Hyperbolic (s,t)-Fibonacci and (s,t)-Lucas Quaternions

HAKAN AKKUS[1], ENGIN OZKAN[2]
[1]Department of Mathematics,
Graduate School of Natural and Applied Sciences,
Erzincan Binali Yıldırım University,
Erzincan
TÜRKIYE

[2]Department of Mathematics,
Faculty of Sciences,
Marmara University,
İstanbul
TÜRKIYE

Abstract: - In this study, we define hyperbolic (s,t)-Fibonacci and (s,t)-Lucas quaternions. For these hyperbolic quaternions, we give the special summation formulas, special generating functions, etc. Also, we calculate the special identities of these hyperbolic quaternions. In addition, we obtain the Binet formulas in two different ways. The first is in the known classical way and the second is with the help of the sequence's generating functions. Moreover, we examine the relationships between the hyperbolic (s,t)-Fibonacci and (s,t)-Lucas quaternions. Finally, the terms of the (s,t)-Fibonacci and (s,t)-Lucas sequences are associated with their hyperbolic quaternion values.

Key-Words: - (s,t) sequence, Fibonacci numbers, Hyperbolic Quaternions, Generating Function, Cassini Identity, Summation formula

Received: April 24, 2024. Revised: September 15, 2024. Accepted: October 16, 2024. Published: November 14, 2024.

1 Introduction

The Fibonacci, Lucas, and Pell sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci sequences have been applied in various fields such as Algebraic Coding Theory [1], [2], graph theory [3], [4], Biomathematics [5], Computer Science [6], and so on. Many generalizations of the Fibonacci sequence have been given. The known examples of such sequences are the Pell, Pell-Lucas, k-Fibonacci, k-Jacobsthal-Lucas, k-Lucas, k-Pell, k-Pell-Lucas, and Modified k-Pell sequences, etc (see for details in [7], [8], [9], [10], [11]).

For $n \in \mathbb{N}$, the Fibonacci numbers F_n, Lucas numbers L_n, Pell numbers p_n, and Jacobsthal-Lucas numbers q_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n,\ L_{n+2} = L_{n+1} + L_n,$$
$$p_{n+2} = 2p_{n+1} + p_n, \text{ and } q_{n+2} = 2q_{n+1} + q_n$$

with the initial conditions $F_0 = 0, F_1 = 1, L_0 = 2, L_1 = 1, p_0 = 0, p_1 = 1$, and $q_0 = 2, q_1 = 2$.

For F_n, L_n, p_n, and q_n the Binet formulas are given by relations, respectively,

$$F_n = \frac{\varphi^n - \omega^n}{\varphi - \omega},\ L_n = \varphi^n + \omega^n,$$
$$p_n = \frac{\lambda^n - \psi^n}{\lambda - \psi}, \text{ and } q_n = \lambda^n + \psi^n$$

where $\varphi = \frac{1+\sqrt{5}}{2}$, $\omega = \frac{1-\sqrt{5}}{2}$, $\lambda = 1+\sqrt{2}$, and $\psi = 1-\sqrt{2}$ are the roots of the characteristic equation $s^2 - s - 1 = 0$ and $v^2 - 2v - 1 = 0$, respectively. Here φ and λ numbers are the known golden ratio and silver ratio.

With the help of the recurrence relation of the Fibonacci sequence, (s,t)-sequences were introduced, and these sequences had an important place in number theory.

In [12], [13], for $n \in \mathbb{N}$, they defined the (s,t)-Fibonacci and (s,t)-Lucas sequences by the recurrence relations, respectively,

$$F_{n+2} = sF_{n+1} + tF_n,$$

$$L_{n+2} = sL_{n+1} + tL_n,$$
with the initial conditions $L_0 = 2$, $L_1 = s$ and $F_0 = 0$, $F_1 = 1$. In addition, they found the Binet formulas and properties of these sequences. In [14], they did applied work on the matrix representations of (s,t) Fibonacci sequences. Also, in [15], she defined the (s,t)-Jacobsthal and (s,t)-Jacobsthal-Lucas sequences and she found the generating function, the Binet formulas and some features of these sequences.
The quaternions were first described by Hamilton in 1843. Then, quaternions used to control rotational movements especially in 3D games and Eulerian angles. In [16], he defined Complex Fibonacci and Fibonacci quaternions and he were found various features of these sequences.
The algebra of hyperbolic quaternions is an algebra that is not related to the elements of the form over the real numbers.
$$q = xi_1 + yi_2 + zi_3 + ti_4, x, y, z, t \in \mathbb{R}$$
In [16], he gave the properties of the q units defined as in Table 1.

$\cdot$	$\boldsymbol{i_1}$	$\boldsymbol{i_2}$	$\boldsymbol{i_3}$	$\boldsymbol{i_4}$
$\boldsymbol{i_1}$	i_1	i_2	i_3	i_4
$\boldsymbol{i_2}$	i_2	i_1	i_4	$-i_3$
$\boldsymbol{i_3}$	i_3	$-i_4$	i_1	i_2
$\boldsymbol{i_4}$	i_4	i_3	$-i_2$	i_1

Table 1. Hyperbolic Quaternions Units

In [17], he did a lot of research on hyperbolic quaternions and their properties. An expression of the general form of hyperbolic quaternions is
$$\hbar = \hbar_1 i_1 + \hbar_2 i_2 + \hbar_3 i_3 + \hbar_4 i_4 = (\hbar_1, \hbar_2, \hbar_3, \hbar_4).$$
Here, the terms of the $\hbar_1, \hbar_2, \hbar_3, \hbar_4$ sequence, $i_1, i_2, i_3 i_4$ are hyperbolic quaternions.
In [18], he defined the hyperbolic k-Fibonacci and k-Lucas quaternions and he found properties of these quaternions. Also, they conducted a study on the hyperbolic Leonardo and Francois quaternions and obtained many features related to these quaternions [19]. In addition, they introduced the Jacobsthal and Jacobsthal-Lucas quaternions [20]. Moreover, they did many studies on hyperbolic quaternions,octonions, and sedenions [21], [22], [23], [24], [25] [26], [27] [28], [29], [30].
As seen above, many generalizations of hyperbolic quaternions of sequences have been given so far. In this study, we give new generalizations inspired by the hyperbolic k-Fibonacci quaternions and Jacobsthal and Jacobsthal-Lucas quaternions. We call these quaternions the hyperbolic (s,t)-Fibonacci and (s,t)-Lucas quaternions and denote them as $\breve{H}F_n(s,t)$, and $\breve{H}L_n(s,t)$, respectively.
We separate the article into three parts.
In chapter 2, we define the hyperbolic (s,t)-Pell and (s,t)-Pell-Lucas quaternions, and the terms of these quaternions are given. Then, we find some properties of these quaternions.
In chapter 3, information is given about the characteristic equations of hyperbolic (s,t)-Fibonacci and (s,t)-Lucas quaternions. Then, we obtain the Binet formulas, generating functions, and sum of terms of these quaternions. In addition, we examine the relationship of hyperbolic (s,t)-Fibonacci and (s,t)-Lucas quaternions. Moreover, we calculate the special identities of these quaternions. Finally, we associate the terms of the (s,t)-Fibonacci and (s,t)-Lucas sequences with their hyperbolic quaternion values.

2 Hyperbolic (s,t)-Fibonacci and (s,t)-Lucas Quaternions

For $n \in \mathbb{N}$, the hyperbolic (s,t)-Fibonacci $\breve{H}F_n(s,t)$ and (s,t)-Lucas quaternions $\breve{H}L_n(s,t)$ are defined by, respectively,
$$\breve{H}F_n(s,t) = F_n i_1 + F_{n+1} i_2 + F_{n+2} i_3 + F_{n+3} i_4 = (F_n, F_{n+1}, F_{n+2}, F_{n+3})$$
and
$$\breve{H}L_n(s,t) = L_n i_1 + L_{n+1} i_2 + L_{n+2} i_3 + L_{n+3} i_4 = (L_n, L_{n+1}, L_{n+2}, L_{n+3})$$
where F_n is n^{th} (s,t)-Fibonacci sequence, L_n n^{th} (s,t)-Lucas sequence and i_1, i_2, i_3 and i_4 are the hyperbolic quaternion units in table 1.
Let us now give some terms of the hyperbolic (s,t)-Fibonacci and (s,t)-Lucas quaternions below.

- $\breve{H}F_0(s,t) = i_2 + si_3 + (s^2+t)i_4$,
- $\breve{H}F_1(s,t) = i_1 + si_2 + (s^2+t)i_3 + (s^3 + 2st)i_4$,
- $\breve{H}F_2(s,t) = si_1 + (s^2+t)i_2 + (s^3+2st)i_3 + (s^4 + 3s^2t + t^2)i_4$,

and

- $\breve{H}L_0(s,t) = 2i_1 + si_2 + (s^2+2t)i_3 + (s^3 + 3st)i_4$,
- $\breve{H}L_1(s,t) = si_1 + (s^2+2t)i_2 + (s^3 + 3st)i_3 + (s^4 + 4s^2t + 2t^2)i_4$,
- $\breve{H}L_2(s,t) = (s^2+2t)i_1 + (s^3+3st)i_2 + (s^4 + 4s^2t + 2t^2)i_3 + (s^5 + 5s^3t + 5st^2)i_4$.

Definition 2.1. For $n \in \mathbb{N}$, the conjugate of (s,t)-Fibonacci $\breve{H}F_n^*(s,t)$ and (s,t)-Lucas $\breve{H}L_n^*(s,t)$ quaternions are defined by, respectively,
$$\breve{H}F_n^*(s,t) = F_n i_1 - F_{n+1} i_2 - F_{n+2} i_3 - F_{n+3} i_4 = (F_n, -F_{n+1}, -F_{n+2}, -F_{n+3})$$
and

$$\breve{H}L_n^*(s,t) = L_n i_1 - L_{n+1} i_2 - L_{n+2} i_3 - L_{n+3} i_4$$
$$= (L_n, -L_{n+1}, -L_{n+2}, -L_{n+3}).$$

Definition 2.2. For $n \in \mathbb{N}$, the norms of the hyperbolic (s,t)-Fibonacci $\breve{H}F_n(s,t)$ and (s,t)-Lucas $\breve{H}L_n(s,t)$ quaternions are defined by, respectively,

$$\|\breve{H}F_n\| = \sqrt{F_n^2 + F_{n+1}^2 + F_{n+2}^2 + F_{n+3}^2}$$

and

$$\|\breve{H}L_n\| = \sqrt{L_n^2 + L_{n+1}^2 + L_{n+2}^2 + L_{n+3}^2}.$$

3 Properties of Hyperbolic (s,t)-Fibonacci and (s,t)-Lucas Quaternions

In this chapter, the relationships between the hyperbolic (s,t)-Fibonacci and (s,t)-Lucas quaternions are examined. In addition, some identities are obtained.

Theorem 3.1. For $n \in \mathbb{N}$, the hyperbolic (s,t)-Fibonacci and (s,t)-Lucas quaternions provide the following recurrence relations.

i. $\breve{H}L_n = \breve{H}F_{n+1} + t\breve{H}F_{n-1}$,
ii. $\breve{H}F_{n+2} = s\breve{H}F_{n+1} + t\breve{H}F_n$,
iii. $\breve{H}L_{n+2} = s\breve{H}L_{n+1} + t\breve{H}L_n$,
iv. $\breve{H}F_{n+2}^* = s\breve{H}F_{n+1}^* - t\breve{H}F_n^*$,
v. $\breve{H}L_{n+2}^* = s\breve{H}L_{n+1}^* + t\breve{H}L_n^*$,
vi. $\breve{H}L_n^* = \breve{H}F_{n+1}^* + t\breve{H}F_{n-1}^*$.

Proof. If the definition of the hyperbolic function is used, we have

i. $\breve{H}F_{n+1} + t\breve{H}F_{n-1} = F_{n+1}i_1 + F_{n+2}i_2 + F_{n+3}i_3$
$+F_{n+4}i_4 + t(F_{n-1}i_1 + F_n i_2 + F_{n+1}i_3 + F_{n+2}i_4)$
$= (F_{n+1} + tF_{n-1})i_1 + (F_{n+2} + tF_n)i_2$
$+(F_{n+3} + tF_{n+1})i_3 + (F_{n+4} + tF_{n+2})i_4.$

Since, $L_n = F_{n+1} + tF_{n-1}$.
Thus, we obtain

$$\breve{H}L_n = \breve{H}F_{n+1} + t\breve{H}F_{n-1}.$$

The proofs of the others can be given in the same way. □

In the following theorem, the Binet formulas of the $\breve{H}F_n$, $\breve{H}F_n{}^*$, $\breve{H}L_n$, and $\breve{H}L_n{}^*$ quaternions are expressed.

Theorem 3.2. Let $n \in \mathbb{N}$. We obtain

i. $\breve{H}F_n = \frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{\alpha-\beta}$, **ii.** $\breve{H}L_n = \overline{\alpha}\alpha^n + \overline{\beta}\beta^n$,

iii. $\breve{H}F_n{}^* = \frac{\overline{\gamma}\alpha^n - \overline{\theta}\beta^n}{\alpha-\beta}$, **iv.** $\breve{H}L_n{}^* = \overline{\gamma}\alpha^n + \overline{\theta}\beta^n$

where

$\overline{\alpha} = i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4 = (1, \alpha, \alpha^2, \alpha^3)$,
$\overline{\gamma} = (1, -\alpha, -\alpha^2, -\alpha^3)$,
$\overline{\beta} = i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4 = (1, \beta, \beta^2, \beta^3)$,
$\overline{\theta} = (1, -\beta, -\beta^2, -\beta^3)$.

Proof. i. With the help of the characteristic equation, the following results are obtained.

$$r^2 - sr - t = 0, r_1 = \alpha = \frac{s+\sqrt{s^2+4t}}{2},$$
$$r_2 = \beta = \frac{s-\sqrt{s^2+4t}}{2}, \alpha + \beta = s,$$

$\alpha - \beta = \delta = \sqrt{s^2+4t}, \alpha^2 + \beta^2 = s^2 + 2t$ and $\alpha\beta = -t$.

The Binet form of the hyperbolic (s,t)-Fibonacci quaternions is

$$\breve{H}F_n = x\alpha^n + y\beta^n.$$

With the initial conditions, the following equations are obtained.

$\breve{H}F_0 = i_2 + si_3 + (s^2+t)i_4 = (0, 1, s, s^2+t)$
$= x + y$

and

$\breve{H}F_1 = i_1 + si_2 + (s^2+t)i_3 + (s^3+2st)i_4$
$= x\alpha + y\beta.$

Thus, we obtain

$$x = \frac{\breve{H}F_1 - \beta\breve{H}F_0}{\alpha-\beta} = \frac{i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4}{\alpha-\beta} = \frac{\overline{\alpha}}{\alpha-\beta}$$

and

$$y = \frac{\breve{H}F_1 - \alpha\breve{H}F_0}{\beta-\alpha} = \frac{i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4}{-(\alpha-\beta)} = \frac{-\overline{\beta}}{\alpha-\beta}.$$

So, we have

$$\breve{H}F_n = \frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{\alpha-\beta}.$$

The proofs of the others can be given in the same way. □

In the next Lemma, we obtain the properties that will be used in the proof of many theorems.

Lemma 3.1. We have

i. $\overline{\alpha} - \overline{\beta} = \delta\breve{H}F_0$,
ii. $\overline{\alpha} + \overline{\beta} = \breve{H}L_0$,
iii. $\overline{\alpha}\overline{\beta} = (-t^3 + t^2 - t + 1, -\delta t^2 + s, s^2$
$+2t - \delta st,\ \delta t + 3st + s^3)$,
iv. $\overline{\beta}\overline{\alpha} = (-t^3 + t^2 - t + 1, \delta t^2 + s, s^2 + 2t$
$+\delta st, -\delta t + 3st + s^3)$
v. $\overline{\alpha}\overline{\beta} + \overline{\beta}\overline{\alpha}$
$= (-2t^3 + 2t^2 - 2t + 2, 2s, 2s^2 + 4t, 6st + 2s^3)$
$= 2\breve{H}L_0 - 2t^3 + 2t^2 - 2t$,
vi. $\overline{\beta}\overline{\alpha} - \overline{\alpha}\overline{\beta} = (0, 2t^2\delta, 2\delta st, -2\delta t)$
$= 2\delta t(0, t, s, -1)$,
vii. $\overline{\alpha}^2 = 2\overline{\alpha} - 1 + \alpha^2 + \alpha^4 + \alpha^6$,
viii. $\overline{\beta}^2 = 2\overline{\beta} - 1 + \beta^2 + \beta^4 + \beta^6$,
ix. $\overline{\alpha}^2 - \overline{\beta}^2 = \delta(2\breve{H}F_0 + F_2 + F_4 + F_6)$,
x. $\overline{\alpha}^2 + \overline{\beta}^2 = 2\breve{H}L_0 - 2L_0 + L_2 + L_4 + L_6$.

Proof. iii. If definition is used, we have

$\overline{\alpha}\overline{\beta} = (i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4)$
$(i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4)$
$= i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4 + \alpha i_2 - t i_1 - t\beta i_4$
$-t\beta^2 i_3 + \alpha^2 i_3 - t\alpha i_4 + t^2 i_1$

$+t^2\beta i_2+\alpha^3 i_4-t\alpha^2 i_3+t^2\alpha i_2-t^3 i_1$
$=(-t^3+t^2-t+1)i_1+(-\delta t^2+s)i_2$
$+(s^2+2t-\delta st)i_3+(\delta t+3st+s^3)i_4$
Thus, we obtain

$$\overline{\alpha}\overline{\beta}=(-t^3+t^2-t+1,-\delta t^2+s,s^2+2t-\delta st,\ \delta t+3st+s^3).$$

The proofs of the others can be given in the same way. □

In the following theorems, we study the summation formulas for these hyperbolic quaternions.

Theorem 3.3. Let $n\in\mathbb{N}$. We obtain

i. $S\breve{H}F_n=\sum_{j=0}^{n}\breve{H}F_j$
$=\frac{1}{t+s-1}[t\breve{H}F_{n-1}+(s+t)\breve{H}F_n+i_1+i_2-si_4]$,

ii. $S\breve{H}L_n=\sum_{j=0}^{n}\breve{H}N_j$
$=\frac{1}{t+s-1}[(s+t)\breve{H}L_n+t\breve{H}L_{n-1}+(2-s)i_1$
$-si_2-si_3-(s^2 2t+2s)i_4]$.

Proof. i. Using the definition, we have
$S\breve{H}F_n=\sum_{j=0}^{n}\breve{H}F_j=i_1\sum_{j=0}^{n}F_j+$
$i_2\sum_{j=0}^{n}F_{j+1}+i_3\sum_{j=0}^{n}F_{j+2}+i_4\sum_{j=0}^{n}F_{j+3}$.
Since $\sum_{j=0}^{n}F_j=\frac{1+tF_{n-1}+(s+t)F_n}{t+s-1}$. We get
$S\breve{H}F_n=\frac{1}{t+s-1}[(1+tF_{n-1}+(s+t)F_n)i_1+(1+$
$tF_n+(s+t)F_{n+1})i_2+(tF_{n+1}+(s+t)F_{n+2})i_3$
$+(tF_{n+1}+(s+t)F_{n+2})i_3$
$+\big((tF_{n+2}+(s+t)F_{n+3}-s)\big)i_4$
So, we obtain
$S\breve{H}F_n$
$=\frac{1}{t+s-1}[t\breve{H}F_{n-1}+(s+t)\breve{H}F_n+i_1+i_2-si_4]$.
The proof of the other can be given in the same way. □

Theorem 3.4. Let $k\in\mathbb{R}$ and $x,y,z\in\mathbb{N}$. We obtain

i. $\sum_{y=0}^{n}\breve{H}F_{xy}=\frac{(-t)^x\breve{H}F_{nx}+\breve{H}F_0-\breve{H}F_{nx+x}+\breve{H}L_0F_x-\breve{H}F_x}{(-t)^x-L_x+1}$,

ii. $\sum_{y=0}^{n}\breve{H}L_{xy}=\frac{(-t)^x\breve{H}L_{nx}+\breve{H}L_0-\breve{H}F_{nx+x}-\breve{H}L_0F_x-\breve{H}F_x}{(-t)^x-L_x+1}$,

iii. $\sum_{y=0}^{n}\breve{H}F_{xy+z}=$

$$\begin{cases}\frac{(-t)^x\breve{H}F_{nx+z}-\breve{H}F_{nx+x+z}-\breve{H}F_z-(-t)^z\breve{H}F_{x-z}}{(-t)^x-L_x+1}, & if\ z<x\\ \frac{(-t)^x\breve{H}F_{nx+z}-\breve{H}F_{nx+x+z}-\breve{H}F_z-(-t)^x\breve{H}F_{z-x}}{(-t)^x-L_x+1}, & othervise\end{cases},$$

iv. $\sum_{y=0}^{n}\breve{H}L_{xy+z}=$

$$\begin{cases}\frac{(-t)^x\breve{H}L_{nx+z}-\breve{H}L_z-\breve{H}L_{nx+x+z}-(-t)^z\breve{H}L_{x-z}}{(-t)^x-L_x+1}, & if\ z<x\\ \frac{(-t)^x\breve{H}L_{nx+z}-\breve{H}L_z-\breve{H}L_{nx+x+z}-(-t)^x\breve{H}L_{z-x}}{(-t)^x-L_x+1}, & othervise\end{cases}.$$

Proof. With the help of definitions, Binet formulas and geometric series, we have

i. $\sum_{y=0}^{n}\breve{H}F_{xy}=\sum_{n=0}^{\infty}\frac{\overline{\alpha}\alpha^{xy}-\overline{\beta}\beta^{xy}}{\alpha-\beta}$

$$=\frac{\overline{\alpha}}{\alpha-\beta}\sum_{y=0}^{n}(\alpha^x)^y-\frac{\overline{\beta}}{(\alpha-\beta)k}\sum_{y=0}^{n}(\beta^x)^y$$

$$=\frac{1}{\alpha-\beta}\Big(\frac{\overline{\alpha}\alpha^{nx+x}-\overline{\alpha}}{\alpha^x-1}-\frac{\overline{\beta}\alpha^{nx+x}-\overline{\beta}}{\beta^x-1}\Big)$$

$$=\frac{1}{\alpha-\beta}\frac{\overline{\alpha}\alpha^{nx+x}\beta^x-\overline{\alpha}\beta^x-\overline{\alpha}\alpha^{nx+x}+\overline{\alpha}-\overline{\beta}\beta^{nx+x}\alpha^x+\overline{\beta}\alpha^x+\overline{\beta}\beta^{nx+x}-\overline{\beta}}{(-t)^x-L_x+1}.$$

Thus, we get

$$\sum_{y=0}^{n}\breve{H}F_{xy}=\frac{(-t)^x\breve{H}F_{nx}+\breve{H}F_0-\breve{H}F_{nx+x}+\breve{H}L_0F_x-\breve{H}F_x}{(-t)^x-L_x+1}.$$

The proofs of the others can be given in the same way. □

Corollary 3.1. We obtain

i. $\sum_{y=0}^{n}\breve{H}F_{2y}=\frac{t^2\breve{H}F_{2n}+\breve{H}F_0-\breve{H}F_{2n+2}+\breve{H}L_0F_2-\breve{H}F_2}{(-t)^x-L_x+1}$,

ii. $\sum_{y=0}^{n}\breve{H}L_{2y}=\frac{t^2\breve{H}L_{2n}+\breve{H}L_0-\breve{H}F_{2n+2}-\breve{H}L_0F_2-\breve{H}F_2}{(-t)^x-L_x+1}$,

iii. $\sum_{y=0}^{n}\breve{H}F_{2y+1}=\frac{t^2\breve{H}F_{2n+1}-\breve{H}F_{2n+3}-\breve{H}F_1-t^2\breve{H}F_1}{(-t)^x-L_x+1}$,

iv. $\sum_{y=0}^{n}\breve{H}L_{2y+1}=\frac{t^2\breve{H}L_{2n+1}-\breve{H}L_1-\breve{H}L_{2n+3}-t^2\breve{H}L_1}{(-t)^x-L_x+1}$.

Proof. i. Let $x=2$ is taken in the relation given by Theorem 3.4. i. We obtain

$$\sum_{y=0}^{n}\breve{H}F_{xy}=\sum_{y=0}^{n}\breve{H}F_{2y}$$
$$=\frac{(t)^2\breve{H}F_{2n}+\breve{H}F_0-\breve{H}F_{2n+2}+\breve{H}L_0F_2-\breve{H}F_2}{(-t)^x-L_x+1}.$$

The proofs of the others are shown similarly to i, using the Theorem 3.4. □

Theorem 3.5. (Generating Functions) The generating functions for hyperbolic (s,t)-Fibonacci and (s,t)-Lucas quaternions are given as follows, respectively,

$$f(x)=\sum_{n=0}^{\infty}\breve{H}F_nx^n=\frac{(1-sx)\breve{H}F_0+\breve{H}F_1}{1-sx-t^2}$$

and

$$g(x)=\sum_{n=0}^{\infty}\breve{H}L_nx^n=\frac{\breve{H}L_0+x\breve{H}L_1-sx\breve{H}L_0}{1-sx-t^2}.$$

Proof. The following equations are written for the hyperbolic (s,t)-Fibonacci sequence.
$f(x)=\sum_{n=0}^{\infty}\breve{H}F_nx^n$
$=\breve{H}F_0+\breve{H}F_1x+\sum_{n=2}^{\infty}\breve{H}F_nx^n$
$=\breve{H}F_0+\breve{H}F_1x+s\sum_{n=2}^{\infty}\breve{H}F_{n-1}x^n$
$+t\sum_{n=2}^{\infty}\breve{H}F_{n-2}x^n$
$=\breve{H}F_0+\breve{H}F_1x+sx\big(-\breve{H}F_0+\sum_{n=0}^{\infty}\breve{H}F_nx^n\big)$
$+tx^2\sum_{n=0}^{\infty}\breve{H}F_nx^n$.
Thus, we have

$$f(x)=\frac{(1-sx)\breve{H}F_0+\breve{H}F_1}{1-sx-t^2}.$$

The proof of the other can be given in the same way. □

In the following theorems, special generating functions for these hyperbolic quaternions are studied. In addition, the Binet formulas are obtained with the help of generating functions.

Theorem 3.6. For $a,b\in\mathbb{N}^+$, $b\geq a$ and $n\in\mathbb{N}$, we obtain

i. $\sum_{n=0}^{\infty}\breve{H}F_{an}x^n=\frac{\breve{H}F_0+(\breve{H}L_0F_a-\breve{H}F_a)x}{1-xL_a+(-t)^ax^2}$,

ii. $\sum_{n=0}^{\infty}\breve{H}L_{an}x^n=\frac{\breve{H}L_0+(-\breve{H}L_0L_a+\breve{H}L_a)x}{1-xL_a+(-t)^ax^2}$,

iii. $\sum_{n=0}^{\infty}\breve{H}F_{an+b}x^n=\frac{\breve{H}F_b-x(-t)^a\breve{H}F_{b-a}}{1-xL_a+(-t)^a x^2}$,

iv. $\sum_{n=0}^{\infty}\breve{H}L_{an+b}x^n=\frac{\breve{H}L_b+x(-t)^a\breve{H}L_a}{1-xL_a+(-t)^a x^2}$,

v. $\sum_{n=0}^{\infty}\frac{\breve{H}F_{bn}}{n!}x^n=\frac{\overline{\alpha}e^{\alpha^b x}-\overline{\beta}e^{\beta^b x}}{\alpha-\beta}$,

vi. $\sum_{n=0}^{\infty}\frac{\breve{H}L_{bn}}{n!}x^n=\overline{\alpha}e^{\alpha^b x}+\overline{\beta}e^{\beta^b x}$.

Proof. With the help of definitions, Binet formulas and geometric series, we have

i. $\sum_{n=0}^{\infty}\breve{H}F_{an}x^n=\sum_{n=0}^{\infty}\frac{\overline{\alpha}\alpha^{an}-\overline{\beta}\beta^{an}}{\alpha-\beta}x^n$

$$=\frac{\overline{\alpha}}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^a)^n x^n-\frac{\overline{\beta}}{\alpha-\beta}\sum_{n=0}^{\infty}(\beta^a)^n x^n$$

$$=\frac{1}{\alpha-\beta}\left(\frac{\overline{\alpha}}{1-\alpha^a x}-\frac{\overline{\beta}}{1-\beta^a x}\right)$$

$$=\frac{\frac{\overline{\alpha}-\overline{\beta}}{\alpha-\beta}+[(\overline{\alpha}+\overline{\beta})\left(\frac{\alpha^a-\beta^a}{\alpha-\beta}\right)-\frac{\overline{\alpha}\alpha^a-\overline{\beta}\beta^a}{\alpha-\beta}]x}{1-xL_a+(-t)^a x^2}.$$

Thus, we get

$$\sum_{n=0}^{\infty}\breve{H}F_{an}x^n=\frac{\breve{H}F_0+(\breve{H}L_0F_a-\breve{H}F_a)x}{1-xL_a+(-t)^a x^2}.$$

The proofs of the others can be given in the same way. □

Theorem 3.7. For $\breve{H}F_n$ and $\breve{H}L_n$ quaternions, the Binet formulas can be obtained with the help of the generating functions.

Proof. With the help of the roots of the characteristic equation of these quaternions, the roots of the $1-sx-tx^2=0$ equation become $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. For $\breve{H}F_n$ quaternions, we obtain

$$\frac{(1-sx)\breve{H}J_0+\breve{H}J_1x}{1-sx-tx^2}=\frac{1}{\alpha-\beta}\overline{\alpha}\frac{1}{1-\alpha x}-\frac{1}{\alpha-\beta}\overline{\beta}\frac{1}{1-\beta x}$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{\alpha-\beta}\overline{\alpha}\alpha^n-\frac{1}{\alpha-\beta}\overline{\beta}\beta^n\right)x^n$$

$$=\sum_{n=0}^{\infty}\breve{H}F_n x^n.$$

Similarly, the Binet formula of the $\breve{H}L_n$ quaternions is found. □

In the following theorems we calculate special identities for these hyperbolic quaternions.

Theorem 3.8. (Cassini Identity) For all $n\geq 1$,

i. $\breve{H}F_{n+1}\breve{H}F_{n-1}-\breve{H}F_n{}^2$
$=(-t)^{n-1}((t^2+1)(t-1),s(1-t^2)$
$,(-s^2-2t-s^2t),(s^3-2st))$,

ii. $\breve{H}L_{n+1}\breve{H}L_{n-1}-\breve{H}L_n{}^2$
$=(-t)^{n-1}\delta^2((t^2+1)(1-t)$
$,s(1-t^2),(s^2+2t-s^2t),s^3+4st)$.

Proof. i. With the Binet formula, we get

$\breve{H}F_{n+1}\breve{H}F_{n-1}-\breve{H}F_n{}^2$

$$=\left(\frac{\overline{\alpha}\alpha^{n+1}-\overline{\beta}\beta^{n+1}}{\alpha-\beta}\right)\left(\frac{\overline{\alpha}\alpha^{n-1}-\overline{\beta}\beta^{n-1}}{\alpha-\beta}\right)-\left(\frac{\overline{\alpha}\alpha^{n}-\overline{\beta}\beta^{n}}{\alpha-\beta}\right)^2$$

$$=\frac{\overline{\alpha}\overline{\beta}\alpha^n\beta^n+\overline{\beta}\overline{\alpha}\beta^n\alpha^n-\overline{\beta}\overline{\alpha}\beta^{n-1}\alpha^{n+1}-\overline{\alpha}\overline{\beta}\alpha^{n-1}\beta^{n+1}}{(\alpha-\beta)^2}$$

$$=\frac{\overline{\alpha}\overline{\beta}(-t)^n\left(\frac{\alpha-\beta}{\alpha}\right)+\overline{\beta}\overline{\alpha}(-t)^n\left(\frac{\beta-\alpha}{\beta}\right)}{(\alpha-\beta)^2}=\frac{(-t)^n}{\alpha-\beta}\frac{\beta\overline{\alpha}\overline{\beta}-\alpha\overline{\beta}\overline{\alpha}}{\alpha\beta}$$

$$=\frac{(-t)^{n-1}}{\alpha-\beta}(\beta\overline{\alpha}\overline{\beta}-\alpha\overline{\beta}\overline{\alpha}).$$

Then, we have

$\beta\overline{\alpha}\overline{\beta}-\alpha\overline{\beta}\overline{\alpha}$
$=(-t)^{n-1}(-(-t^3+t^2-t+1)\delta,s\delta(1-t^2),\delta(-s^2-2t-s^2t),\delta s(s^3-2st))$.

So, we obtain

$\breve{H}F_{n+1}\breve{H}F_{n-1}-\breve{H}F_n{}^2$
$=(-t)^{n-1}((t^2+1)(t-1),s(1-t^2)$
$,(-s^2-2t-s^2t),(s^3-2st))$,

The proof of the other can be given in the same way. □

Theorem 3.9. (Catalan Identity) For natural numbers n and r, we have

i. $\breve{H}F_{n+r}\,\breve{H}F_{n-r}-\breve{H}F_n{}^2$
$=(-t)^{n-r}F_r((t^2+1)(t-1)F_r,-sF_r-t^2L_r$
$,(-s^2-2t)F_r-stL_r,(-s^3-3st)F_r+tL_r)$,

ii. $\breve{H}L_{n+r}\,\breve{H}L_{n-r}-\breve{H}L_n{}^2$
$=(-t)^{n-r}\delta^2F_r\big((t^2+1)(1-t)F_r,-t^2L_r+sF_r,(s^2+2t)F_r-stL_r,(s^3+3st)F_r+tL_r\big)$.

Proof. i. With the Binet formula, we have

$\breve{H}F_{n+r}\,\breve{H}F_{n-r}-\breve{H}F_n{}^2$

$$=\left(\frac{\overline{\alpha}\alpha^{n+r}-\overline{\beta}\beta^{n+r}}{\alpha-\beta}\right)\left(\frac{\overline{\alpha}\alpha^{n-r}-\overline{\beta}\beta^{n-r}}{\alpha-\beta}\right)-\left(\frac{\overline{\alpha}\alpha^{n}-\overline{\beta}\beta^{n}}{\alpha-\beta}\right)^2$$

$$=\frac{\overline{\alpha}\overline{\beta}\alpha^n\beta^n+\overline{\beta}\overline{\alpha}\beta^n\alpha^n-\overline{\beta}\overline{\alpha}\beta^{n-r}\alpha^{n+r}-\overline{\alpha}\overline{\beta}\alpha^{n-r}\beta^{n+r}}{(\alpha-\beta)^2}$$

$$=\frac{\overline{\alpha}\overline{\beta}(-t)^n\left(\frac{\alpha^r-\beta^r}{\alpha^r}\right)+\overline{\beta}\overline{\alpha}(-t)^n\left(\frac{\beta^r-\alpha^r}{\beta^r}\right)}{(\alpha-\beta)^2}$$

$$=\frac{(-t)^n}{(\alpha-\beta)^2}\frac{\beta^r\overline{\alpha}\overline{\beta}-\alpha^r\overline{\beta}\overline{\alpha}}{(\alpha\beta)^r}$$

$$=\frac{(-t)^{n-r}F_r}{\alpha-\beta}(\beta^r\overline{\alpha}\overline{\beta}-\alpha^r\overline{\beta}\overline{\alpha}).$$

So, we can write

$\beta^r\overline{\alpha}\overline{\beta}-\alpha^r\overline{\beta}\overline{\alpha}=(t^2+1)(t-1)(\beta^r-\alpha^r)i_1$
$+[-\delta t^2(\alpha^r+\beta^r)-s(\alpha^r-\beta^r)]i_2$
$+[-s^2(\alpha^r-\beta^r)-2t(\alpha^r-\beta^r)-\delta st(\alpha^r$
$+\beta^r)]i_3+[\,\delta t(\alpha^r+\beta^r)-3st(\alpha^r-\beta^r)$
$-s^3(\alpha^r-\beta^r)]i_4$.

Thus, we get

$\breve{H}F_{n+r}\,\breve{H}F_{n-r}-\breve{H}F_n{}^2$
$=(-t)^{n-r}F_r((t^2+1)(t-1)F_r,-sF_r-t^2L_r,(-s^2-2t)F_r-stL_r,(-s^3-3st)F_r+tL_r)$.

The proof of the other can be given in the same way. □

Theorem 3.10. (Vajda's Identity) For natural numbers i and j, we obtain

i. $\breve{H}F_{n+i}\breve{H}F_{n+j}-\breve{H}F_n\breve{H}F_{n+i+j}$
$=(-t)^nF_i((t^2+1)(1-t)F_j,-sF_j+t^2L_j,(s^2+2t)F_j+stL_j,(s^3+3st)F_j-tL_j)$,

ii. $\breve{H}L_{n+i}\breve{H}L_{n+j}-\breve{H}L_n\breve{H}L_{n+i+j}$

$= (-t)^n\delta^2 F_i\Big((t^2+1)(t-1)F_j, -sF_j -$
$t^2L_j, (-s^2-2t\Big)F_j - stL_j, (-s^3-3st)F_j + tL_j).$

Proof. i. With the Binet formula, we get

$\breve{H}F_{n+i}\breve{H}F_{n+j} - \breve{H}F_n\breve{H}F_{n+i+j}$
$= \frac{\overline{\alpha}\alpha^{n+i}-\overline{\beta}\beta^{n+i}}{\alpha-\beta})\left(\frac{\overline{\alpha}\alpha^{n}-\overline{\beta}\beta^{n}}{\alpha-\beta}\right)$
$-(\frac{\overline{\alpha}\alpha^{n}-\overline{\beta}\beta^{n}}{\alpha-\beta})(\frac{\overline{\alpha}\alpha^{n+i+j}-\overline{\beta}\beta^{n+i+j}}{\alpha-\beta})$
$= \frac{\overline{\alpha}\overline{\beta}\alpha^n\beta^{n+i+j}+\overline{\beta}\overline{\alpha}\beta^n\alpha^{n+i+j}-\overline{\beta}\overline{\alpha}\beta^{n+i}\alpha^{n+j}-\overline{\alpha}\overline{\beta}\alpha^{n+i}\beta^{n+j}}{(\alpha-\beta)^2}$
$= \frac{-\overline{\alpha}\overline{\beta}\alpha^n\beta^{n+j}(\alpha^i-\beta^i)+\overline{\beta}\overline{\alpha}\alpha^{n+j}\beta^n(\alpha^i-\beta^i)}{(\alpha-\beta)^2}$
$= \frac{(-t)^nF_i}{\alpha-\beta}(-\beta^j\overline{\alpha}\overline{\beta} + \alpha^j\overline{\beta}\overline{\alpha}).$

So, we can write

$-\beta^j\overline{\alpha}\overline{\beta} + \alpha^j\overline{\beta}\overline{\alpha}$
$= (t^2+1)(1-t)(\alpha^j-\beta^j)i_1$
$+[\delta t^2(\alpha^j+\beta^j) + s(\alpha^j-\beta^j)]i_2$
$+[(s^2+2t)(\alpha^j-\beta^j) + \delta st(\alpha^r+\beta^r)]i_3$
$+[-\delta t(\alpha^j+\beta^j) + (3st+s^3)(\alpha^j-\beta^j)]i_4.$

Thus, we have

$\breve{H}F_{n+i}\breve{H}F_{n+j} - \breve{H}F_n\breve{H}F_{n+i+j}$
$= (-t)^nF_i((t^2+1)(1-t)F_j, -sF_j +$
$t^2L_j, (s^2+2t)F_j + stL_j, (s^3+3st)F_j - tL_j).$

The proof of the other can be given in the same way. □

Theorem 3.11. (D'ocagne Identity) For natural numbers n, m, and $n+1 \le m$, we have

i. $\breve{H}F_m\breve{H}F_{n+1} - \breve{H}F_{m+1}\breve{H}F_n$
$= (-t)^n[((t^2+1)(1-t)F_{m-n}, sF_{m-n}$
$-t^2L_{m-n}, (s^2+2t)F_{m-n} - stL_{m-n}$
$, (s^3+3st)F_{m-n} + tL_{m-n})],$

ii. $\breve{H}L_m\breve{H}L_{n+1} - \breve{H}L_{m+1}\breve{H}L_n$
$= (-t)^n\delta^2[((t^2+1)(t-1)F_{m-n}, -sF_{m-n}$
$+t^2L_{m-n}, -(s^2+2t)F_{m-n}$
$+stL_{m-n}, -(s^3+3st)F_{m-n} - tL_{m-n})].$

Proof. i. Binet formulas are used for proofs. We obtain

$\breve{H}F_m\breve{H}F_{n+1} - \breve{H}F_{m+1}\breve{H}F_n$
$= \left(\frac{\overline{\alpha}\alpha^{m}-\overline{\beta}\beta^{m}}{\alpha-\beta}\right)\left(\frac{\overline{\alpha}\alpha^{n+1}-\overline{\beta}\beta^{n+1}}{\alpha-\beta}\right)$
$-(\frac{\overline{\alpha}\alpha^{m+1}-\overline{\beta}\beta^{m+1}}{\alpha-\beta})(\frac{\overline{\alpha}\alpha^{n}-\overline{\beta}\beta^{n}}{\alpha-\beta})$
$= \frac{\overline{\alpha}\overline{\beta}\alpha^{m+1}\beta^n+\overline{\beta}\overline{\alpha}\beta^{m+1}\alpha^n-\overline{\beta}\overline{\alpha}\beta^m\alpha^{n+1}-\overline{\alpha}\overline{\beta}\alpha^m\beta^{n+1}}{(\alpha-\beta)^2}$
$= \frac{\alpha^n\beta^n(\alpha^{m-n}\overline{\alpha}\overline{\beta}-\beta^{m-n}\overline{\beta}\overline{\alpha})}{\alpha-\beta}.$

So, we have

$\breve{H}F_m\breve{H}F_{n+1} - \breve{H}F_{m+1}\breve{H}F_n$
$= (-t)^n[((t^2+1)(1-t)F_{m-n}, sF_{m-n}$
$-t^2L_{m-n}, (s^2+2t)F_{m-n}$
$-stL_{m-n}, (s^3+3st)F_{m-n} + tL_{m-n})].$

The proof of the other can be given in the same way. □

In the following theorems we examine the relationships between these hyperbolic quaternions.

Theorem 3.12. For any integer $n \le m$, we obtain

i. $\breve{H}F_m\breve{H}F_n - \breve{H}F_n\breve{H}F_m$
$= (-t)^nF_{m-n}(0, -2t^2, -2st, 2t),$

ii. $\breve{H}L_m\breve{H}L_n - \breve{H}L_n\breve{H}L_m$
$= (-t)^n\delta^2F_{m-n}(0, 2t^2, +2st, -2t).$

Proof. i. Using the Binet Formula, we have

$\breve{H}F_m\breve{H}F_n - \breve{H}F_n\breve{H}F_m$
$= (\frac{\overline{\alpha}\alpha^{m}-\overline{\beta}\beta^{m}}{\alpha-\beta}\frac{\overline{\alpha}\alpha^{n}-\overline{\beta}\beta^{n}}{\alpha-\beta})\ (\frac{\overline{\alpha}\alpha^{n}-\overline{\beta}\beta^{n}}{\alpha-\beta}\frac{\overline{\alpha}\alpha^{m}-\overline{\beta}\beta^{m}}{\alpha-\beta})$
$= \frac{\overline{\alpha}\overline{\beta}(\alpha^n\beta^m-\alpha^m\beta^n)-\overline{\beta}\overline{\alpha}(-\beta^n\alpha^m+\beta^m\alpha^n)}{(\alpha-\beta)^2}$
$= \frac{\alpha^n\beta^n(\alpha^{m-n}-\beta^{m-n})(\overline{\alpha}\overline{\beta}-\overline{\beta}\overline{\alpha})}{(\alpha-\beta)^2}.$

Thus, we obtain

$\breve{H}F_m\breve{H}F_n - \breve{H}F_n\breve{H}F_m$
$= (-t)^nF_{m-n}(0, -2t^2, -2st, 2t).$

The proof of the other can be given in the same way. □

Lemma 3.2. We have

i. $\alpha^{2i} = \frac{F_{2i}}{s}\alpha\delta - t\frac{L_{2i-1}}{s},$

ii. $\beta^{2i} = -\frac{F_{2i}}{s}\beta\delta - t\frac{L_{2i-1}}{s},$

iii. $\alpha^{2i+1} = -\frac{F_{2i}}{s}t\delta + \alpha\frac{L_{2i+1}}{s},$

iv. $\beta^{2i+1} = \frac{F_{2i}}{s}t\delta + \beta\frac{L_{2i+1}}{s}.$

Proof. i. Using the Binet Formula, we have

$\frac{F_{2i}}{s}\alpha\delta - t\frac{L_{2i-1}}{s} = \frac{\alpha^{2i}-\beta^{2i}}{(\alpha-\beta)s}\alpha\delta - t\frac{\alpha^{2i-1}+\beta^{2i-1}}{s}$
$= \frac{\alpha^{2i+1}-\alpha\beta^{2i}-t\alpha^{2i-1}-t\beta^{2i-1}}{s}$
$= \frac{\alpha^{2i}\left(\alpha-\frac{t}{\alpha}\right)+\beta^{2i}(-\alpha-\frac{t}{\beta})}{s}.$

Thus, we can write

$$\alpha^{2i} = \frac{F_{2i}}{s}\alpha\delta - t\frac{L_{2i-1}}{s}.$$

The proofs of the others can be given in the same way. □

Theorem 3.13. If n, m natural number and $n \le m$, we have

i. $\breve{H}F_{m+n} + (-t)^p\breve{H}F_{m-n} = \breve{H}F_mL_n,$

ii. $\breve{H}F_{m+n} - (-t)^p\breve{H}F_{m-n} = \breve{H}L_mF_n,$

iii. $\breve{H}L_{m+n} + (-t)^p\breve{H}L_{m-n} = \breve{H}L_mL_n.$

Proof. The proofs of the theorem are shown using the definition, lemma 3.1-3.2, and the Binet formulas.

In the following theorems, the terms of the (s,t)-Fibonacci and (s,t)-Lucas sequences are associated with their hyperbolic quaternion values.

Theorem 3.14. For all $m, n \ge 1$, we get

i. $\breve{H}F_{m+2n} = \frac{F_{2n}}{s}\breve{H}F_{m+2} - t^2\frac{F_{2n-2}}{s}\breve{H}F_m,$

ii. $\breve{H}L_{m+2n} = \frac{F_{2n}}{s}\breve{H}L_{m+2} - t^2\frac{F_{2n-2}}{s}\breve{H}L_m,$

iii. $\breve{H}F_{m+2n} = \frac{F_{2n}}{s}\breve{H}L_{m+1} - \frac{tL_{2n-1}}{s}\breve{H}F_m$,

iv. $\breve{H}L_{m+2n} = \frac{F_{2n}}{s}\delta^2\breve{H}F_{m+1} - \frac{tL_{2n-1}}{s}\breve{H}L_m$.

Proof. iv. Using the Binet Formula, we have

$$\frac{F_{2n}}{s}\delta^2\breve{H}F_{m+1} - \frac{tL_{2n-1}}{s}\breve{H}L_m$$
$$= \frac{\alpha^{2n}-\beta^{2n}}{s(\alpha-\beta)}(\alpha-\beta)^2\left(\frac{\overline{\alpha}\alpha^{m+1}-\overline{\beta}\beta^{m+1}}{\alpha-\beta}\right)$$
$$-t\frac{(\alpha^{2n-1}+\beta^{2n-1})}{s}(\overline{\alpha}\alpha^m+\overline{\beta}\beta^m)$$
$$= \frac{\overline{\alpha}\alpha^{m+2n}\left(\alpha-\frac{t}{\alpha}\right)+\overline{\beta}\beta^{m+2n}(\beta-\frac{t}{\beta})}{s}$$
$$= \frac{\overline{\alpha}\alpha^{m+2n}s+\overline{\beta}\beta^{m+2n}s}{s}.$$

Thus, we obtain

$$\breve{H}L_{m+2n} = \frac{F_{2n}}{s}\delta^2\breve{H}F_{m+1} - \frac{tL_{2n-1}}{s}\breve{H}L_m.$$

The proofs of the others can be given in the same way. □

Theorem 3.15. For all $m, n \geq 1$, we have

i. $\breve{H}F_{m+2n+1} = \frac{L_{2n+1}}{s}\breve{H}F_{m+1} - \frac{tF_{2n}}{s}\breve{H}L_m$,

ii. $\breve{H}L_{m+2n+1} = \frac{L_{2n+1}}{s}\breve{H}L_{m+1} - t\delta^2\frac{F_{2n}}{s}\breve{H}F_m$,

iii. $\breve{H}F_{m+2n+1} = \frac{L_{2n+1}}{s(s^2+3t)}\breve{H}F_{m+3} - t^3\frac{F_{2n-2}}{s(s^2+3t)}\breve{H}L_m$,

iv. $\breve{H}L_{m+2n+1}$
$$= \frac{L_{2n+1}}{s(s^2+3t)}\breve{H}L_{m+3} - t^3\delta^2\frac{F_{2n-2}}{s(s^2+3t)}\breve{H}F_m.$$

Proof. iii. Using the Binet Formula, we have

$$\frac{L_{2n+1}}{s(s^2+3t)}\breve{H}F_{m+3} - t^3\frac{F_{2n-2}}{s(s^2+3t)}\breve{H}L_m$$
$$= \frac{\alpha^{2n+1}+\beta^{2n+1}}{s(s^2+3t)}\frac{\overline{\alpha}\alpha^{m+3}-\overline{\beta}\beta^{m+3}}{(\alpha-\beta)}$$
$$-\frac{\alpha^{2n-2}-\beta^{2n-2}}{(\alpha-\beta)s(s^2+3t)}(\overline{\alpha}\alpha^m+\overline{\beta}\beta^m)$$
$$= \frac{\overline{\alpha}\alpha^{m+2n+1}(\alpha^3-\frac{t^3}{\alpha^3})+\overline{\beta}\beta^{m+2n+1}(\beta^3-\frac{t^3}{\beta^3})}{s(s^2+3t)(\alpha-\beta)}.$$

So, we get

$$\breve{H}F_{m+2n+1} = \frac{L_{2n+1}}{s(s^2+3t)}\breve{H}F_{m+3} - t^3\frac{F_{2n-2}}{s(s^2+3t)}\breve{H}L_m.$$

The proofs of the others can be given in the same way. □

Theorem 3.16. For all $k, m, n, p \geq 1$, we obtain

i. $\breve{H}F_{m+n} = F_m\breve{H}F_{n+1} + tF_{m-1}\breve{H}F_n$,

ii. $\breve{H}F_{2m+n} = L_m\breve{H}F_{m+n} - (-t)^m\breve{H}F_n$,

iii. $\breve{H}F_{km+n} = \frac{F_{km}}{F_m}\breve{H}F_{m+n} - (-t)^m\frac{F_{km-m}}{F_m}\breve{H}F_n$,

iv. $(-t)^{km}F_{m(p-k)}\breve{H}F_n$
$$= \breve{H}F_{km+n}F_{mp} - \breve{H}F_{mp+n}F_{km}.$$

Proof. The proofs are shown in the same way as theorem 3.15. □

4 Conclusion

In this study, we defined the hyperbolic (s,t)-Jacobsthal and (s,t)-Jacobsthal-Lucas quaternions. Then, we obtained some properties of these quaternions. Also, we examined the relationships between these quaternions. In addition, we calculated the special identities of these quaternions. Moreover, we found the terms of the (s,t)-Jacobsthal and (s,t)-Jacobsthal-Lucas sequences are associated with their hyperbolic quaternion values. In the future, we can spread a new approach to hyperbolic (s,t)-Jacobsthal and (s,t)-Jacobsthal-Lucas octonions and sedenions.

References:

[1] Kaymak, Ö. Ö., Coding theory for h (x)-Fibonacci polynomials. *Balıkesir Üniversitesi Fen Bilimleri Enstitüsü Dergisi*, Vol.26, No.1, 2024, pp. 226-236. DOI: https://doi.org/10.25092/baunfbed.1347379

[2] Zhang, X., Liu, M., & Tian, J., Multiple-image encryption algorithm based on Sarrus rule and 3D Fibonacci matrix. *Physica Scripta*, Vol.98, No.5, 2023, pp. 055208. DOI: 10.1088/1402-4896/acc905.

[3] Ignatius, F., & Kaspar, S., A new graph labeling with Tribonacci, Fibonacci and Triangular numbers. *Discover Sustainability*, Vol.5, No.1, 2024, pp. 1-16. DOI:https://doi.org/10.1007/s43621-024 00325-z

[4] Sulayman, J. E., & Pedrano, A. C., On Fibonacci Cordial Labeling of Some Snake Graphs, *European Journal of Mathematics and Statistics*, Vol.4, No.2, 2023, pp. 29-35. DOI: 10.24018/ejmath.2202.4.2.193.

[5] Avazzadeh, Z., Hassani, H., Agarwal, P., Mehrabi, S., Javad Ebadi, M., & Hosseini Asl, M. K., Optimal study on fractional fascioliasis disease model based on generalized Fibonacci polynomials, *Mathematical methods in the applied sciences*, Vol.46, No.8, 2023, pp. 9332-9350. DOI: 10.1002/mma.9057.

[6] Chavan, A. D., & Suryawanshi, C. V., Correlation of fibonacci sequence and golden ratio with its applications in engineering and science. *International Journal of Engineering and Management Research*, Vol.10, No.3, 2020, pp. 31-36. DOI: 10.31033/ijemr.10.3.5.

[7] Özkan, E., & Akkus, H., On K-Chebsyhev Sequence, *Wseas Transactions on Mathematics*, Vol.22, 2023, pp. 503-507. DOI: 10.37394/23206.2023.22.56

[8] Özkan, E., & Akkuş, H., Copper ratio obtained by generalizing the Fibonacci sequence, *AIP Advances*, Vol.14, No.7, 2024, pp. 1-11. DOI: https://doi.org/10.1063/5.0207147.

[9] Falcon, S., Plaza, A., On the Fibonacci k-numbers, *Chaos, Solitons & Fractals*, Vol.32,

No.5, 2007, pp. 1615-1624. DOI: https://doi.org/10.1016/j.chaos.2006.09.022

[10] Akkuş, H., Deveci, Ö., Özkan, E., & Shannon, A. G., Discatenated and lacunary recurrences, *Notes on Number Theory and Discrete Mathematics*, Vol.30, No.1, 2024, pp. 8-19. DOI: 10.7546/nntdm.2024.30.1.8-19

[11] Prasad, K., Mohanty, R., Kumari, M., & Mahato, H., Some new families of generalized k-Leonardo and Gaussian Leonardo Numbers, *Communications in Combinatorics and Optimization*, Vol.9, No.3, 2024, pp. 539-553. DOI: 10.22049/cco.2023.28236.1485.

[12] Civciv, H., & Turkmen, R., On the (s, t)-Fibonacci and Fibonacci matrix sequences, *Ars Combinatoria*, Vol.87, 2008, pp. 161-174.

[13] Civciv, H., & Türkmen, R., Notes on the (s, t)-Lucas and Lucas matrix sequences, *Ars Combinatoria*, Vol.89, 2008, pp. 271-285.

[14] Ipek, A., Ari, K., & Turkmen, R., The Generalized-(s, t) Fibonacci and Fibonacci Matrix Sequences, *Transylvanian Journal of Mathematics and Mechanics*, Vol.7, No.2, 2015, pp. 137-148.

[15] Uygun, S., The (s, t)-Jacobsthal and (s, t)-Jacobsthal Lucas sequences, *Applied Mathematical Sciences*, Vol.70, No.9, 2015, pp. 3467-3476. DOI: 10.12988/ams.2015.52166.

[16] Horadam, A. F., Complex Fibonacci numbers and Fibonacci quaternions, *The American Mathematical Monthly*, Vol.70, No.3, 1963, pp. 289-291.

[17] Macfarlane, A., Hyperbolic quaternions, *Proceedings of the Royal Society of Edinburgh*, Vol.23, 1902, pp. 169-180.

[18] Godase, A. D., Hyperbolic k-Fibonacci and k-Lucas quaternions, *The Mathematics Student*, Vol.90, No.1-2, 2021, pp. 103-116.

[19] Dışkaya, O., Menken, H., & Catarino, P. M. M. C., On the hyperbolic leonardo and hyperbolic francois quaternions, *Journal of New Theory*, Vol.42, 2023, pp. 74-85. DOI: https://doi.org/10.53570/jnt.1199465.

[20] Szynal-Liana, A., & Włoch, I., A note on Jacobsthal quaternions, *Advances in Applied Clifford Algebras*, Vol.26, 2016, pp. 441-447. DOI: 10.1007/s00006-015-0622-1.

[21] Dagdeviren, A., A generalization of complex, dual and hyperbolic quaternions: Hybrid quaternions, *Filomat*, Vol.37, No.25, 2023, pp. 8441-8454. DOI: 10.2298/FIL2325441D.

[22] Yağmur, T., A note on hyperbolic (p, q)-Fibonacci quaternions, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, Vol.69, No.1, 2020, pp. 880-890. DOI: https://doi.org/10.31801/cfsuasmas.582674.

[23] Prasad, K., Kumari, M., & Kızılateş, C., Some properties and identities of hyperbolic generalized k-Horadam quaternions and octonions, *Communications in Combinatorics and Optimization*, Vol.9, No.4, 2024, pp. 707-724. DOI: 10.22049/cco.2023.28272.1495.

[24] Szynal-Liana, A., & Włoch, I., Generalized commutative quaternions of the Fibonacci type, *Boletín de la Sociedad Matemática Mexicana*, Vol.28, No.1, 2022, pp. 1. DOI: 10.1007/s40590-021-00386-4.

[25] Dasdemir, A., On hyperbolic Lucas quaternions, *Ars Combinatoria*, Vol.150, 2020, pp. 77-84.

[26] Öztürk, İ., & Özdemir, M., On geometric interpretations of split quaternions, *Mathematical Methods in the Applied Sciences*, Vol.46, No.1, 2023, pp. 408-422. DOI: 10.1002/mma.8518.

[27] Alagöz, Y., & Özyurt, G., Hyper-Number Generalized Quaternions, *Computational Mathematics and Mathematical Physics*, Vol.64, No.5, 2024, pp. 908-917. DOI: 10.1134/S0965542524700337.

[28] Daşdemir, A., On recursive hyperbolic fibonacci quaternions, *Communications in Advanced Mathematical Sciences*, Vol.4, No.4, 2021, pp. 198-207. DOI: https://doi.org/10.33434/cams.997824.

[29] Aydın, F. T., Circular-hyperbolic Fibonacci quaternions, *Notes on Number Theory and Discrete Mathematics*, Vol.26, No.2, 2020, pp. 167-176. DOI: 10.7546/nntdm.2020.26.2.167-176.

[30] Erişir, T., & Yıldırım, E., On the fundamental spinor matrices of real quaternions, *Wseas Transactions on Mathematics*, Vol22, 2023, pp. 854-866. DOI: 10.37394/23206.2023.22.93.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)
-ENGIN OZKAN carried out the introduction and the main result of the article.
-Hakan AKKUS has improved Chapter 2 and Chapter 3.
-All authors read and approved the final manuscript.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
This study did not receive any funding in any form.

Conflict of Interest
The authors have no conflict of interest to declare.