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Total Graphic Topology on the Vertex Sets of Directed Graphs

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Abstract: In this work, we introduce a topology $\mathcal{T}_G^{tot}$, called the total graphic topology, for the vertices set of a directed graph $G = (V, E)$. We prove many properties of this topology and we give some open sets and some closed ones. We prove that $\mathcal{T}_G^{tot}$ is an Alexandroff topology. In addition, we investigate functions between directed graphs, the connectedness of this topology for some strongly connected graphs and we give an example for each case.

Key-Words: Directed graph, topology, minimal bases, connected components, homeomorphism.

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1 Introduction

Graph theory attires attention sice the resolving of the problem of the Königsberg seven bridges, [1]. It becomes one of discrete mathematics structures.
Graphs are simple to understand and can be used for representing many of mathematical combinations. Today, graph theory becomes a fundamental mathematical tool for many domains as chemistry, marketing and computers network. If we add topology to the graph, we can use them to solve economic and the traffick flow problems, [2], [3], [4], as in medical application and blood circulation, [5], [6], [7], [8].
A topology is called an Alexandroff topology if any intersection of open sets is also an open set, [9], [10]. Such topology is very interesting because we have a minimal bases and the characteristic properties can be studied by using this minimal bases or its subbases.
In, [11], the graphic topology for undirected graph was introduced on the vertices set. After that many topologies are introduced on undirected graphs.
In, [12], the authors investigated the graphic topology and solved partially an open problem mentioned in, [11]. After that, in 2023 the $\mathcal{Z}$-graphic topology was introduced in order to answer the first open problem in, [11], and bypass it, see, [13]. Graphic topology was also defined on fuzzy graphs in, [14].
For directed graphs in, [15], two topologies on the edges set are given. In this paper, we consider directed graph and introduce the total graphic topology on the vertices set.
This work has five sections in addition to the introduction and conclusion. Section 2 is devoted to some useful preliminaries in directed graph theory and topology. We give a set of subset of the vertices set V of a directed graph $G = (V, E)$ which is will be the subbases of our graphic topology. In section 3, we prove a lot of typical and preliminary results as proving that the total graphic topology is an Alexandroff topology, we prove some characterizations of minimal open sets. Section 4 is devoted to some advantaged results and we give some examples of open and closed sets. In Section 5, we study functions between digraphs and their relation with continuous and homeomorphism maps. The last section is devoted to total graphic topology and connectedness.

2 Preliminaries

In this section, we will recall some definitions and properties of directed graph theory and topology, [16], [17], [18]. Then, we introduce a new topology for the graph which will be have the total graphic topology as name.

Definition 2.1 A directed graph $G = (V, E)$ is a pair of sets: a nonempty set V and a set E such

that $E \subset V \times V$. More precisely, if $e = (x, y) \in E$, e has a direction from x to y. We also say e is an edge from x to y.
A directed graph $G = (V, E)$ is called simple if $\forall x \in V,\ \ (x, x) \notin E$ and $\forall x,\ y \in V$ there is no multiple edges from x to y.

Definition 2.2 A digraph $G = (V, E)$ is called complete if it is simple and for any distinct $x, y \in V$, there exist a unique edge from x to y and a unique edge from y to x.

Definition 2.3 Let $G = (V, E)$ be a digraph. G is called an oriented graph if at most one of the two edges (x, y) and (y, x) is in E, for all x and y vertices of G.
If G is a simple graph and $(x, y) \in E$ if and only if $(y, x) \notin E$, for all $x, y \in V$, we call G is a tournament.

Definition 2.4 Let $G = (V, E)$ be a simple digraph. The digraph $\overline{G} = (V, \overline{E})$ defined by $(x, y) \in \overline{E}$ if and only if $(x, y) \notin E$ is called the complement of G.

Definition 2.5 Consider a directed graph $G = (V, E)$.
In G, a directed path P from a_0 to a_n is a sequence of the form $P : a_0, e_0, a_1, e_1, \cdots, a_{n-1}, e_{n-1}, a_n$, where $a_k \in V$ and e_k an edge from a_k to a_{k+1}, $k = 0, \cdots, n-1$.
We say that a and b are connected in G if there is a directed path from a to b and a directed path from b to a. Also, G is called strongly connected if any two distinct vertices are connected in G.

Let x a vertex of a simple directed graph $G = (V, E)$. We define the out-neighborhood set of x as

$$\mathcal{KH}_x = \{y \in V, (x, y) \in E\}. \tag{1}$$

and the int-neighborhood set of x as

$$\mathcal{D}_x = \{y \in V, (y, x) \in E\}. \tag{2}$$

It is clear from (1) and (2) that

$$y \in \mathcal{KH}_x \quad \text{if and only if} \quad x \in \mathcal{D}_y.$$

Let

$$\mathcal{M}_x = \mathcal{KH}_x \cup \mathcal{D}_x. \tag{3}$$

The cardinal of the out-neighborhood $\mathcal{KH}_x$ of x is called the out-degree of x, we denote

$$d^+(x) = card(\mathcal{KH}_x), \tag{4}$$

the cardinal of int-neighborhood $\mathcal{D}_x$ of x is named int-degree of the vertex x and we set

$$d^-(x) = card(\mathcal{D}_x) \tag{5}$$

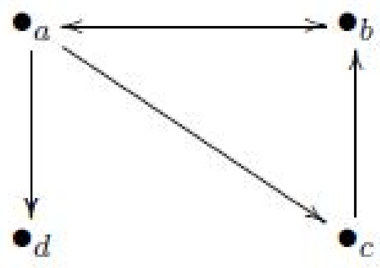

Figure 1: A directed graph $G = (V, E)$

and we donate

$$d^t(x) = card(\mathcal{M}_x) \tag{6}$$

the total degree of the vertex x.
We set the minimum out-degree, the minimum int-degree and the minimum total degree of a digraph $G = (V, E)$ as

$$\delta^+(G) = \min\{d^+(x),\ x \in V\}, \tag{7}$$

$$\delta^-(G) = \min\{d^-(x),\ x \in V\} \tag{8}$$

and

$$\delta^t(G) = \min\{d^t(x),\ x \in V\} \tag{9}$$

However, the maximum out-degree, the maximum int-degree and the maximum total degree of G are respectively given by

$$\Delta^+(G) = \max\{d^+(x),\ x \in V\}, \tag{10}$$

$$\Delta^-(G) = \max\{d^-(x),\ x \in V\} \tag{11}$$

and

$$\Delta^t(G) = \max\{d^t(x),\ x \in V\}. \tag{12}$$

Example 2.1 For the graph G given by Fig. 1, we have $\mathcal{M}_a = \{b, c, d\}$, $\mathcal{M}_b = \{a, c\}$, $\mathcal{M}_c = \{a, b\}$ and $\mathcal{M}_d = \{a\}$.

Definition 2.6 Suppose that $G = (V, E)$ is a digraph. A vertex $x \in V$ is called isolated if $\mathcal{M}_x = \emptyset$.

Definition 2.7 Let $G = (V, E)$ be a digraph. We say that G is locally finite if $\mathcal{M}_x$ is a finite set for all $x \in V$.

It is clear that a finite digraph is locally finite one. We pass to give some definitions and notations for topological spaces that we will need them later.

Definition 2.8 Let V be a non empty set and let τ be a family of subsets of V. If the following conditions

(1) $\emptyset, \quad V \in \tau$;

(2) $\forall\ U_1,\ U_2 \in \tau$, we have $U_1 \cap U_2 \in \tau$;

(3) $\forall\ \{U_i\}_{i\in I}$ a family of elements in τ, the union $\cup_{i\in I} U_i \in \tau$

are satisfied, we say τ is a topology for V or (V, τ) is a topological space.
An element of τ is named an open set of V.

Definition 2.9 Suppose that (V, τ) is a topological space and $U \subset V$.

(i) The set $U^c = V \setminus U$ is called the complement of U in V.

(ii) The set U is called a closed set of V if and only if U^c is an open set.

(iii) We denote $\overline{U}$ the smallest closed set of V containing U. $\overline{U}$ is called the closure of U in V.

Next, suppose that the digraph $G = (V, E)$ is simple and without isolated vertices. Consider the set

$$\mathcal{S}_G^{tot} = \{\mathcal{M}_x; \quad x \in V\}, \tag{13}$$

where $\mathcal{M}_x$ is given by (3).

Theorem 2.1 Let $G = (V, E)$ be a simple directed graph without isolated point. Then, $\mathcal{S}_G^{tot}$ is a subbases for a topology of the vertices set V.

Proof. Since $\bigcup_{x\in V} \mathcal{M}_x \subset V$, we have to prove that $V \subset \bigcup_{x\in V} \mathcal{M}_x$.
Let $z \in V$, Since $\mathcal{M}_z \neq \emptyset$, there exists $x \in \mathcal{M}_z$. So, $z \in \mathcal{M}_x$ and we get the result.

□

The topology induced by the subbases $\mathcal{S}_G^{tot}$ is called the total graphic topology of G and it is denoted $\mathcal{T}_G^{tot}$.

Example 2.2 For the graph in the Example 2.1, we have $\mathcal{S}_G^{tot} = \Big\{\emptyset, \{b, c, d\}, \{a, c\}, \{a, b\}, \{a\}\Big\}$, $\mathcal{B} = \Big\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c, d\}\Big\}$ and $\mathcal{T}_G^{tot} = \Big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}\Big\}$.

Throughout this paper, a digraph means a simple locally finite digraph without isolated vertex.

3 Preliminary Results

An Alexandroff space is a topological space satisfying any intersection of open sets is an open set. With an Alexandroff space and so Alexandroff topology we have a minimal bases of the topology and we can use it to study the properties of the topological space as we will do in the rest of this paper.

Theorem 3.1 suppose that $G = (V, E)$ is a directed graph. Then $(V, \mathcal{T}_G^{tot})$ is an Alexandroff space.

Proof. The total graphic topology $\mathcal{T}_G^{tot}$ is constructed from the subbases $\mathcal{S}_G^{tot}$, so it is sufficient to prove that any intersection of elements in the subbases is an open set.
Consider $\cap_{x\in A} \mathcal{M}_x$, where $A \subset V$, and suppose that

$$\bigcap_{x\in A} \mathcal{M}_x \neq \emptyset.$$

Suppose that $y \in \cap_{x\in A} \mathcal{M}_x$. Then, $y \in \mathcal{M}_x$, for all $x \in A$.
We get for all $x \in A$, $x \in \mathcal{M}_y$. Therefore this means $A \subset \mathcal{M}_x$ and so A is finite. Hence $\cap_{x\in A} \mathcal{M}_x$ is an open set.

As consequence of the above theorem, Let G be a digraph, then the total graphic topology $\mathcal{T}_G^{tot}$ of a directed graph $G = (V, E)$ has a minimal basis which we denote

$$\mathcal{B}_G = \{M_x; \quad x \in V\}, \tag{14}$$

where M_x is the intersection of all open sets containing x, it is the smallest open set containing the vertex x. We can characterise the smallest open sets by using the subbases as follows.

Theorem 3.2 Suppose that $G = (V, E)$ is a digraph and x is a vertex of G. Then,

$$M_x = \bigcap_{z\in \mathcal{M}_x} \mathcal{M}_z. \tag{15}$$

Proof. Since x is a vertex of G, $\mathcal{M}_x$ is a nonempty set. Consider $z \in \mathcal{M}_x$, then $x \in \mathcal{M}_z$ and so the open set $\cap_{z\in\mathcal{M}_x} \mathcal{M}_z$ contains x and so

$$M_x \subset \bigcap_{z\in \mathcal{M}_x} \mathcal{M}_z.$$

Conversely, since $\mathcal{T}_G^{tot}$ has $\mathcal{S}_G^{tot}$ as subbases there exists $A \subset V$ such that $M_x = \cap_{z\in A} \mathcal{M}_z$.
For all $z \in A$, $x \in \mathcal{M}_z$. Therefore, for all $z \in A$, $z \in \mathcal{M}_x$. Then, $A \subset \mathcal{M}_x$ and so,

$$\bigcap_{z\in \mathcal{M}_x} \mathcal{M}_z \subset \bigcap_{z\in A} \mathcal{M}_z = M_x.$$

□

Remark 3.1 For a directed $G = (V, E)$, each minimal open set M_x is a finite set since each set $\mathcal{M}_z$ is finite et

$$M_x = \bigcap_{z\in \mathcal{M}_x} \mathcal{M}_z.$$

Corollary 3.1 Let G be a digraph and let x and a two distinct vertices of G.

(i) If $\mathcal{M}_x = \{a\}$, then $M_x = \mathcal{M}_a$.

(ii) If $a \in \mathcal{M}_x$, then $M_x \subset \mathcal{M}_a$.

(iii) If $M_a \subset \mathcal{M}_x$, then $M_x \subset \mathcal{M}_a$.

Proof.

(i) If $\mathcal{M}_x = \{a\}$, from the fact that $M_x = \cap_{z\in\mathcal{M}_x}\mathcal{M}_z$ we get $M_x == \mathcal{M}_a$.

(ii) From the last theorem, $M_x = \cap_{z\in\mathcal{M}_x}\mathcal{M}_z$, so $M_x \subset \mathcal{M}_z$ for all $z \in \mathcal{M}_x$. In particular, $M_x \subset \mathcal{M}_a$.

(iii) When $M_a \subset \mathcal{M}_x$, we have $a \in M_a \subset \mathcal{M}_x$. The result follows From (ii).

The following result follows from the Theorem 3.2, but it very useful when dealing with the minimal bases.

Theorem 3.3 When $G = (V, E)$ is a digraph and a a vertex of G. Then,

$$M_a = \{x \in V;\ \ \mathcal{M}_a \subset \mathcal{M}_x\}.$$

Proof. We have

$$M_a = \bigcap_{z\in\mathcal{M}_a} \mathcal{M}_z,$$

so we have $x \in M_a$ if and only if $x \in \mathcal{M}_z, \forall z \in \mathcal{M}_a$ which is equivalent to for all $z \in \mathcal{M}_a$, $z \in \mathcal{M}_x$ and this is true if and only if $\mathcal{M}_a \subset \mathcal{M}_x$.

Corollary 3.2 Suppose that G is a digraph and let a be a vertex of G. Then, $M_a \cap \mathcal{M}_a = \emptyset$. Also, if $M_x \subset \mathcal{M}_a$, we have $M_x \cap M_a = \emptyset$.

Proof. (i) By contradiction, suppose that there exists $y \in M_a \cap \mathcal{M}_a$.
$y \in M_a$ gives $\mathcal{M}_a \subset \mathcal{M}_y$ from Theorem 3.3.
$y \in \mathcal{M}_a$ implies $y \in \mathcal{M}_y$, contradiction since G is a simple directed graph.
(ii) If $M_x \subset \mathcal{M}_a$, then $M_x \cap M_a \subset \mathcal{M}_x \cap M_a = \emptyset$.

Theorem 3.4 For any vertex a in a directed graph $G = (V, E)$, we have

$$\overline{\{a\}} = \{x \in V;\ \ \mathcal{M}_x \subset \mathcal{M}_a\}.$$

Proof. $x \in \overline{\{a\}}$ if and only if $A \cap \{x\} \neq \emptyset$, for all open set A containing x. Using minimal bases, this is equivalent to $M_x \cap \{a\} \neq \emptyset$, this means $a \in M_x$. That is, by Theorem3.3

$$x \in \overline{\{a\}} \Leftrightarrow \mathcal{M}_x \subset \mathcal{M}_a.$$

4 Some Properties of Graphic Topology

Theorem 4.1 For a directed graph $G = (V, E$, the space $(V, \mathcal{T}_G^{tot})$ is a compact topological space if and only if the vertices set V is finite.

Proof. Recall that the topological space $(V, \mathcal{T}_G^{tot})$ is said compact if every open cover of the space $V \subset \bigcup_{x\in V} U_x$ has a finite subcover.
If V is a finite set, then from any open cover of V, we have a finite subcover by definition of the compactness.
Conversely, suppose that $(V, \mathcal{T}_G^{tot})$ is a compact topological space. Consider the minimal basis $\mathcal{B}_G$. The family $\mathcal{B}_G$ is an open cover of V, so there exists a finite subcover of $\mathcal{B}_G$. Since it is minimal as basis, $\mathcal{B}_G$ is equal to this subcover. Since from (14), we have

$$\mathcal{B}_G = \{M_x;\ \ x \in V\},$$

we conclude that V is finite.

Proposition 4.1 Let $G = (V, E)$ be a digraph. Then, $A = \{x \in V,\ \ d^t(x) = \Delta^t(G)\}$ is an open set for the total graphic topology of G.

Proof. Let $x \in A$. We will prove that $M_x \subset A$.
Let $y \in M_x$, we have $\mathcal{M}_x \subset \mathcal{M}_y$ (from Theorem 3.3). We obtain

$$card(\mathcal{M}_x) \leq card(\mathcal{M}_y) \leq \Delta^t(G).$$

Then $d^t(x) = \Delta^t(G) = d^t(y)$ and hence $y \in A$.
We get $x \in M_x \subset A$, for all $x \in A$ and the result is proved.

Proposition 4.2 Let $G = (V, E)$ be a digraph. Then the following set

$$B = \{x \in V,\ \ d^t(x) = \delta^t(G)\} \tag{16}$$

is a closed set for the total graphic topology of G.

Proof. We have $B \subset \overline{B}$. We will prove the inverse inclusion.
Let $x \in \overline{B}$, since M_x is an open set containing x, we have $M_x \cap B \neq \emptyset$.
Set $z \in M_x \cap B$, we obtain the two facts:
$\mathcal{M}_x \subset \mathcal{M}_z$ and $d^t(z) = \delta^t(G)$.
So,

$$card(\mathcal{M}_x) \leq card(\mathcal{M}_z) = \delta^t(G).$$

Therefore $d^t(x) = \delta^t(G)$ and we get $x \in B$.

Proposition 4.3 Suppose that $G = (V, E)$ is a finite directed graph. Then the following set

$$\mathcal{T}_G^c = \{A;\ \ A^c \in \mathcal{T}_G^{tot}\} \tag{17}$$

is a topology for V and if G is an oriented graph such that $\mathcal{T}_G^{tot} = \mathcal{T}_G^c$, then $\mathcal{T}_G^{tot}$ is the discrete topology.

Proof. We have $\emptyset, V \in \tau$ since their complements are in $\mathcal{T}_G^{tot}$.
When A and B are in $\mathcal{T}_G^c$, we have $(A \cap B)^c = A^c \cup B^c$ and so $(A \cap B)^c \in \mathcal{T}_G^{tot}$. Hence $A \cap B \in \mathcal{T}_G^c$.
Now, if we have a countable family $\{A_i\}$ of elements of $\mathcal{T}_G^c$, we know that

$$(\cup_i A_i)^c = \cap_i A_i^c.$$

But $\mathcal{T}_G^{tot}$ is an Alexandroff topology, we deduce that $\cup_i A_i \in \mathcal{T}_G^c$.
Then $\mathcal{T}_G^c$ is a topology for V.

Next, suppose that G is an oriented graph and $\mathcal{T}_{\overline{G}}^{tot} = \mathcal{T}_G^c$. Let $x \in V$, we have $\Big(\mathcal{M}_x \cup \{x\}\Big)^c$ as the set of all vertices adjacent to x in $\overline{G}$. Then, $\Big(\mathcal{M}_x \cup \{x\}\Big)^c \in \mathcal{T}_{\overline{G}}^{tot}$since it is an element of its subbases.
Therefore, $\Big(\mathcal{M}_x \cup \{x\}\Big)^c \in \mathcal{T}_G^c$ and so,

$$\mathcal{M}_x \cup \{x\} \in \mathcal{T}_G^{tot}.$$

Since M_x is the minimal open set containing x, we get

$$M_x \subset \mathcal{M}_x \cup \{x\}.$$

Since $M_x \cap \mathcal{M}_x = \emptyset$ (Corollary 3.2), we obtain $M_x = \{x\}$. We have so $\mathcal{T}_G^{tot}$ is the discrete topology.

5 Isomorphic Digraphs and Homeomorphic Graphic Topologies

Definition 5.1 Un homomorphism h from a digraph $G = (V, E)$ to a digraph $G' = (V', E')$ is a function $h : V \to V'$ satisfying

$$\forall (x, y) \in E, (h(x), h(y)) \in E'.$$

h is called isomorphism if $h : V \to V'$ is bijective and $\forall (x, y) \in V^2$, we have

$$(x, y) \in E \ \text{ if and only if } \ (h(x), h(y)) \in E'.$$

We say that the two graphs are isomorphic.

Definition 5.2 An homeomorphism h from a topological space (V, T) to a topological space (V', T') is a continuous bijective map $h : V \to V'$ such that its inverse is also continuous. In this case, the two spaces or the two topologies are called homeomorphic.

Our first result in this section is the following.

Theorem 5.1 Suppose that two directed graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic and $h : V \to V'$ is an isomorphism. Then, their total graphic topologies are homeomorphic.

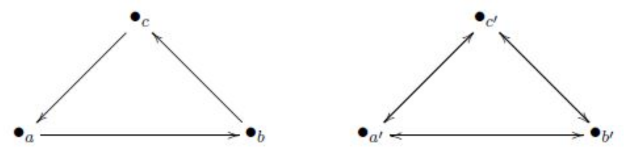

Figure 2: These two graphs have the discrete topology as total graphic topology but they are not isomorphic.

Proof. It is sufficient to prove that for all $A \in \mathcal{S}_{G'}^{tot}$, $h^{-1}(A)$ is in $\mathcal{T}_G^{tot}$. For this, let $z \in V'$ satisfying $A = \mathcal{M}_z$ and let $x = h^{-1}(z)$.
In this case, we get

$$\begin{aligned} h^{-1}(A) &= \{a \in V; \ \ h(a) \in \mathcal{M}_z\} \\ &= \{a \in V; \ \ (z, h(a)) \in E' \ \text{ or } \ (h(a), z) \in E'\} \\ &= \{a \in V; \ \ (h(x), h(a)) \in E' \ \text{ or } \ (h(a), h(x)) \in E'\} \\ &= \{a \in V; \ \ (x, a) \in E \ \text{ or } \ (a, x) \in E\} \\ &= \mathcal{M}_x \in \mathcal{T}_G^{tot}. \end{aligned}$$

Hence the bijective function $h : V \to V'$ is continuous. In a similar way, we prove that h^{-1} is continuous.

For the converse, see Fig. 2. The two graphs have the the discrete topology as total graphic topology but they are not isomorphic.

Theorem 5.2 Suppose that $G = (V, E)$ and $G = (V', E')$ are two digraphs and $h : V \to V'$ is a function. Then, h is continuous if and only if

$$\forall \ \ y, z \in V, \ \ \mathcal{M}_y \subset \mathcal{M}_z \Longrightarrow \mathcal{M}_{h(y)} \subset \mathcal{M}_{h(z)}. \tag{18}$$

Proof. If the function h is a continuous function and $y, z \in V$ such that

$$\mathcal{M}_y \subset \mathcal{M}_z.$$

In order to get $\mathcal{M}_{h(y)} \subset \mathcal{M}_{h(z)}$, we are going to prove that $h(z) \in M_{h(y)}$ and the result follows from Theorem 3.3.
Now, consider the minimal open set $M_{h(y)}$. such that $y \in h^{-1}\Big(M_{h(y)}\Big)$ and M_y the smallest open set containing y, we get $M_y \subset h^{-1}\Big(V_{h(y)}\Big)$.
As $\mathcal{M}_y \subset \mathcal{M}_z$, we obtain $z \in M_y$ and so $z \in h^{-1}\Big(M_{h(y)}\Big)$, this means, $h(z) \in M_{h(y)}$.

For the converse, Suppose that the property (18) is satisfied and we have to prove that the function h is continuous. Let A an open set of V'

and consider $y \in h^{-1}(A)$. We have $h(y) \in A$ and so $M_{h(y)} \subset A$.
Let $z \in M_y$, then $\mathcal{M}_y \subset \mathcal{M}_z$ from Theorem 3.3. Therefore, $\mathcal{M}_{h(y)} \subset \mathcal{M}_{h(z)}$ and hence $h(z) \in M_{h(y)}$. Or, we have $M_{h(y)} \subset A$ then $h(z) \in A$ and so $z \in h^{-1}(A)$. We get $M_y \subset h^{-1}(A)$, for all $y \in h^{-1}(A)$ and the result follows.

Theorem 5.3 Suppose that $G = (V, E)$ and $G' = (V', E')$ are two digraphs and $h : V \to V'$ a function. Then, $h : (V, \mathcal{T}_G^{tot}) \to (V', \mathcal{T}_{G'}^{tot})$ is an homeomorphism if and only if

$$\forall \ \ y, z \in V, \ \ \mathcal{M}_y \subset \mathcal{M}_z \Longleftrightarrow \mathcal{M}_{h(y)} \subset \mathcal{M}_{h(z)}. \tag{19}$$

Proof. First, by the Theorem 5.2, we have: h continuous if and only if

$$\forall \ \ y, z \in V, \ \ \mathcal{M}_y \subset \mathcal{M}_z \Longrightarrow \mathcal{M}_{h(y)} \subset \mathcal{M}_{h(z)}.$$

Using Theorem 5.2 for the function h^{-1}, we get: h^{-1} continuous if and only if

$$\forall \ y', z' \in V', \ \ \mathcal{M}_{y'} \subset \mathcal{M}_{z'} \Longrightarrow \mathcal{M}_{h^{-1}(y')} \subset \mathcal{M}_{h^{-1}(z')}.$$

Since h is bijective, we get: h^{-1} continuous if and only if

$$\forall \ \ y, z \in V, \ \ \mathcal{M}_{h(y)} \subset \mathcal{M}_{h(z)} \Longrightarrow \mathcal{M}_y \subset \mathcal{M}_z$$

and so, the result follows.

6 Graphic Topology and Connectedness

Being connected, is a property can be defined for a topological space as for a graph. Here, we will consider the strongly connectivity for directed graph, [16], [17], [18]. Let us recall the definitions.

Definition 6.1 Let $(V, \mathcal{T})$ be a topological space. The space V is said connected if whenever $V = A \cup B$ such that $A \cap B = \emptyset$, we have necessary $A = \emptyset$ or $B = \emptyset$. That is, V can not written as the union of two disjoint proper open sets.

Definition 6.2 Let $G = (V, E)$ be a digraph. G is called strongly connected if for all $a, b \in V$ there exist at least two paths joining a and b: one from a to b and one from b to a.

In general, a digraph $G = (V, E)$ does not have to be strongly connected, so we can define their connected components.

Definition 6.3 Suppose that $G = (V, E)$ is a digraph. Let $U_1, U_2, \cdots$ be subsets of V such that

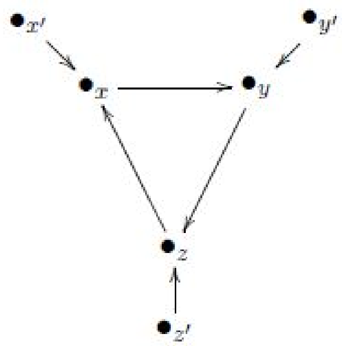

Figure 3: This is a non strongly connected digraph but its total graphic topology is connected.

(i) $V = \cup_i U_i$;

(ii) $U_i \cap U_j = \emptyset$, for all $i \neq j$;

(iii) For $i = 1, 2, \cdots$, for all $a, b \in U_i$, there exist a path from a to b and a path from b to a.

(iv) For all $a \in U_i$, $b \in U_j$ and $i \neq j$, if there exists a path from a to b, then there is no path from b to a.

Then, each subset U_i is called connected component of the digraph G.

It is clear that a strongly connected digraph has one connected component. and a finite digraph has a finite connected components. When the graph is undirected and disconnected, the graphic topological is disconnected and this is due to the connected components are open sets [11], but for directed graph this result no longer true. Our first example is a proof for this fact.

Example 6.1 For the digraph given by Fig. 3, we have: $\mathcal{M}_x = \{x', y, z\}, \mathcal{M}_y = \{y', x, z\}, \mathcal{M}_z = \{z', x, y\}$, $\mathcal{M}_{x'} = \{x\}, \mathcal{M}_{y'} = \{y\}, \mathcal{M}_{z'} = \{z\}$
$M_x = \{x'\}, M_y = \{y'\}, M_z = \{z'\}$, $M_{x'} = \{x', y, z\}, M_{y'} = \{y', x, z\}, M_{z'} = \{z', x, y\}$.

Example 6.2 The digraph given by Fig. 4 is a non strongly connected digraph with disconnected total graphic topology.

Example 6.3 The digraph, given by Fig. 5, is strongly connected and its connected total graphic topology.

Example 6.4 In the Fig. 6, the graph is non strongly connected digraph and its total graphic topology is disconnected.

In the rest of this paper, we prove some elementary results about connectedness of the total graphic topology.

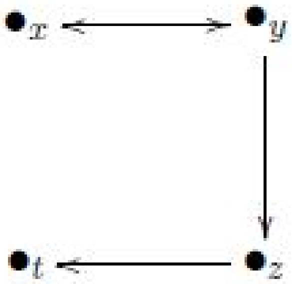

Figure 4: An example of digraph which is not strongly connected and its total graphic topology is disconnected.

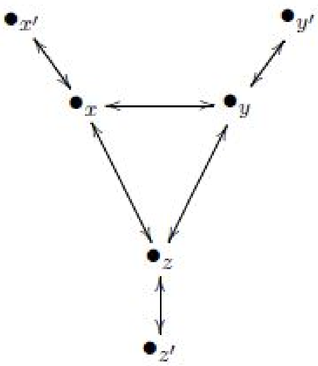

Figure 5: This is a strongly connected digraph with strongly connected total graphic topology.

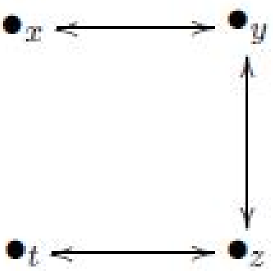

Figure 6: This is an example of a non strongly connected digraph with a disconnected total graphic topology.

Theorem 6.1 Let $G = (V, E)$ be a bipartite digraph. The total graphic topology $\mathcal{T}_G^{tot}$ of G is a disconnected topology.

Proof. G is a bipartite graph means there exist two disjoint subsets B_1 and B_2 of V such that $V = B_1 \cup B_2$ and if $(x, y) \in E$ and $(x, y) \notin B_1 \times B_2$ then $(x, y) \in B_2 \times B_1$.
Consider

$$U_1 = \bigcup_{x \in B_1} \mathcal{M}_x \text{ and } U_2 = \bigcup_{x \in B_2} \mathcal{M}_x.$$

then, U_1 and U_2 are nonempty disjoint open sets of V satisfying $U_1 \subset B_2$, $U_2 \subset B_1$ and $V = U_1 \cup U_2$. So, $(V, \mathcal{T}_G^{tot})$ is a disconnected topological space.

In fact, the last proof confirms the following result.

Theorem 6.2 Let $G = (V, E)$ be a strongly connected bipartite digraph. The total graphic topology $\mathcal{T}_G^{tot}$ of G is a disconnected topology.

Theorem 6.3 Suppose that $G = (V, E)$ is a finite one sense directed cycle, that is, $V = \{x_1, \cdots, x_n\}$ and $E = \{(x_{j-1}, x_j), i = 2 \ldots, n\} \cup \{(x_n, x_1)\}$. The total graphic topology $\mathcal{T}_G^{tot}$ of G is a disconnected topology.

Proof. For all $x_i \in V$, $i \neq 1$, we have $\mathcal{M}_{x_i} = \{x_{i-1}\}$ and $\mathcal{M}_{x_1} = \{x_n\}$. Then, for all $x \in V$, $\{x\}$ is an open set and so $\mathcal{T}_G^{tot}$ is discrete.

7 Conclusion

In this paper, we introduce the total graphic topology $\mathcal{T}_G^{tot}$ on the vertices set of a directed graph $G = (V, E)$ by using a subbases. The elements of this subbases are the sets of all neighbors in any direction of all vertices of the graph. That is, a neighborhood of a vertex is the set of all out-neighbors and int-neighbors. For this reason, the obtained topology are called total graphic topology. We prove that this topology is an Alexandroff topology that is any intersection of open set is also an open set. The existing of minimal bases follows. We give some characterizations of minimal open sets using the subbases. In addition, we investigate the relation between isomorphic graphs and their graphic total topologies and prove that they will be homeomorphic. The problem of connectedness was investigated through some examples. As future work, we have the following question: are there some necessary and sufficient conditions for the connectivity of $(V, \mathcal{T}_G^{tot})$?

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