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Computational method to find solutions of mathematical problems with error and stability analysis

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Abstract: This paper provides stability theorems and residual correction procedures for a realistic optimization problem using an accurate computational method. An example illustration has shown optimal performance with high-accuracy orders.

Key-Words: error analysis, stability analysis, B-Polynomials, operational matrices

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1 Introduction

One of the most significant computational techniques is the Bernstein polynomial method. The main strategy of this method is to transform every problem term into a matrix form. The fractional Bernstein method is a generalization of the standard Bernstein method in writing. Numerous authors have used the Bernstein polynomial method to solve significant problems in applied mathematics and physics.; see, [1], [2], [3]. Fractional integral equations have received a lot of attention (FIDEs) for intensity, physical system, [4], radio astronomy, [5], and dynamical system, [6]. This paper will investigate stability and error correction procedures while solving a class of integral differential equations using the Bernstein polynomial method.

This paper examines a class of fractional order integro-differential equations (FIDEs) and discusses some of them.

$$y^{\alpha}(x) = y(x) + \mu_1 \int_0^x g(x,t) y^{(\alpha)}(t)dt + h(x) \tag{1}$$

under the initial condition

$$y^{(\alpha)}(\epsilon) = y_i. \quad n-1 < \alpha \leq n, \quad n \in N. \tag{2}$$

2 Method of Solutions

To create the method, let us use the operational matrices of derivatives with applications. The objective is to approximate the solution of the problem as the Bernstein series solution, which can be accomplished by using the operational matrix of differentiation.

$$y(x) = \mathbf{B}_n(x)\mathbf{C} \tag{3}$$

$\mathbf{C}$ is the matrix of unknown coefficients. It is possible to express the method solution and its fractional derivative using similar arguments as

$$y(x) = \mathbf{X}(x)\mathbf{D}^{\mathbf{T}}\mathbf{C}. \tag{4}$$

then

$$D^{\alpha}y(x) = \left[D^{\alpha}\mathbf{X}(x)\right]\mathbf{D}^{\mathbf{T}}\mathbf{C}, \tag{5}$$

The same way. According to Caputo's definition, the relationship between the matrix $\mathbf{X}(x)$ and its derivative $D^{\alpha}[\mathbf{X}(x)]$ can be presented as

$$D^{\alpha}[\mathbf{X}(x)] = \begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{1-\alpha} & \frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha} & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha} \end{bmatrix}. \quad (6)$$

The relation (6) can be presented as

$$D^{\alpha}[\mathbf{X}(x)] = [1 \quad x \quad x^2 \ldots x^n] \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{-\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{-\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{-\alpha} \end{pmatrix} \quad (7)$$

from the matrix form (7) we have

$$D^{\alpha}[\mathbf{X}(x)] = \mathbf{X}(x)\psi(x), \quad (8)$$

Thus, we are able to write the relationship in (5) as

$$y^{(\alpha)}(x) = \mathbf{X}(x)\psi(x)\mathbf{D}^{\mathbf{T}}\mathbf{C}, \quad (9)$$

For the part $\mu_1 \int_0^x g(x,t)u^{(\alpha)}(t)dt$ using formula, [7].

$$\mu_1 \int_0^x g(x,t)y^{(\alpha)}(t)dt = \mu_1 V_x \psi(x)\mathbf{D}^{\mathbf{T}}\mathbf{C}, \quad (10)$$

Substituting the relations in (9) and (10) into Eq. (1) The significant matrix equation is derived.

$$\left[\mathbf{X}(x)\psi(x)\mathbf{D}^{\mathbf{T}} - \mathbf{X}(x)\mathbf{D}^{\mathbf{T}} - \mu_1 V_x \psi(x)\mathbf{D}^{\mathbf{T}}\right]\mathbf{C} = \mathbf{H}(x). \quad (11)$$

By substituting collocation points $\{x_i : 0 \leq i \leq n\}$ into Eq. (11), the matrix $\mathbf{W}_{(n+1)\times(n+1)}$ will be obtained. Hence, the relation matrix in (11) become

$$\left[\mathbf{X}\psi\mathbf{D}^{\mathbf{T}} - \mathbf{X}\mathbf{D}^{\mathbf{T}} - \mu_1 V\psi\mathbf{D}^{\mathbf{T}}\right]\mathbf{C} = \mathbf{H}, \quad (12)$$

it is able to formulate (12) as the basic matrix form provided by

$$\mathbf{W}\mathbf{C} = \mathbf{H}, \quad (13)$$

where

$$\mathbf{W} = \mathbf{X}\psi\mathbf{D}^{\mathbf{T}} - \mathbf{X}\mathbf{D}^{\mathbf{T}} - \mu_1 V\psi\mathbf{D}^{\mathbf{T}}. \quad (14)$$

Then

$$\mathbf{C} = (\mathbf{W})^{-1}\mathbf{H}$$

The initial conditions can be written as corresponding matrix forms as follow

$$\mathbf{X}(\delta)\mathbf{D}^{\mathbf{T}}\mathbf{C} = [y_i], \quad 0 \leq \delta \leq R, \quad i = 0, 1,$$

2.1 Stability analysis of method and residual analysis

In this section, we will constitute the stability estimation related to the linear systems see, [8].

To construct the residual correction procedure for the problem, Let R_n be defined as follows.

$$\mathrm{R}_n(x) := y_n^{\alpha}(x) - y_n(x) - \mu_1 \int_0^x g(x,t)y_n^{(\alpha)}(t)dt.$$

Then, adding and subtracting the term R_n from Eq. (1) gives the following problem for the absolute error

$$e_n^{\alpha}(x) = e_n(x) + \mu_1 \int_0^x g(x,t)e_n^{(\alpha)}(t)dt + h(x) \quad (15)$$

where $e_n = y - y_n$ with the following initial condition

$$e_n^{(\alpha)}(\delta) = 0. \quad (16)$$

See, [8].

3 Results and discussion in numbers

Example 1.

The fractional integro differential equation will be examined, [9].

$$y^{(0.75)}(x) + \frac{1}{5}x^2 e^x u(x) - \int_0^x e^x t u(t)dt = \frac{6x^{2.25}}{\Gamma(3.25)} \quad (17)$$

The initial condition is

$$y(0) = 0.$$

The exact solution of this problem is

$$y(x) = x^3.$$

By applying the technique in Section 2, with n = 3, the fundamental matrices for Eq. (17) are obtained as

$$C=\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{W} = \begin{pmatrix} -0.5205 & -0.7962 & -1.4405 & 1.3748 \\ -0.6009 & -0.7085 & 0.5194 & 1.2883 \\ -1.7617 & 0.1926 & 0.7150 & 0.845 \\ 2.1568 & 1.0257 & 0.1671 & 0.0015 \end{pmatrix}, \quad \mathbf{H} = \begin{bmatrix} 2.1568 \\ 1.0257 \\ 0.1671 \\ 0.0015 \end{bmatrix}.$$

By applying all the matrices founded above in relation (4):

$$y(x) = \mathbf{X}(x)\mathbf{D}^{\mathbf{T}}\mathbf{C}. \tag{18}$$

we have

$$y(x) = x^3.$$

which is the exact solution.

Our method produced an exact solution, whereas the methods used dealt with example 1 had been found to include errors (see, [9]). In this case, using the method with different values of $n = 2, 4$ caused a few errors in our results. As a result, we improved our results using an error correction procedure. (see Fig. 1) and stability tests to validate the stability of the method. Table 1 shows the findings.

	$n = 2$	$n = 3$	$n = 4$
$cond(W)$	8.84	17	25.10
$\|\Delta A\|$	0	0	1.15×10^{-16}
$\|A\|$	10^{-16}	10^{-16}	4.68×10^{-11}
$\|\Delta G\|$	10^{-16}	10^{-16}	10^{-16}
$\|G\|$	2.01	2.15	2.22
Upper Bound	1.54×10^{-15}	6.30×10^{-15}	1.64×10^{-14}
$\left\|u_n - u_n^p\right\|$	1.0×10^{-16}	2.0×10^{-16}	3.0×10^{-16}

Table 1: Stability results of the system obtained by present method for example 1.

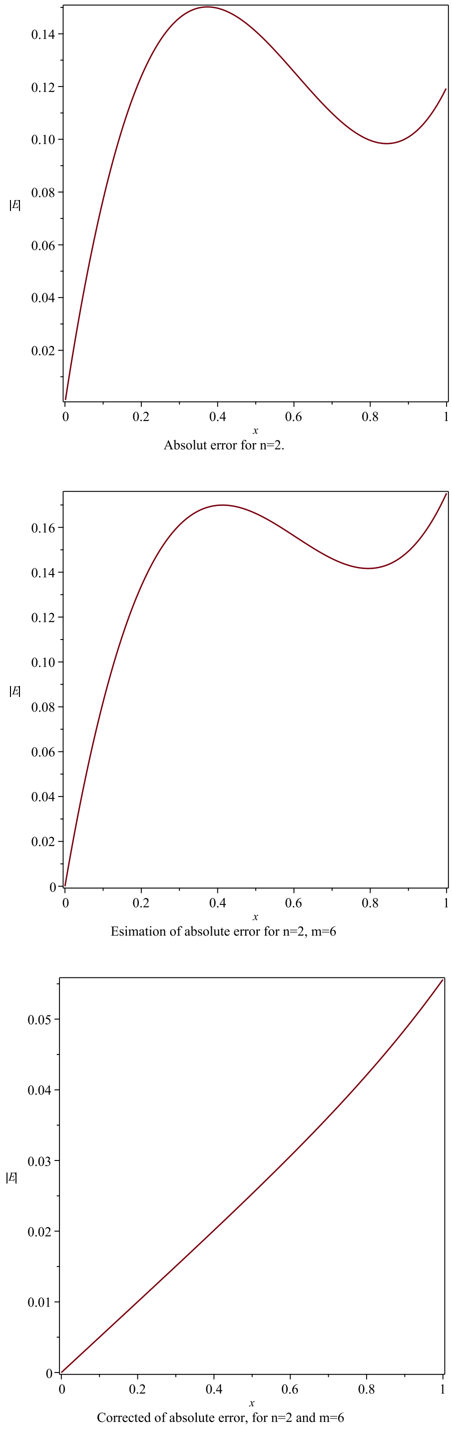

Figure 1: Error correction procedure for example 1, with $n = 2$ and $m = 6$.

4 Conclusions

In this work, we suggest a numerical method to solve a kind of fractional-order integro-differential equations by using operational matrices based on B-polynomials. To study the stability results based on the techniques and estimate the absolute error, we presented the residual correction procedure for the methods. We also tested the effectiveness of the suggested approaches in a few examples. The numerical experiments show that theoretical predictions and numerical results agree very well.

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Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.