is isometrically isomorphic to a norm closed subalgebra of*** $LB\left(L^{p(\cdot)}(\Omega, \Sigma, \mu)\right)$, ***is called a variable exponent*** $L^{p(\cdot)}$ ***operator algebra.***

The Banach algebra A ***is*** $p(\cdot)$-***incompressible if for each variable exponential*** $L^{p(\cdot)}$ ***space*** E***, each contractive injective homomorphism*** $\rho: \quad \mathrm{A} \to LB(E)$ ***is isometry.***

Banach algebras $LB\left(L^{p(\cdot)}(\Omega, \Sigma, \mu)\right)$ and $LK\left(L^{p(\cdot)}(\Omega, \Sigma, \mu)\right)$ are examples of a variable exponent $L^{p(\cdot)}$ operator algebras. We will always assume that A is $p(\cdot)$-incompressible.

Assume $B(X)$ is the C^*-algebra of all linear bounded operators $X \to X$. Assume the map $\pi : \mathrm{A} \to LB(X)$ is a representation of A. The representation π is called σ-finite if the measure μ is σ-finite. Let $v : G \to LB(X)$ be an isometric group representation such that $g \mapsto v(g)\varphi$ is strongly continuous for all $\varphi \in X$ and equality

$$v(g)\pi(a)v(g)^{-1} = \pi(\omega(g, a)) \tag{9}$$

holds for all $g \in G, \quad a \in \mathrm{A}$.

With each covariant representation (π, v) of the dynamical system (G, A, ω), we uniquely associate a representation

$$v \times \pi : \quad C_C(G, \mathrm{A}, \omega) \to LB(X) \tag{10}$$

given by

$$(v \times \pi)(f)(x) = \int_G \pi(f(h)) v(h)(x) d\mu(h) \tag{11}$$

for all $f \in C_C(G, \mathrm{A}, \omega)$ and all $x \in X$.

Definition 7. ***The completion of*** $C_C(G, \mathrm{A})$ ***with respect to the enveloping norm defined by***

$$\|f\|_{env} = \sup_{(\pi, v)} \left\{ \left\| (v \times \pi)(f) \right\|_{LB(X)} : \quad f \in C_C(G, \mathrm{A}, \omega) \right\} \tag{12}$$

is called the full-crossed product $G \times_\omega \mathrm{A} \equiv F^{p(\cdot)}(G, \mathrm{A}, \omega)$ **.**

The representation $v \times \pi : \quad C_C(G, \mathrm{A}, \omega) \to LB(X)$ can be uniquely and continuously extended to the representation

$$v \times \pi : \quad F^{p(\cdot)}(G, \mathrm{A}, \omega) \to LB(X)$$

since $G \times_\omega \mathrm{A}$ is enveloping C^*-algebra of the Banach algebra $L^1(G, \mathrm{A})$.

Definition 8. ***Let*** (G, A, ω) ***be a dynamical system, and let*** $R^{p(\cdot)}(G, \mathrm{A}, \omega)$ ***be the class of nondegenerate*** σ***-finite contractive regular covariant representations, then we define the norm of the reduced space by***

$$\|f\|_{F_R^{p(\cdot)}(G,\,\mathrm{A},\,\omega)} = \sup_{(v,\,\pi)\in R^{p(\cdot)}(G,\,\mathrm{A},\,\omega)} \left\|(v\times\pi)(f)\right\|_{LB\left(L^{p(\cdot)}(\Omega,\,\Sigma,\,\mu)\right)}. \quad (13)$$

The reduced $L^{p(\cdot)}$ operator-crossed product $F_R^{p(\cdot)}(G,\mathrm{A},\omega)$ is the completion of $C_C(G,\mathrm{A},\omega)$ in the norm $\|\cdot\|_{F_R^{p(\cdot)}(G,\,\mathrm{A},\,\omega)}$.

Proposition 3. ***Let*** $p(\cdot)\in P^{\log}(E)$, $1<p_m\le p(x)\le p_S<\infty$. ***Let*** E ***be a measurable set with countable measures on*** E***. Then, algebra*** $M_E^{\;p(\cdot)}\equiv\bigcup_{finite\;\;F\subset E} M_F^{\;p(\cdot)}$ ***is a subalgebra of*** $LB\left(l^{p(\cdot)}(E)\right)$***, the closed algebra*** $clos\left(M_E^{\;p(\cdot)}\right)$ ***is subalgebra of*** $LB\left(l^{p(\cdot)}(E)\right)\subset LK\left(l^{p(\cdot)}(E)\right)$.

The proof follows from the finiteness of all elements of subalgebra $M_E^{\;p(\cdot)}$.

In our previous investigations, we showed that $clos\left(M_E^{\;p(\cdot)}\right)$ coincides with $LK\left(l^{p(\cdot)}(E)\right)$. Let function $f\in C_0(\Omega)$, we define an operator $(m(f)\psi)(x)=f(x)\psi(x)$ for all $x\in\Omega$ and all $\psi\in L^{p(\cdot)}(\Omega,\Sigma,\mu)$, then the mapping $f\mapsto m(f)$ is an isometric isomorphism from $C_0(\Omega)$ into closed subalgebra of $LB\left(L^{p(\cdot)}(\Omega,\Sigma,\mu)\right)$.

In variable exponent Lebesgue spaces $L^{p(\cdot)}(G,\Sigma,\mu)$, the Young convolution inequality $\|f*\varphi\|_{L^{p(\cdot)}}\le c\|f\|_1\|\varphi\|_{L^{p(\cdot)}}$ holds if and only if p is a constant.

Let $p(\cdot),r(\cdot),s(\cdot)\in P^{\log}(G)$ such that

$$\frac{1}{s(g)}=\left|\frac{1}{p(g)}-\frac{1}{p_\infty}\right|,$$

$$\frac{1}{t_1}=1-\frac{1}{p_\infty}+\frac{1}{r_\infty},$$

$$\frac{1}{t_2}=1-\frac{1}{p_m}+\frac{1}{r_S},$$

then the convolution operation $*:\ L^{p(\cdot)}(G)\times\left(L^{t_1}(G)\cap L^{t_2}(G)\right)\to L^{r(\cdot)}(G)\cap L^{r_S}(G)$ is bounded.

Let

$$\lambda_{p(\cdot)}:\ L^{p(\cdot)}(G)\times\left(L^{t_1}(G)\cap L^{t_2}(G)\right)\to \\ \to LB\left(L^{p(\cdot)}(G,\Sigma,\mu)\right)$$

be the left regular representation.

Proposition 4. ***Let*** G ***be a communicative locally compact group, and let*** $f\in L^{p(\cdot)}(G)\times\left(L^{t_1}(G)\cap L^{t_2}(G)\right)$***. Then, we have***

1) $$\left\|\lambda_{p(\cdot)}(f)\right\|_{L^{p(\cdot)}}= \\ =\left\|\lambda_{q(\cdot)}(f)\right\|_{L^{q(\cdot)}},\quad q(g)=\frac{p(g)}{p(g)-1}; \quad (14)$$

2) assume $2\le r(g)\le p(g)\le p_S$ ***or*** $1<p_m\le p(g)\le r(g)\le 2$ ***then*** $\left\|\lambda_{r(\cdot)}(f)\right\|_{L^{r(\cdot)}}\le\left\|\lambda_{p(\cdot)}(f)\right\|_{L^{p(\cdot)}}$.

Preposition 5. ***Let*** G ***be a locally compact commutative group, let*** $p(\cdot)\in P^{\log}(E)$ ***and*** p ***does not identically equal*** 2***. Then, we have two statements:***

1) the Gelfand transform $\Upsilon:\ F_\lambda^{p(\cdot)}(G)\to C_0(G)$ ***is a dense range, injective contractive mapping;***

2) $\Upsilon:\ F_\lambda^{p(\cdot)}(G)\to C_0(G)$ ***is surjective if and only if the group*** G ***is finite.***

Proof. Assume $1<p_m\le p(g)\le r(g)\le 2$ then we have $\|f\|_{L^{r(\cdot)}}\le\|f\|_{L^{p(\cdot)}}$ since $\left\|\lambda_{r(\cdot)}(f)\right\|_{L^{r(\cdot)}}\le\left\|\lambda_{p(\cdot)}(f)\right\|_{L^{p(\cdot)}}$, so there exists a

PROOF
DOI: 10.37394/232020.2025.5.5

contractive homomorphism $F_\lambda^{p(\cdot)}(G) \to F_\lambda^{r(\cdot)}(G)$. Assuming the group G is finite then the homomorphism $F_\lambda^{p(\cdot)}(G) \to F_\lambda^{r(\cdot)}(G)$ is surjection with dense range since $F_\lambda^{r(\cdot)}(G)$ is finite dimensional. Assuming the group G is infinite then $F_\lambda^{p(\cdot)}(G) \to F_\lambda^{r(\cdot)}(G)$ cannot be surjective since even if p is constant different from 2 mapping $F_\lambda^{p(\cdot)}(G) \to F_\lambda^{r(\cdot)}(G)$ is not surjective. If $2 \le r(g) \le p(g) \le p_S$ the arguments are similar.

4. The Takai duality for variable exponential $L^{p(\cdot)}$ operator crossed products

The Pontryagin Duality theory states the existence of canonical isomorphism ev_G between a locally compact commutative group G and its double-dual $\hat{\hat{G}}$ so that $ev_G(g)(\chi) = \chi(g) \in S^1$. Analogously, let (G, A, ω) be a dynamical system, the classical Takai duality theory asserts the existence of an isomorphism Φ between $\hat{G} \times_{\hat{\omega}} (G \times_\omega \mathrm{A})$ and $KL(L^2(G, \Sigma, \mu)) \otimes_{\max} \mathrm{A}$ such that the isomorphism Φ is equivariant for $\hat{\hat{\omega}} : G \to \hat{G} \times_{\hat{\omega}} (G \times_\omega \mathrm{A})$ and $\omega \otimes Ad(\rho) : G \to KL(L^2(G, \Sigma, \mu)) \otimes \mathrm{A}$.

The important issue is to extend the Takai duality theory to variable exponent $L^{p(\cdot)}$- operator crossed products. Let A be a separable unital variable exponent $L^{p(\cdot)}$ operator algebra with variable exponent $L^{p(\cdot)}$ operator matrix norms, and let (G, A, ω) be a variable exponent $L^{p(\cdot)}$ operator algebra dynamical system. We consider the homomorphism

$$\begin{aligned}\Phi^{p(\cdot)} : \quad & F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right) \to \\ & LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}\end{aligned}, \qquad (15)$$

which we present in the form of the composition $\Phi^{p(\cdot)} = \Phi_3 \circ \Phi_2 \circ \Phi_1$ of

$$\begin{aligned}\Phi_1 : \quad & F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right) \to \\ & F^{p(\cdot)}\left(G, F^{p(\cdot)}\left(\hat{G}, \mathrm{A}, \vartheta\right), \hat{\vartheta} \otimes \omega\right)\end{aligned}, \qquad (16)$$

$$\begin{aligned}\Phi_2 : \quad & F^{p(\cdot)}\left(G, F^{p(\cdot)}\left(\hat{G}, \mathrm{A}, \vartheta\right), \hat{\vartheta} \otimes \omega\right) \to \\ & F^{p(\cdot)}\left(G, C_0(G, \mathrm{A}), lt \otimes \omega\right)\end{aligned}, \qquad (17)$$

$$\begin{aligned}\Phi_3 : \quad & F^{p(\cdot)}\left(G, C_0(G, \mathrm{A}), lt \otimes \omega\right) \to \\ & LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}\end{aligned}, \qquad (18)$$

where we denote $\vartheta : \hat{G} \to \mathrm{A}$ a trivial action, and a dual action is given by

$$\hat{\omega} : \quad \hat{G} \to Aut\left(F^{p(\cdot)}(G, \mathrm{A}, \omega)\right).$$

In this case, group G is a discrete communicative and A is a variable exponential $L^{p(\cdot)}$ operator separable algebra with $L^{p(\cdot)}$ operator matrix norm. Then, homomorphisms Φ_1 and Φ_3 are isometric isomorphisms, and Φ_2 is isometric isomorphism if and only if the group G is finite. When the group G is finite, the isometric isomorphism $\Phi^{p(\cdot)} = \Phi_3 \circ \Phi_2 \circ \Phi_1$ is equivariant for the $\hat{\hat{\omega}} : G \to F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right)$ and $\omega \otimes Ad(\rho) : G \to LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}$.

We formulate the main theorem of this paper.

Theorem 1. ***Let G be a discrete communicative group and let A be a variable exponential $L^{p(\cdot)}$ operator separable algebra with $L^{p(\cdot)}$ operator matrix norm. Then, there exists an isometric isomorphism***

$$\begin{aligned}\Phi^{p(\cdot)} : \quad & F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right) \to \\ & LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A},\end{aligned}$$

which is equivariant for actions $\hat{\hat{\omega}} : G \to F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right)$ and

$\omega \otimes Ad(\rho): G \to LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}$ ***if and only if the group*** G ***is finite.***

Proof. We define the homomorphism $\Phi_1(F)(h,\chi) = \chi(h) F(\chi, h)$, which acts from $C_C\left(\hat{G} \times G, \mathrm{A}\right)$ to $C_C\left(G \times \hat{G}, \mathrm{A}\right)$. So defined isomorphism
$\Phi_1: \ C_C\left(\hat{G} \times G, \mathrm{A}\right) \to C_C\left(G \times \hat{G}, \mathrm{A}\right)$ extends to the isometric isomorphism

$$\Phi_1: \ F^{p(\cdot)}\left(\hat{G}, F^{p(\cdot)}(G, \mathrm{A}, \omega), \hat{\omega}\right) \to F^{p(\cdot)}\left(G, F^{p(\cdot)}\left(\hat{G}, \mathrm{A}, \vartheta\right), \hat{\vartheta} \otimes \omega\right).$$

The isomorphism Φ_3 can be presented in the form of the combination $\Phi_3 = \Upsilon_2 \circ \Upsilon_1$, where isometric isomorphism

$$\Upsilon_1: \ F^{p(\cdot)}\left(G, C_0(G, \mathrm{A}), lt \otimes \omega\right) \to F^{p(\cdot)}\left(G, C_0(G, \mathrm{A}), lt \otimes id\right)$$

is given by $\Upsilon_1(F)(g,h) = \omega^{-1}\left(h, F(g,h)\right)$ for all $g, h \in G$ and $F \in C_C(G \times G, \mathrm{A})$; and isometric isomorphism

$$\Upsilon_2: \ F^{p(\cdot)}\left(G, C_0(G, \mathrm{A}), lt \otimes id\right) \to LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}$$

is defined by

$$\Upsilon_2(F)(\zeta)(g) = \int_G F(h, g)\, \zeta\left(h^{-1} g\right) d\mu(h)$$

for all $F \in C_C(G \times G, \mathrm{A})$, $\zeta \in C_C(G) \subset L^{p(\cdot)}(G)$, and all $g \in G$.

Isometric isomorphisms Υ_1 and Υ_2 extend to isomorphism from $F^{p(\cdot)}\left(G, C_0(G, \mathrm{A}), lt \otimes \omega\right)$ to $F^{p(\cdot)}\left(G, C_0(G, \mathrm{A}), lt \otimes id\right)$, and from $F^{p(\cdot)}\left(G, C_0(G, \mathrm{A}), lt \otimes id\right)$ to $LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}$, respectively. The combination $\Upsilon_2 \circ \Upsilon_1$ defines isomorphism from $F^{p(\cdot)}\left(G, C_0(G, \mathrm{A}), lt \otimes \omega\right)$ to $LK\left(l^{p(\cdot)}(G)\right) \otimes_{p(\cdot)} \mathrm{A}$.

Thus, for arbitrary discrete communicative groups G, we obtain that Φ_1 and Φ_3 are isometric isomorphisms. We define a mapping φ_2 from $F \in C_C\left(G \times \hat{G}, \mathrm{A}\right)$ to $C_C\left(G, C_0(G, \mathrm{A})\right)$ given by

$$\varphi_2(F)(g,h) = \int_{\hat{G}} F(g, \chi)\, \overline{\chi(h)}\, d\hat{\mu}(\chi)$$

for all $F \in C_C\left(G \times \hat{G}, \mathrm{A}\right)$.

The Gelfand transform $\Upsilon: \ F_\lambda^{p(\cdot)}(G) \to C_0(G)$ coincides with the Fourier transform on $L^1\left(\hat{G}\right)$, therefore, from proposition 5, we have that in order for the mapping

$$\varphi_2 \times id \equiv \Phi_2: \ F^{p(\cdot)}\left(G, F^{p(\cdot)}\left(\hat{G}, \mathrm{A}, \vartheta\right), \hat{\vartheta} \otimes \omega\right) \to F^{p(\cdot)}\left(G, C_0(G, \mathrm{A}), lt \otimes \omega\right)$$

to be isometric isomorphism it is necessary and sufficient that the group G is finite.

References:

[1] C. Arhancet, *Constructively decomposable projections on noncommutative Lp -spaces*. J. Math. Anal. Appl. 533 (2024), no. 2, Paper No. 128017, 36 pp. 3.
[2] D. Blecher and N. C. Phillips, *Lp operator algebras with approximate identities I*, Pacific J. Math. 303, (2019), no. 2, pp. 401-457.
[3] S. J. Bernau and H. E. Lacey, *The range of a contractive projection on an Lp-space*. Pacific J. Math., 53, 21-41, (1974).
[4] Y. C. Chung and K. Li, *Rigidity of l p Roe-type algebras*, Bull. Lond. Math. Soc. 50 (2018), no. 6, 1056–1070.
[5] I. Doust, *Well-bounded and scalar-type spectral operators on Lp-spaces*. J. London Math. Soc. (2), 39, (3), 525–534, (1989).

[6] I. Doust, *Contractive projections on Lebesgue-Bochner spaces.* In Function spaces, pages 101–109. Dekker, New York, (1995).
[7] E. Gardella, *A modern look at algebras of operators on L p spaces,* Exo. Math. 39 (2021), no. 3, pp. 420-453.
[8] H. Guo, V. Proietti, H. Wang, *A geometric Elliott invariant and noncommutative rigidity of mapping tori,* (2022). arxiv preprint arXiv:2207.06840.
[9] G. Hong, L. Xudong, and X. Bang, *Maximal singular integral operators acting on noncommutative L p-spaces.* Mathematische Annalen 386.1 (2023), pp. 375-414.
[10] G. Hong, L. Wei, and X. Bang, *Quantitative mean ergodic inequalities: Power bounded operators acting on one single noncommutative Lp space.* Journal of Functional Analysis 286, no. 1, (2024), 110190.
[11] N. C. Phillips, *Crossed products of Lp operator algebras and the K-theory of Cuntz algebras on L p spaces*, Preprint (arXiv:1309.6406), (2013).
[12] X.L. Fan, D. Zhao, *On the spaces Lp(x)(Ω) and Wk,p(x) (Ω),* J. Math. Anal. Appl. 263, (2001), pp. 424–446.
[13] X. Li, *K-theory for semigroup C*-algebras and partial crossed products*, Comm. Math. Phys. 390, (2022), no. 1, 1–32.
[14] H. Takai, *On a duality for crossed products of C ∗ -algebras*, J. Funct. Anal. 19, (1975), pp. 25-39.
[15] M. Takesaki, *Duality for crossed products and the structure of von Neumann algebras of type III,* Acta Math. 131, (1973), pp. 249-310.
[16] Z. Wang and Y. Zeng, *Gelfand theory of reduced group L p operator algebra*, Ann. Funct. Anal. 13, 14 (2022).
[17] Z. Wang and S. Zhu, *On the Takai duality for L p operator, crossed products*, Math. Z. 304 (2023), no. 4, Paper no. 54, 23 pp.
[18] J.F. Zhao, *Structure Theory of Banach Spaces*, Wuhan University Press, Wuhan, (1991).
[19] M. Yaremenko, *General Periodic Functions and Generalization of Fourier analysis*, International Journal of Pure Mathematics, vol. 9, pp. 103-109, (2022).
[20] Q. Wang, and W. Zhen, *Notes on the ℓ p-Toeplitz algebra on ℓ p (ℕ).* Israel Journal of Mathematics 245.1, (2021) pp. 153-163.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.