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Adequate Criteria for Strongly Starlikeness and Strongly Convexity of Functions

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Abstract: - T The purpose of this paper is to investigate adequate criteria that ensure the reciprocal of analytic functions to be both strongly starlike and strongly convex within the open unit disk U. Our study leads to the discovery of novel findings.

Key-Words: - Analytic Function, Strongly Starlike Function, Strongly Convex Function.

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1 Introduction

Denote by H the set of analytic functions in the unit disk $U = \{z : |z| < 1\}$, and let $A \subset H$ be the subset of normalized analytic functions f in U such that $f(0) = f'(0) - 1 = 0$. Consider $n \geq 1$,

We define A_n as the set of analytic functions in U of the form

$$f(z) = z + a_{n+1}z^{n+1} \dots \quad (1)$$

For $n \geq 1$. Additionally, we set $A_1 = A$. Consider an element $f(z) \in A$ that satisfies:

$$\left|arg\frac{zf\prime(z)}{f(z)}\right| < \frac{\alpha\pi}{2} \qquad (z \in U). \quad (2)$$

For $0 < \alpha \leq 1$, a function $f(z)$ is considered strongly starlike of order within U. This class of functions is denoted by $\overline{S}^*(\alpha)$, with the special case of $\bar{S}^*(1) = S$. Additionally, if a function $f(z) \in A$ satisfies:

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2}, \qquad (z \in U). \quad (3)$$

According to the definition of $\overline{K}(\alpha)$, for some $0 < \alpha \leq 1$, $f(z)$ is a strongly convex function of order α in U. Keep in mind that $\overline{K}(1) = K$ and that $f(z) \in \overline{K}(\alpha)$ if $zf'(z) \in \overline{S}^*(\alpha)$.

It is obvious that the scenario (2) is similar to

$$\frac{zf\prime(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \quad (4)$$

where $\prec$ denotes the usual subordination.

Each of the aforementioned classes is a subclass of the univalent functions in U.

Numerous studies have produced numerous findings about the necessary requirements for functions $f(z)$ that are analytic in U to be starlike, convex, highly starlike, and strongly convex functions. (See [1], [6], [8], [9], [10], [12], [15]).

Here, we introduce a requirement that must be met in order for $f \in A_n$ to satisfy:

$$f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1 + \lambda z, \quad 0 < \mu < 1, 0 < \lambda < 1, \quad (5)$$

is in $S(\alpha)$. We also take an integrated transformation into account.

2 Several necessary conditions for strongly starlikeness

It can be noted that the values for in (5) were originally established by the author in [11], implying starlikeness in U. Subsequently, Ponnusamy and Singh [14] identified the condition that characterizes strongly starlikeness of order α (excluding $\mu < 0$) in (5).

We take strongly starlikeness into account in the case (5) using a similar methodology as in [14].

The following lemma is the first one we cite.

Lemma 1. Consider $\varrho \in H$ satisfying the subordination condition.

$$Q(z) \prec 1 + \lambda_1 z, \qquad Q(0) = 1, \quad (6)$$

We have $0 < \lambda_1 \leq 1$. For $0 < \alpha \leq 1$, let $p\epsilon H, p(0) = 1$ and p satisfies the following condition:

$$Q(z)p^{\alpha}(z) \prec 1 + \lambda z, \quad 0 < \lambda \leq 1. \quad (7)$$

Then for

$$\sin^{-1}\lambda + \sin^{-1}\lambda_1 \leq \frac{\alpha\pi}{2}, \quad (8)$$

Consequently, we have $Re\{p(z)\} > 0$ in U.
The lemma mentioned in [14] is more general and includes this specific case.
The following two lemmas are crucial for our findings to be comprehensive.

Lemma 2. Assume that $p \in H$, $p(z) = 1 + p_n z^n + \cdots, n \geq 1$, satisfy the requirement

$$p(z) - \frac{1}{\mu} zp(z) \prec 1 + \lambda z, \quad 0 < \mu < 1, 0 < \lambda \leq 1 \tag{9}$$

Then

$$p(z) \prec 1 + \lambda_1 z, \tag{10}$$

where

$$\lambda_1 = \frac{\lambda\mu}{n-\mu}. \tag{11}$$

The demonstration of this lemma is presented in [11] for n=1. Alternatively, Jack's lemma [3] can be utilized for any $n \in N$.

Lemma 3. If $0 < \mu < 1, 0 < \lambda \leq 1$, and $Q \in H$ satisfy

$$Q(z) \prec 1 + \frac{\lambda\mu}{n-\mu} z, \qquad Q(0) = 1, \quad n \in N, \tag{12}$$

and if $p\epsilon H$, $p(0) = 1$, and satisfies

$$Q(z)p^{\alpha}(z) \prec 1 + \lambda z, \tag{13}$$

Such that

$$0 < \lambda \leq \frac{(n-\mu)\sin\left(\frac{\pi\alpha}{2}\right)}{\left|\mu + (n-\mu)e^{\frac{i\pi\alpha}{2}}\right|}, \tag{14}$$

then $\mathrm{Re}\{p(z)\} > 0$ is achieved in U.
Proof: λ_(1)= λμ/((n-μ)) in Lemma (1), that the condition (8) is equivalent to
Proof: By setting $\lambda_1 = \frac{\lambda\mu}{(n-\mu)}$ in Lemma (1), it can be shown then the condition (8) is equivalent to

$$\sin^{-1}\lambda + \sin^{-1}\frac{\lambda\mu}{(n-\mu)} \leq \frac{\alpha\pi}{2}. \tag{15}$$

This disparity is equal to or similar to the disparity

$$\sin^{-1}\left(\lambda\sqrt{1 - \frac{\lambda^2\mu^2}{(n-\mu)^2}} + \frac{\lambda\mu}{(n-\mu)}\sqrt{1-\lambda^2}\right) \leq \frac{\alpha\pi}{2}, \tag{16}$$

or to the inequality:

$$\lambda\left[\sqrt{(n-\mu)^2 - \lambda^2\mu^2} + \mu\sqrt{1-\lambda^2}\right] \leq (n - \mu)\sin\left(\frac{\alpha\pi}{2}\right). \tag{17}$$

After some adjustments, we arrive at the following equivalent inequality

$$\left\{[\mu^2 + (n-\mu)^2]^2 - 4\mu^2(n-\mu)^2 cos^2\left(\frac{\alpha\pi}{2}\right)\right\}\lambda^4 - 2(n-\mu)^2[\mu^2 + (n-\mu)^2]sin^2\left(\frac{\alpha\pi}{2}\right)\lambda^2 + (1-\mu)^4 sin^4\left(\frac{\alpha\pi}{2}\right) \geq 0, \tag{18}$$

which is equivalent to the condition (14).
$\mathrm{Re}\{p(z)\} > 0$ in U is shown by Lemma (1).

Theorem 1. Suppose that $f \in A_n, 0 < \mu < 1$ and f is met by the subordination:

$$f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1 + \lambda z, \tag{19}$$

where

$$0 < \lambda \leq \frac{n-\mu}{\sqrt{\mu^2 + (n-\mu)^2}}. \tag{20}$$

Then $f \in S^*$.

Proof. If we set $\varrho(z) = \left(\frac{z}{f(z)}\right)^{\mu} = (1 + q_n z^n + \cdots)$, then after some calculations, we get

$$Q(z) - \frac{1}{\mu} zQ'(z) = f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1 + \lambda z. \tag{21}$$

Lemma (2) gives us

$$Q(z) \prec 1 + \lambda_1 z,\ \lambda_1 = \frac{\lambda\mu}{n-\mu}. \tag{22}$$

We skip the specifics of the remainder of the proof because it is comparable to the case where $n = 1$ (see [11, Theorem1]).

Theorem 2. Let $0 < \mu < 1$, and , $0 < \alpha \leq 1$. If $f \in A_n$ satisfies

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} - 1\right| < \frac{(n-\mu)\sin\left(\frac{\alpha\pi}{2}\right)}{\left|\mu + (n-\mu)e^{\frac{i\pi\alpha}{2}}\right|}, (z \in U), \tag{23}$$

then $f \in S(\alpha)$.

Proof. If we put $\lambda = \frac{(n-\mu)\sin\left(\frac{\alpha\pi}{2}\right)}{\left|\mu + (n-\mu)e^{\frac{i\pi\alpha}{2}}\right|}$, $0 < \alpha \leq 1$ then, we have

$0 < \lambda \leq \frac{(n-\mu)}{\sqrt{\mu^2 + (n-\mu)^2}}$, and by theorem 2, $f \in S^*$.

Later, the function $Q(z) = \left(\frac{z}{f(z)}\right)^{\mu} = 1 + q_n z^n + \cdots$ is analytical in U and meets:

$$Q(z) \prec 1 + \lambda_1 z, \lambda_1 = \frac{\lambda\mu}{(n-\mu)}.$$

If we were to define

$$p(z) = \left(\frac{zf\prime(z)}{f(z)}\right)^{\frac{1}{\alpha}}, \tag{24}$$

Since $p(0) = 1$ and p is analytical in U, condition (23) is equivalent to

$$Q(z)p^{\alpha}(z) \prec 1 + \lambda z. \tag{25}$$

Lemma (2) finally leads to the conclusion

$$\left(\frac{zf\prime(z)}{f(z)}\right)^{\frac{1}{\alpha}} \prec \frac{1+z}{1-z}\left(\Leftrightarrow \frac{zf\prime(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}\right),$$

that is, $f \in S(\alpha)$.
We observe that the statement of Theorem 2 is present when $\alpha = 1$.
The following corollary is obtained for $n = 1, \mu = \frac{1}{2}, \alpha = \frac{2}{3}$.

Corollary 1. Let $f \in A$ and let

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{3/2}-1\right|<\frac{1}{2}, \quad (z \in U). \quad (26)$$

Then

$$\left|arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{3}, z \in U, \quad (27)$$

Consequently, $f \in S(2/3)$.

Theorem 3. Let $0<\mu<1$, $\operatorname{Re}\{c\}>-\mu$, and $0<\alpha \leq 1$. If $f \in A_n$ fulfilled

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}-1\right|< \left|\frac{n+c-\mu}{c-\mu}\right|\frac{(n-\mu)\sin\left(\frac{\alpha\pi}{2}\right)}{\left|\mu+(n-\mu)e^{i\pi\alpha/2}\right|}, (z \in U), \quad (28)$$

then the function

$$F(z)=z\left[\frac{c-\mu}{z^{c-\mu}}\int_0^z\left(\frac{t}{f(t)}\right)^{\mu}t^{c-\mu-1}\,dt\right]^{-1/\mu}, \quad (29)$$

$S(\alpha)$ is the owner.

Proof. Assuming that λ is equal to the right-hand side of the aforementioned inequality (28), the following argument holds.

$$Q(z)=F'(z)\left(\frac{z}{F(z)}\right)^{1+\mu}=(1+q_n+z^n+\cdots), \quad (30)$$

then from (28) and (29), we obtain

$$Q(z)+\frac{1}{c-\mu}zQ'(z)=f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu} \prec 1+\lambda z. \quad (31)$$

As a result, we can conclude utilizing Hallenbeck and Ruscheweyh [3, Theorem1], we have that

$$Q(z) \prec 1+\lambda_1 z, \lambda_1=\frac{|(c-\mu)\lambda|}{|n+c-\mu|}=\frac{(n-\mu)\sin\left(\frac{\pi\alpha}{2}\right)}{\left|\mu+(n-\mu)e^{i\pi\alpha/2}\right|}, \quad (32)$$

and Theorem 2's easy application yields the desired outcome.

Remark 1. We have the similar conclusion from [11] for $\alpha=1$ and $n=1$.
we have $c=\mu+1$ for this.

Corollary 2. Let $0<\mu<1, 0<\alpha \leq 1$. If $f \in A_n$ meets the prerequisite:

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{1+\mu}-1\right|< \left|\frac{n+c-\mu}{c-\mu}\right|\frac{(n-\mu)\sin\left(\frac{\alpha\pi}{2}\right)}{\left|\mu+(n-\mu)e^{i\pi\alpha/2}\right|}, (z \in U), \quad (33)$$

after that, the action the function

$$F(z)=z\left[\frac{c-\mu}{z^{c-\mu}}\int_0^z\left(\frac{t}{f(t)}\right)^{\mu}t^{c-\mu-1}\,dt\right]^{-1/\mu} \quad (34)$$

pertaining to $S(\alpha)$.

3 New sufficient conditions for the reciprocal of the strongly starlikeness and strongly convex functions.

The new required requirements for substantially starlikeness and strongly convex functions that Frasin [2] obtained using a different reciprocal formula are discussed in this section. Additionally, we'll analyze in a way that suggests that $f(z)$ belongs to one of the classes $\overline{S}^*(\alpha)$ and $\overline{K}(\alpha)$ previously described in terms of the reciprocal formula.
We will require the following lemmas in order to demonstrate our key findings.

Lemma 4. ([15]) if $f(z) \in A$ is satisfied

$$|f'(z)-1|<2a\sqrt{\frac{5-4\sqrt{1-a^2}}{16a^2+9}} \quad (z \in U), \quad (35)$$

then where
$a=\sin(\alpha\pi/2)$, $0<\alpha \leq 1$, then $f(z) \in \overline{S}^*(\alpha)$.

Lemma 5. ([5]) If $f(z) \in A_n$, is satisfied

$$|f'(z)-1|<\frac{(1-\alpha)(n+1)}{\alpha+\sqrt{(n+1)^2+1}}, \quad (z \in U) \quad (36)$$

where $0<\alpha \leq 1$, then $f(z) \in \overline{S}^*(\alpha)$.

Lemma 6. ([13]) If $f(z) \in A_n$, is satisfied

$$|f'(z)+\alpha z f''(z)-1|<\frac{(1-\alpha)(n+1)}{\alpha+\sqrt{(n+1)^2+1}}, \quad (z \in U)$$

where $\alpha>2$, then $f(z) \in K$.
Using the same approach as Nunokawa et al. [9], we prove the following.

Theorem 4. If $f(z) \in A$ satisfies

$$\left|\frac{1}{f''(z)}\right| \leq \frac{1}{2a}\sqrt{\frac{16a^2+9}{5-4\sqrt{1-a^2}}}, \quad (z \in U;\ 0<\alpha<1), \quad (37)$$

where
$a=\sin(\alpha\pi/2)$, $0<\alpha<1$, After that $f(z) \in \overline{S}^*(\alpha)$.

Proof. Noting that

$$\left|\frac{1}{f'(z)-1}\right|=\left|\frac{1}{\int_0^z f''(\sigma)d\sigma}\right| \leq \frac{1}{\int_0^{|z|}|f''(te^{i\theta})dt|}$$

$$\leq \frac{1}{2a}\sqrt{\frac{16a^2+9}{5-4\sqrt{1-a^2}}}\ \frac{1}{\int_0^{|z|}dt}$$

$$=\frac{1}{2a}\sqrt{\frac{16a^2+9}{5-4\sqrt{1-a^2}}}\ \frac{1}{|z|}<\frac{1}{2a}\sqrt{\frac{16a^2+9}{5-4\sqrt{1-a^2}}}$$

As a result, we deduce from Lemma 4 that $f(z) \in \bar{S}^*(\alpha)$.

Corollary 3. Let $f(z) \in A, z \in U$. Then

1. $|f"(z)| \le \sqrt{\frac{13}{(5-2\sqrt{3})}} = 2.9093129113$ implies $f(z) \in \bar{S}^*(1/2)$;
2. $|f"(z)| \le \frac{3}{2}\sqrt{\frac{17}{(10-4\sqrt{2})}} = 2.9676557273$ implies $f(z) \in \bar{S}^*(1/3)$; and
3. $|f"(z)| \le \frac{3}{4}\sqrt{\frac{145/9}{5-4\left(\frac{\sqrt{5}}{3}\right)}} =$ 2.1188560554 implies $f(z) \in \bar{S}^*(2/3)$.

Now, we derive

Theorem 5. If $f(z) \in A$ satisfies

$$\left|\frac{1}{f"(z)}\right| \le \frac{1}{a}\sqrt{\frac{16a^2+9}{5-4\sqrt{1-a^2}}} \quad (z \in U;\ 0 < \alpha < 1), \tag{38}$$

where

$a = \sin(\alpha\pi/2)$, $0 < \alpha < 1$, Then $f(z) \in \bar{K}(\alpha)$.

Proof. Thus, it follows

$$\begin{aligned}\left|\frac{1}{(zf'(z))' - 1}\right| &= \left|\frac{1}{f'(z) + zf"(z) - 1}\right| \\ &\le \left|\frac{1}{f'(z) - 1}\right| + \left|\frac{1}{zf"(z)}\right| \\ &\le \int_0^{|z|} \left|\frac{1}{f"(t)dt}\right| + \frac{1}{a}\sqrt{\frac{16a^2+9}{5-4\sqrt{1-a^2}}}\ \frac{1}{|z|} \\ &\le \frac{1}{2a}\sqrt{\frac{16a^2+9}{5-4\sqrt{1-a^2}}}\frac{1}{|z|} < \frac{1}{a}\sqrt{\frac{16a^2+9}{5-4\sqrt{1-a^2}}}\ .\end{aligned}$$

As a result, by applying Lemma (4), we can see that $zf'(z) \in \bar{S}^*(\alpha)$, or $f(z) \in \bar{K}(\alpha)$.

Corollary 4. Assume that$f(z) \in A, z \in U$.

1. $\left|\frac{1}{f"(z)}\right| \le 2\sqrt{\frac{13}{(5-2\sqrt{3})}} = 5.8186258226$ implies $f(z) \in \bar{K}(1/2)$;
2. $\left|\frac{1}{f"(z)}\right| \le 3\sqrt{\frac{17}{(10-4\sqrt{2})}} = 5.9353114545$ implies $f(z) \in \bar{K}(1/3)$; and
3. $\left|\frac{1}{f"(z)}\right| \le \frac{3}{2}\sqrt{\frac{145/9}{5-4\left(\frac{\sqrt{5}}{3}\right)}} =$ 4.2377121107 implies $f(z) \in \bar{K}(2/3)$.

By utilizing the same technique as in the proof of Theorem (4) and replacing Lemma 4 with Lemma 5, we can establish the following theorem.

Theorem 6. If $f(z) \in A_n$, satisfies

$$\left|\frac{1}{f'(z)}\right| \le \frac{\alpha+\sqrt{(n+1)^2+1}}{(1-\alpha)(n+1)}\ , \quad (z \in U; 0 < \alpha < 1), \tag{39}$$

then $f(z) \in \bar{S}^*(\alpha)$.

In Theorem 6, if we assume that $\alpha = 0$, we get

Corollary 5. If $f(z) \in A_n$, then

$\left|\frac{1}{f'(z)}\right| \le \frac{\sqrt{(n+1)^2+1}}{(n+1)}$, $(z \in U)$, $f(z) \in S^*$.

Are satisfied.

We conclude by proving.

Theorem 7. $f(z) \in A_n$, is satisfies if

$$\left|\frac{1}{f"(z)}\right| < \frac{\alpha(\alpha+1)(n+1)}{(\alpha-2)(n\alpha+1)}, \quad (z \in U), \tag{40}$$

where $\alpha > 2$, then $f(z) \in K$.

Proof. We have

$$\begin{aligned}\left|\frac{1}{f'(z) + \alpha zf"(z) - 1}\right| &\le \left|\frac{1}{f'(z) - 1}\right| + \frac{1}{\alpha|zf"(z)|} \\ &\le \int_0^{z} \left|\frac{1}{f"(t)dt}\right| + \frac{1}{\alpha|zf"(z)|} \\ &\le \int_0^{|z|} \frac{1}{|f"(t)dt|} + \frac{\alpha(\alpha+1)(n+1)}{(\alpha-2)(n\alpha+1)}\ \frac{1}{|z|} \\ &\le \frac{(\alpha+1)(n+1)}{(\alpha-2)(n\alpha+1)}\ \frac{1}{|z|} < \frac{\alpha(n+1)}{(\alpha-2)(n\alpha+1)}\,.\end{aligned}$$

Using Lemma 6, we have $f(z) \in K$.

By allowing $\alpha \to \infty$ in Theorem 7, Mocanu [7] was able to arrive at the following conclusion.

Corollary 6. If $f(z) \in A_n$, satisfied

$$\left|\frac{1}{f"(z)}\right| \le 1 + \frac{1}{n}, \quad (z \in U), \tag{41}$$

so's $f(z) \in K$.

We get the following result from Obradovic [12] if we assume that $n = 1$ in Corollary 6.

Corollary 7. If $f(z) \in A$, satisfied

$$\left|\frac{1}{f"(z)}\right| < 2, \quad (z \in U), \tag{42}$$

so's $f(z) \in K$.

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