Quasi hemi Slant Submanifolds of Lorentzian Concircular Structures

TOUKEER KHAN[1], SHEEBA RIZVI[1], OĞUZHAN BAHADIR[2]
[1]Department of Liberal Education,
Era University, Lucknow-226003,
INDIA

[2]Department of Mathematics, Faculty of Sciences,
Kahramanmaras Sutcu Imam University, Kahramanmaras,
TURKEY

Abstract: - In this manuscript, we introduce and explore quasi hemi-slant submanifolds, extending the concepts of slant submanifolds, semi-slant submanifolds, and hemi-slant submanifolds within Lorentzian concircular structures- manifolds $(LCS)_n$ -manifolds. We establish necessary and sufficient conditions for the integrability of distributions relevant to defining quasi hemi-slant submanifolds within Lorentzian concircular structures-manifolds or $(LCS)_n$- manifolds. Additionally, we investigate the conditions under which quasi hemi-slant submanifolds of Lorentzian concircular structures can be totally geodesic and analyze the geometric properties of foliations determined by the associated distribution.

Key-Words: - Quasi hemi-slant submanifolds, Lorentzian concircular structures – manifolds or $(LCS)_n$-manifolds, totally geodesic, Lorentzian manifold, Submanifold theory, Almost contact metric manifold .

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1 Introduction

Submanifold theory is an important concept used in the fields of physics and geometry. The term Quasi Hemi Slant Submanifolds plays a very important role in differential geometry and submanifold theory. This concept is of great importance in differential geometry. Quasi Hemi Slant Manifolds have some important features specific to the geometry of that manifold. These properties can be detailed with mathematical expressions. Practical applications and real-world examples of this theoretical concept can be examined. Ways in which this concept can be applied in physics, engineering, or other scientific fields can be highlighted.

To understand Lorentz Circular Structures, it is crucial to have a foundation in Lorentz geometry. Circular structures in Lorentz manifolds involve certain geometric relationships between vectors. The interaction between Lorentz geometry and circular structures has notable applications in theoretical physics, especially in the context of spacetime curvature and relativistic effects. Like any mathematical concept, Lorentzian Concircular Structures present challenges and areas for further exploration. Lorentzian Concircular Structures offer a fascinating intersection of geometry and physics, providing a deeper understanding of spacetime within the context of special relativity.

[1], introduced and studied the concept of Lorentzian almost paracontact manifolds. Then after, many other authors studied the Lorentzian almost paracontact manifolds, [2], [3], [4], [5], [6], [7], [8], [9], [10]. The application of $(LCS)_n$ -manifolds was investigated by [11], in the field of general theory of relativity and cosmology. [12], defined and introduced the geometry of slant submanifolds naturally generalizes both holomorphic and totally real immersions. Then many more expert mathematicians of geometry previous more than the last two decades studies this interesting topic, [13], [14], [15]). [16], studied submanifolds of $(LCS)_n$- manifolds. This concept was studied by many authors in differentiable manifolds, [17], [18], [19], [20], [21], [22]. In [11], the authors examine the geometry of hemi-slant $\xi^{\perp}$-Lorentzian submersions from $(LCS)_n$- manifolds. [23], also studied slant and pseudo slant submanifolds of $(LCS)_n$ - manifolds. In [24], author's focus to studied quasi-hemi-slant submanifolds of $(\alpha,\beta)-$ type almost contact manifolds and allow an description of a submanifold with a quasi-hemi-slant factor and discusses its use in the field of number theory.

2 Lorentzian Concircular Structures – Manifolds or $(LCS)_n$- Manifolds

A non-zero vector $u \in T_pM$ is said to be time-like (non-space-like, null, space-like) if it fulfils the condition $g_p(u,u) < 0$), [6].

Definition 2.1. In a Lorentzian manifold $(\bar{M}, g)$, a vector field P given by:

$$g(X,P) = A(X) \tag{1}$$

for any $X \in \Gamma(TM)$, is said to be a concircular vector field if:

$$(\bar{\nabla}_X A)Y = \sigma\big(g(X,Y) + A(X)A(Y)\big) \tag{2}$$

where σ is a non-zero scalar and $\bar{\nabla}$ denotes the covariant differentiation operator with respect to the Lorentzian metric g.

Let the Lorentzian manifold $\bar{M}$ admit a unit time-like concircular vector filed ξ, called the characteristic vector field of manifold, then we have:

$$g(\xi,\xi) = -1 \tag{3}$$

Since ξ is unit concircular vector field, it follows that there exist 1-form η which is a non-zero such that:

$$g(X,\xi) = \eta(X) \tag{4}$$

for which the equation of the following form holds:

$$(\bar{\nabla}_X \eta)Y = \alpha\big(g(X,Y) + \eta(X)\eta(Y)\big) \tag{5}$$

for all vector fields X and Y, where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfying

$$\bar{\nabla}_X \eta = (X\alpha) = d\alpha(X) = \rho\eta(X) \tag{6}$$

where ρ being a definitive scalar function given by

$$\rho = -(\xi\alpha) \tag{7}$$

If we substitute, then

$$\phi X = \frac{1}{\alpha}\bar{\nabla}_X \xi \tag{8}$$

then using (2.5) and (2.8), we get

$$\phi X = X + \eta(X)\xi \tag{9}$$

where ϕ is symmetric $(1,1)$ tensor field and is called structure tensor field of the manifold.

In $(LCS)_n$ - manifold $\bar{M}(n > 2)$, the following relation holds:

$$\phi^2 X = X + \eta(X)\xi \tag{10}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{11}$$

$$g(\phi X, Y) = g(X, \phi Y) = \psi(X,Y) \tag{12}$$

$$\phi\xi = 0, \eta(\xi) = -1, \eta(\phi X) = 0 \tag{13}$$

$$(\bar{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X] \tag{14}$$

$$(X\rho) = d\rho(X) = \beta\eta(X) \tag{15}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(X)Y - \eta(Y)X] \tag{16}$$

$$R(\xi,Y)Z = (\alpha^2 - \rho)[g(Y,Z)\xi - \eta(Z)Y] \tag{17}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X) \tag{18}$$

$$R(X,Y)Z = \phi R(X,Y)Z + (\alpha^2 - \rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \tag{19}$$

for all $X, Y, Z \in \Gamma(TM)$, [16].

For a special case, if we consider $\alpha = 1$, then we can obtain the LP-Sasakian structure [24].

From (14), putting $Y = \xi$ and using (8) and (13), we have:

$$\eta(\bar{\nabla}_X \xi)\xi = 0. \tag{20}$$

Let M be a submanifold of a $(LCS)_n$- manifold $\bar{M}$ with induced metric g. Also, let ∇ and $\nabla^\perp$ be the induced connections on TM and $T^\perp M$ of M respectively, then Gauss and Weingarten formula are given by:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{21}$$

$$\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V \tag{22}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where h is second fundamental form and A_V is shape operator. These are related as follows:

$$g(A_V X, Y) = g(h(X,Y), V) \tag{23}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

The mean curvature vector H of M is defined by:

$$H = \frac{1}{n} trace(h) = \frac{1}{n} \tag{24}$$

where $n$is the dimension of M and $e_i, i = 1 \dots . n$is a local orthogonal frame of M.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H \qquad (25)$$

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be totally geodesic if $h(X,Y) = 0$, and minimal submanifold if $H = 0$, for each $X, Y \in \Gamma(TM)$.

For any $X \in TM$, we can write:

$$\phi X = TX + NX \qquad (26)$$

where TX and NX are known as tangential components and normal component of ϕX on M, respectively.

Similarly, for any $V \in T^{\perp}M$, we have:

$$\phi U = tV + nV \qquad (27)$$

where tU and nU are the tangential component and normal component of ϕVon M, respectively.
The covariant derivative of projection morphisms in (26) and (27) are defines as:

$$\begin{cases} (\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \\ (\bar{\nabla}_X N)Y = \nabla_X^{\perp} NY - N\nabla_X Y, \\ (\bar{\nabla}_X t)V = \nabla_X tV - t\nabla_X V, \\ (\bar{\nabla}_X n)V = \nabla_X^{\perp} nV - n\nabla_X V \end{cases} \qquad (28)$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

3 Quasi Hemi-Slant Submanifolds of Lorentzian Concircular Structures – Manifolds or $(LCS)_n$- manifolds

Quasi hemi-slant submanifolds stand out as a significant concept in the field of differential geometry. Representing a broad class that includes various types of submanifolds such as semi-slant, slant, hemi-slant, invariant, anti-invariant, and semi-invariant submanifolds, this article delves into the fundamental features of quasi hemi-slant submanifolds and their importance in terms of generalization. Quasi Hemi-Slant Submanifolds are associated with Lorentzian Concircular Structures and other concepts in differential geometry. These submanifolds represent a generalization concept with specific geometric properties, encompassing different classes of submanifolds.

In this part of an article, we introduce and study the definition of quasi-hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$- manifolds.

Definition 3.1. A submanifold M of Lorentzian concircular structures – manifolds or $(LCS)_n$ - manifolds $\bar{M}$ is known as quasi-hemi-slant submanifold if there exist distributions D, D^{θ} and$D^{\perp}$such that

(i) TM admits the orthogonal direct decomposition as:

$$TM = D \oplus D^{\theta} \oplus D^{\perp} \oplus < \xi > \qquad (29)$$

(ii) The distribution D is ϕ invariant, i.e., $\phi D = D$.
(iii) The distribution D^{θ} is slant with an angle θ. The angle θ is known as slant angle.
(iv) The distribution $D^{\perp}$ is ϕ anti-invariant, i.e., $\phi D^{\perp} \subseteq T^{\perp}M$.

If this situation is met, we call the angle θ quasi-hemi-slant angle of M. Supposen_1, n_2 and n_3 are dimension of distributions D, D^{θ} and $D^{\perp}$ respectively. In this situation, We obtain a classification as follows:

(i) If $n_1 = 0$, then M is called hemi-slant submanifold.
(ii) If $n_2 = 0$, then M is called semi-invariant submanifold
(iii) If $n_3 = 0$, then M is called semi-slant submanifold

We can express that M is proper if $D \neq 0, D^{\perp} \neq \{0\}$ and $\theta \neq \frac{0 \wedge \pi}{2}$.
The statement suggests that above submanifolds serve as examples of quasi hemi-slant submanifolds, highlighting their role as instances and demonstrating the concept's generalization. .

Remark 3.2. We can generalize this concept by considering the following expressions :
$TM = D \oplus D^{\theta_1} \oplus D^{\theta_2} \dots \dots \dots \oplus D^{\theta_k} \oplus D^{\perp} \oplus < \xi >$ This structure are called multi-slant submanifolds,

Let M be a quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$ - manifolds $\bar{M}$. Let P, Q and R re the projection of the distributions D, D^{θ} and $D^{\perp}$ respectively. Then we can write:

$$X = PX + QX + RX + \eta(X)\xi \qquad (30)$$

For all $X \in \Gamma(TM)$
Now we put,

$$\phi X = TX + NX \qquad (31)$$

where TX and NX are called tangential component and normal component of ϕX on M.
By using equations (3.2) and (3.3), we get

$$\phi X = TPX + NPX + TQX + NQX + TRX + NRX. \tag{32}$$

As $\phi D = D$ and $\phi D^{\perp} \subseteq T^{\perp}M$, we have $NPX = 0$and $TRX = 0$.So, we get:

$$\phi X = TPX + TQX + NQX + NRX \tag{33}$$

Then for any $X \in \Gamma(TM)$, we can write:
$TX = TPX + TQX$,
and

$$NX = NQX + NRX.$$

So, from (33), we get the following decomposition

$$\phi(TM) = D \oplus TD^{\theta} \oplus ND^{\theta} \oplus ND^{\perp} \tag{34}$$

where, '$\oplus$'denotes orthogonal direct sum. Since $ND^{\theta} \subset T^{\perp}M$ and $ND^{\perp} \subset T^{\perp}M$, we have:

$$T^{\perp}M = ND^{\theta} \oplus ND^{\perp} \oplus \mu \tag{35}$$

where μ is known as the orthogonal complement of $ND^{\theta} \oplus ND^{\perp}$ in $\Gamma(T^{\perp}M)$ and it is invariant with respect to ϕ. For non-zero vector filed $V \in \Gamma(T^{\perp}M)$, then:

$$\phi V = tV + nV \tag{36}$$

where, $tV \in \Gamma(D^{\theta} \oplus D^{\perp})$ & $nV \in \Gamma(\mu)$.

Proposition 3.3. Let M be a quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$- manifolds $\bar{M}$, then for any $X \in \Gamma(TM)$, we have:

$$\nabla_X TY - A_{NY}X - T\nabla_X Y - t\sigma(X,Y) = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X],$$

$$h(X,TY) + \nabla_X^{\perp} NY - N\nabla_X Y - nh(X,Y) = 0,$$

and, $TD = D, TD^{\theta} = D^{\theta}, TD^{\perp} = \{0\}, tND^{\theta} = D^{\theta}, tND^{\perp} = D^{\perp}$.

Proof. Using equations (14), (21), (22), (26) and (27) and equating tangential component and normal component.

Proposition 3.4. Let M be a quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$ - manifolds $\bar{M}$ Then the endomorphism T, N, t and n in the tangent bundle of M satisfy the following identities.

$T^2 + tN = I + \eta \otimes \xi$ on TM,
$NT + nN = 0$ on TM,
$Nt + n^2 = I$ on $T^{\perp}M$,
$Tt + tn = 0$ on $T^{\perp}M$,

where I is the identity.

Proof. Using the equations (31) and (36) and using the fact that $\phi^2 = I + \eta \otimes \xi$, then on comparing tangential component and normal component.

Lemma 3.5. Let M be a quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$- manifolds$\bar{M}$, then:

$$T^2X = (cos^2\theta)X,$$
$$g(TX,TY) = (cos^2\theta)g(X,Y),$$
$$g(NX,NY) = (sin^2\theta)g(X,Y),$$

for any $X, Y \in D^{\theta}$.

Proof. The proof is analogous to Proposition 2.8 in [19].

Proposition 3.6. Let M be a quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$- manifolds$\bar{M}$, then:

$$(\bar{\nabla}_X T)Y = A_{NY}X + th(X,Y) + \alpha[g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi],$$
$$(\bar{\nabla}_X N)Y = nh(X,Y) - h(X,TY),$$
$$(\bar{\nabla}_X t)V = A_{nV}X - TA_V X$$
$$(\bar{\nabla}_X n)V = -h(X,tV) - NA_V X.$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. By utilizing equations (14), (21), (22), (26), (27), and (28) and equating their tangential and normal components."

Proposition 3.7. Let M be a quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$- manifolds$\bar{M}$, then
$h(X,\xi) = \alpha NX$ and $\nabla_X \xi = \alpha TX$ for any $X \in \Gamma(TM)$.

Proof. The proof is completed by equating the tangential and normal components using (8), (21) and (22).

Lemma 3.8. Let M be a quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$- manifolds$\bar{M}$, then

$$T([Z,W]) = A_{\phi Z}W - A_{\phi w}Z,$$
$$N([Z,W]) = \nabla_Z^{\perp}\phi W - \nabla_W^{\perp}\phi Z$$

for all and $Z, W \in D^{\perp}$.

Proof. For all and $Z, W \in D^{\perp}$ and using covariant differentiation in (14), we have

$$\bar{\nabla}_Z \phi W - \phi(\bar{\nabla}_Z W) = \alpha[g(Z,W)\xi + 2\eta(Z)\eta(W)\xi + \eta(W)Z]$$

Using (21) and (22), we have

$$-A_{\phi W}Z + \nabla_Z^{\perp}\phi W - \phi\big(\nabla_Z W + h(Z,W)\big) = \alpha g(Z,W)\xi$$

$$-A_{\phi W}Z + \nabla_Z^{\perp}\phi W - \phi(\nabla_Z W) - \phi h(Z,W) = \alpha g(Z,W)\xi$$

Using (26) and (27), we have

$$-A_{\phi W}Z + \nabla_Z^{\perp}\phi W - T(\nabla_Z W) - N(\nabla_Z W) - th(Z,W) - nh(Z,W) = \alpha g(Z,W)\xi$$

Using tangential and normal of this equation, we obtain:

$$A_{\phi W}Z + T(\nabla_Z W) + th(Z,W) = -\alpha g(Z,W)\xi \tag{37}$$

$$N(\nabla_Z W) - \nabla_Z^{\perp}\phi W + nh(Z,W) = 0 \tag{38}$$

Interchanging Z and W in (37) and (38), we can easily get the required outcomes.

Lemma 3.9. Let M be a quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$- manifolds$\bar{M}$, then

(3.11) $g([Y,Z],\xi) = 0,$

(3.12) $g(\bar{\nabla}_Y Z,\xi) = -\alpha g(TY,Z)$

for all $Y,Z \in \Gamma\big(D \oplus D^{\theta} \oplus D^{\perp}\big)$.

Proof. The equations (8) and (26) give us the proof.

4 Integrability of Distribution

When a differential distribution is integrable, it signifies that there is a specific order or structure defined on the submanifold. This concept has found application in geometry, mathematical physics, control theory, and various other mathematical subdisciplines. For example, it is often used to understand how a submanifold varies under a particular distribution.

Theorem 4.1. Let M be a proper quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$ - manifolds $\bar{M}$, then the distribution D is integrable if and only if

$$(\nabla_X TY - \nabla_Y TX, TQZ) + g(h(X,TY) - h(Y,TX), NQZ + NRZ) = 0 \tag{39}$$

for all $X,Y \in \Gamma(D)$ and $Z \in \Gamma\big(D^{\theta} \oplus D^{\perp}\big)$.

Proof. For all $X,Y \in \Gamma$ and $Z = QZ + RZ \in \Gamma\big(D^{\theta} \oplus D^{\perp}\big)$, we know that

$$g([X,Y],Z) = g(\bar{\nabla}_X Y,Z) - g(\bar{\nabla}_Y X,Z)$$

Using (11) in above equation, we have

$$g([X,Y],Z) = g(\phi\bar{\nabla}_X Y,\phi Z) - g(\phi\bar{\nabla}_Y X,\phi Z).$$

Using (14),we have

$$g(\phi(\widetilde{\nabla}_X Y),\phi Z) = g(\widetilde{\nabla}_X \phi Y - \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X],\phi Z) \tag{40}$$

As $\phi Z \in T^{\perp}M$, then from above

$$g(\phi(\widetilde{\nabla}_X Y),\phi Z) = g(\widetilde{\nabla}_X \phi Y,\phi Z).$$

Using (40), we get

$$g([X,Y],Z) = g(\bar{\nabla}_X \phi Y,\phi Z) - g(\bar{\nabla}_Y \phi X,\phi Z) \tag{41}$$

Using (21) in (41), we have:

$$g([X,Y],Z) = g(\nabla_X \phi Y + \sigma(X,\phi Y),\phi Z) - g(\nabla_Y \phi X + \sigma(Y,\phi X),\phi Z).$$

Using (26) in above equation , we have:

$$g([X,Y],Z) = g(\nabla_X TY + h(X,TY),\phi Z) - g(\nabla_Y TX + h(Y,TX),\phi Z)$$

Using equation (31) and as $\phi D^{\perp} \in T^{\perp}M$, so, $TRZ = 0$, then from above, we have:

$$g([X,Y],Z) = g(\nabla_X TY, TQZ) + g(h(X,TY), NQZ + NRZ) - g(\nabla_Y TX, TQZ) - g(h(Y,TX), NQZ + NRZ)$$

$$g([X,Y],Z) = g(\nabla_X TY - \nabla_Y TX, TQZ) + g(h(X,TY) - h(Y,TX), NQZ + NRZ)$$

As D is integrable, so we have the required outcomes.

Theorem 4.2. Let M be a proper quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$- manifolds$\bar{M}$, then the slant distribution D^{θ} is integrable if and only if:

$$g(A_{NTZ}Y - A_{NTY}Z, W) + g(A_{NY}Z - A_{NZ}Y, TPW) + g(\nabla_Y^{\perp}NZ - \nabla_z^{\perp}NY, NRW) = 0 \tag{42}$$

for all $Y,Z \in \Gamma\big(D^{\theta}\big)$ and $W \in \Gamma(D \oplus D^{\perp})$.

Proof. For all $Y,Z \in \Gamma\big(D^{\theta}\big)$ and $W = PW + RW \in \Gamma(D \oplus D^{\perp})$, we know that

$$g([Y,Z],W) = g(\bar{\nabla}_Y Z,W) - g(\bar{\nabla}_Z Y,W)$$

By employing equations (3) and (4), along with the concept of covariant differentiation, we obtain.

$$g([Y,Z],W) = g(\bar{\nabla}_Y \phi Z,\phi W) - g(\bar{\nabla}_Z \phi Y,\phi W) \tag{43}$$

Using (11) in (43), we have:

$$g([Y,Z],W) = g(\bar{\nabla}_Y(TZ+NZ),\phi W) - g(\bar{\nabla}_Z(TY+NY),\phi W)$$

$$g([Y,Z],W) = g(\bar{\nabla}_Y TZ,\phi W) + g(\bar{\nabla}_Y NZ,\phi W) - g(\bar{\nabla}_Z TY,\phi W) - g(\bar{\nabla}_Z NY),\phi W$$

$$g([Y,Z],W) = -g(\bar{\nabla}_Y \phi TZ,W) + g(\bar{\nabla}_Z \phi TY,W) + g(\bar{\nabla}_Y NZ,\phi W) - g(\bar{\nabla}_Z NY,\phi W)$$

Substituting equations (22) and (26) into the aforementioned equation, we obtain:

$$g([Y,Z],W) = -g$$

$$g([Y,Z],W) = -g(\bar{\nabla}_Y T^2 Z,W) - g(\bar{\nabla}_Y NTZ,W) + g(\bar{\nabla}_Z T^2 Y,W) + g(\bar{\nabla}_Z NTY,W) + g(-A_{NZ}Y + \nabla_Y^{\perp} NZ,\phi W) - g(-A_{NY}Z + \nabla_Z^{\perp} NY,\phi W)$$

$$g([Y,Z],W) = g(A_{NY}Z,\phi W) - g(A_{NZ}Y,\phi W) + g(\nabla_Y^{\perp} NZ,\phi W) - g(\nabla_Z^{\perp} NY,\phi W) - g(\bar{\nabla}_Y T^2 Z,W) + g(\bar{\nabla}_Z T^2 Y,W) + g(\bar{\nabla}_Z NTY,W) - g(\bar{\nabla}_Y NTZ,W)$$

Employing equation (22) in the preceding equation yields:

$$g([Y,Z],W) = g(A_{NY}Z,\phi W) - g(A_{NZ}Y,\phi W) + g(\nabla_Y^{\perp} NZ,\phi W) - g(\nabla_Z^{\perp} NY,\phi W) - g(\bar{\nabla}_Y T^2 Z,W) + g(\bar{\nabla}_Z T^2 Y,W) + g(-A_{NTY}Z + \nabla_Z^{\perp} NTY,W) - g(-A_{NTZ}Y + \nabla_Y^{\perp} NTZ,W)$$

$$g([Y,Z],W) = g(A_{NY}Z - A_{NZ}Y,\phi W) + g(\nabla_Y^{\perp} NZ - \nabla_Z^{\perp} NY,\phi W) - g(\bar{\nabla}_Y T^2 Z - \bar{\nabla}_Z T^2 Y,W) - g(A_{NTY}Z,W) + g(A_{NTZ}Y,W)$$

$$g([Y,Z],W) = g(A_{NY}Z - A_{NZ}Y,\phi W) + g(\nabla_Y^{\perp} NZ - \nabla_Z^{\perp} NY,\phi W) - g(\bar{\nabla}_Y T^2 Z - \bar{\nabla}_Z T^2 Y,W) + g(A_{NTZ}Y - A_{NTY}Z,W)$$

Applying equation (31) and Lemma 3.5 to the previous equation, we get:

$$g([Y,Z],W) = g(A_{NY}Z - A_{NZ}Y, TPW + TRW + NPW + NRW) + g(\nabla_Y^{\perp} NZ - \nabla_Z^{\perp} NY, TPW + TRW + NPW + NRW) - cos^2\theta g([Y,Z],W) + g(A_{NTZ}Y - A_{NTY}Z,W)$$

$$(1+cos^2\theta)g([Y,Z],W) = g(A_{NY}Z - A_{NZ}Y, TPW) + g(\nabla_Y^{\perp} NZ - \nabla_Z^{\perp} NY, NRW) + g(A_{NTZ}Y - A_{NTY}Z,W)$$

Since distribution D^θ is integrable, we have obtained the desired result.

Theorem 4.3. Let M be a proper quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$ - manifolds $\bar{M}$, then the antiderivative $D^\perp$ is integrable if and only if f:

$$\nabla_Z^{\perp} NY - \nabla_Y^{\perp} NZ \in ND^\theta \oplus \mu,$$

$$A_{NTZ}Y - A_{NTY}Z \in D^\theta, \text{ and}$$

$$A_{NZ}Y - A_{NY}Z \in D^\perp \oplus D^\theta$$

for all $Y, Z \in \Gamma(D^\theta)$, then the slant distribution D^θ is integrable.

Theorem 4.4. If M be a proper quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$ - manifolds $\bar{M}$, then the antiderivative distribution $D^\perp$ is integrable if:

$$g(T[Z,W],TY) + g(N[Z,W],NQY) = 0 \quad (44)$$

for all $Z, W \in \Gamma(D^\perp)$ and $Y \in \Gamma(D \oplus D^\theta)$.

Proof. For all $Z, W \in \Gamma(D^\perp)$ and $Y = PY + QY \in \Gamma(D \oplus D^\theta)$, as we know that

$$g([Z,W],Y) = g(\bar{\nabla}_Z W,Y) - g(\bar{\nabla}_W Z,Y)$$

Applying equations (3) and (4) to the equation above, we obtain:

$$g([Z,W],Y) = g(\bar{\nabla}_Z \phi W,\phi Y) - g(\bar{\nabla}_W \phi Z,\phi Y)$$

Using (7) in above we have:

$$g([Z,W],Y) = g(-A_{\phi W}Z + \nabla_Z^{\perp}\phi W,\phi Y) - g(-A_{\phi Z}W + \nabla_W^{\perp}\phi Z,\phi Y)$$

$$g([Z,W],Y) = g(A_{\phi Z}W - A_{\phi W}Z,\phi Y) - g(\nabla_W^{\perp}\phi Z - \nabla_Z^{\perp}\phi W,\phi Y)$$

Applying equation (31) in above, we obtain:

$$g([Z,W],Y) = g(A_{\phi Z}W - A_{\phi W}Z, TPY + TQY + NPY + NQY) - g(\nabla_W^{\perp}\phi Z - \nabla_Z^{\perp}\phi W, TPY + TQY + NPY + NQY)$$

$$g([Z,W],Y) = g\big(A_{\phi Z}W - A_{\phi W}Z, TPY + TQY\big) - g(\nabla^{\perp}_{W}\phi Z - \nabla_{Z}^{\perp}\phi W, NQY)$$

Using result of Lemma (3.8) in above, we have:

$$g([Z,W],Y) = g(T([Z,W]),TY) - g(-N([Z,W]),NQY)$$
$$g([Z,W],Y) = g(T([Z,W]),TY) - g(N([W,Z]),NQY)$$

Since the antiderivative distribution $D^{\perp}$ is integrable, therefore we have the desired result.

5 Totally Geodesic Foliations

Geodesicness and foliations are significant geometric notions. In this section, we will investigate the geometry of foliations of quasi hemi-slant submanifolds of Lorentzian concircular structures – manifolds or $(LCS)_n$- manifolds, also, some conditions are given for the totally Geodesicness.

Theorem 5.1. If M is a proper quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$ - manifolds $\bar{M}$, then M is totally geodesic if

$$g(h(X,PY) - cos^2\theta h(X,QY),V) + g(\nabla^{\perp}_{X}NY, nV) = g\big(A_{NQY}X + A_{NRY}X, tV\big) + g(\nabla^{\perp}_{X}NTQY, V) \quad (45)$$

For every $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. For all $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$ and using (30), we have

$$g(\bar{\nabla}_X Y, V) = g(\bar{\nabla}_X(PY + QY + RY + \eta(Y)\xi), V)$$

$$g(\bar{\nabla}_X Y, V) = g(\bar{\nabla}_X PY, V) + g(\bar{\nabla}_X QY, V) + g(\bar{\nabla}_X RY, V) \quad (46)$$

Using (11), we have:

$$g(\phi(\bar{\nabla}_X Y), \phi V) = g(\bar{\nabla}_X Y, V) \quad (47)$$

From (2.14), we have:

$$\bar{\nabla}_X \phi Y - \phi$$
$$\phi(\bar{\nabla}_X Y) = \bar{\nabla}_X \phi Y - \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X]$$

$$g(\phi(\bar{\nabla}_X Y), \phi V) = g(\bar{\nabla}_X \phi Y, \phi V) - \alpha g(X,Y)g(\xi,\phi V) - 2\alpha\eta(X)\eta(Y)g(\xi,\phi V) - \alpha\eta(Y)g(X,\phi V)$$
$$g(\phi(\bar{\nabla}_X Y), \phi V) = g(\bar{\nabla}_X \phi Y, \phi V) \quad (48)$$

Utilizing equations (30), (47) and (48) in (46), we obtain:

$$g(\phi(\bar{\nabla}_X Y), \phi V) = g(\bar{\nabla}_X \phi Y, \phi V)$$
$$g(\bar{\nabla}_X Y, V) = g(\bar{\nabla}_X PY, V) + g(\bar{\nabla}_X \phi QY, \phi V) + g(\bar{\nabla}_X \phi RY, \phi V)$$

Utilizing equations (21) and (26) in above equation, we have:

$$g(\bar{\nabla}_X Y, V) = g(\nabla_X PY + h(X,PY), V) + g(\bar{\nabla}_X(TQY + NQY), \phi V) + g(\bar{\nabla}_X(TRY + NRY), \phi V)$$
$$g(\bar{\nabla}_X Y, V) = g(h(X,PY), V) + g(\bar{\nabla}_X TQY, \phi V) + g(\bar{\nabla}_X NQY, \phi V) + g(\bar{\nabla}_X NRY, \phi V)$$
$$g(\bar{\nabla}_X Y, V) = g(h(X,PY), V) - g(\bar{\nabla}_X \phi TQY, V) + g(\bar{\nabla}_X NQY, \phi V) + g(\bar{\nabla}_X NRY, \phi V)$$
$$g(\bar{\nabla}_X Y, V) = g(h(X,PY), V) - g(\bar{\nabla}_X(T^2QY + NTQY), V) + g(\bar{\nabla}_X NQY, \phi V) + g(\bar{\nabla}_X NRY, \phi V)$$
$$g(\bar{\nabla}_X Y, V) = g(h(X,PY), V) - g(\bar{\nabla}_X T^2QY, V) - g(\bar{\nabla}_X NTQY, V) + g(\bar{\nabla}_X NQY, \phi V) + g(\bar{\nabla}_X NRY, \phi V)$$

Applying equation (22) and Lemma 3.5 in above equation, we get:

$$g(\bar{\nabla}_X Y, V) = g(h(X,PY), V) - cos^2\theta g(\bar{\nabla}_X QY, V) - g\big(-A_{NTQY}X + \nabla^{\perp}_{X}NTQY, V\big) + g\big(-A_{NQY}X + \nabla^{\perp}_{X}NQY, \phi V\big) + g(-A_{NRY}X + \nabla^{\perp}_{X}NRY, \phi V)$$

Utilizing (21) in above equation, we have:

$$g(\bar{\nabla}_X Y, V) = g(h(X,PY), V) - cos^2\theta g(\nabla_X QY + h(X,QY), V) - g(\nabla^{\perp}_{X}NTQY, V) - g\big(A_{NQY}X, \phi V\big) - g(A_{NRY}X, \phi V) + g(\nabla^{\perp}_{X}NQY, \phi V) + g(\nabla^{\perp}_{X}NRY, \phi V)$$

$$g(\bar{\nabla}_X Y, V) = g(h(X, PY), V) - cos^2\theta g(h(X, QY), V) - g(\nabla_X^{\perp} NTQY, V) - g\big(A_{NQY}X + A_{NRY}X, \phi V\big) + g(\nabla_X^{\perp} NQY + \nabla_X^{\perp} NRY, \phi V)$$

Applying (27) in above equation, we have

$$g(\bar{\nabla}_X Y, V) = g(h(X, PY) - cos^2\theta h(X, QY), V) - g(\nabla_X^{\perp} NTQY, V) - g\big(A_{NQY}X + A_{NRY}X, tV + nV\big) + g(\nabla_X^{\perp} NQY + \nabla_X^{\perp} NRY, tV + nV)$$

As $NY = NPY + NQY + NRY$ and $NPY = 0$, thus we have

$$g(\bar{\nabla}_X Y, V) = g(h(X, PY) - cos^2\theta h(X, QY), V) - g(\nabla_X^{\perp} NTQY, V) - g\big(A_{NQY}X + A_{NRY}X, tV\big) + g(\nabla_X^{\perp} NY, nV)$$

Therefore, the proof follows.

Theorem 5.2. Let M be a proper quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$ - manifolds $\bar{M}$, then the distribution D defines a totally geodesic foliation on M iff

$$g(\nabla_X TY, TQZ) + g(h(X, TY), NQZ + NRZ) = 0 \quad (49)$$

$$g(\nabla_X TY, tV) + g(h(X, TY), nV) = 0 \quad (50)$$

for all $X, Y \in \Gamma(D)$, $Z \in \Gamma\big(D^{\theta} \oplus D^{\perp}\big)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. For any $X, Y \in \Gamma(D)$, $Z = QZ + RZ \in \Gamma\big(D^{\theta} \oplus D^{\perp}\big)$ and using (11) and (14), we have:

$$g(\bar{\nabla}_X Y, Z) = g(\bar{\nabla}_X \phi Y, \phi Z) \quad (51)$$

Utilizing (26) in (51), we get:

$$g(\bar{\nabla}_X Y, Z) = g(\bar{\nabla}_X TY, \phi Z)$$

Utilizing (21) and (31), we get:

$$g(\bar{\nabla}_X Y, Z) = g(\nabla_X TY + h(X, TY), TQZ + TRZ + NQZ + NRZ)$$

$$g(\bar{\nabla}_X Y, Z) = g(\nabla_X TY, TQZ) + g(h(X, TY), NQZ + NRZ) \quad (52)$$

Now, for any $X, Y \in \Gamma(D)$, $V \in \Gamma(T^{\perp}M)$ and using (11), (14) and (31), we have

$$g(\bar{\nabla}_X Y, V) = g(\bar{\nabla}_X TY, \phi V) \quad (53)$$

Using (21) and (27) in (53), we have:

$$g(\bar{\nabla}_X Y, V) = g(\nabla_X TY + h(X, TY), tV + nV)$$

$$g(\bar{\nabla}_X Y, V) = g(\nabla_X TY, tV) + g(h(X, TY), nV) \quad (54)$$

Since distribution D defines a totally geodesic foliation on M, So, from (52) and (54), we obtained the desired results.

Theorem 5.3. Let M be a proper quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$ - manifolds $\bar{M}$, then the distribution $D^{\perp}$ defines a totally geodesic foliation on M if and only if

$$g(A_{NZ}Y, TPW + TQW) = g(\nabla_Y^{\perp} NZ, NQW) \quad (55)$$

$$g(A_{NZ}Y, tV) = g(\nabla_Y^{\perp} NZ, nV) \quad (56)$$

for any $Y, Z \in \Gamma(D^{\perp})$, $W \in \Gamma\big(D \oplus D^{\theta}\big)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. For any $Y, Z \in \Gamma(D^{\perp})$, $W = PW + QW \in \Gamma\big(D \oplus D^{\theta}\big)$ and using (11) and (14), we have:

$$g(\bar{\nabla}_Y Z, W) = g(\bar{\nabla}_Y \phi Z, \phi W)$$

Applying (26), in above, we have:

$$g(\bar{\nabla}_Y Z, W) = g(\bar{\nabla}_Y NZ, \phi W)$$

Utilizing (22) and (31), we have:

$$g(\bar{\nabla}_Y Z, W) = g(-A_{NZ}Y + \nabla_Y^{\perp} NZ, TPW + TQW + NPW + NQW)$$

$$g(\bar{\nabla}_Y Z, W) = -g(A_{NZ}Y, TPW + TQW) + g(\nabla_Y^{\perp} NZ, NQW) \quad (57)$$

Now, for any $Y, Z \in \Gamma(D^{\perp})$, $V \in \Gamma(T^{\perp}M)$ and using (11), (14) and(31), we have:

$$g(\bar{\nabla}_Y Z, V) = g(\bar{\nabla}_Y NZ, \phi V)$$

Using (22) and (27), we have:

$$g(\bar{\nabla}_Y Z, V) = g(-A_{NZ}Y + \nabla_Y^{\perp} NZ, tV + nV)$$

$$g(\bar{\nabla}_X Y, V) = -g(A_{NZ}Y, tV) + g(\nabla_Y^{\perp} NZ, nV) \quad (58)$$

Given that the distribution $D^{\perp}$ defines a totally geodesic foliation on M, thus, from (57) and (58), we obtain the desired results.

Theorem 5.4. If M be a proper quasi hemi-slant submanifold of Lorentzian concircular structures – manifolds or $(LCS)_n$ - manifolds $\bar{M}$, then the distribution D^θ defines a totally geodesic foliation on M iff

$$g(\nabla_Z^\perp NW, NRX) = g(A_{NW}Z, TPX) - g(A_{NTW}Z, X) \quad (59)$$

$$g(A_{NW}Z, tV) = g(\nabla_Z^\perp NW, nV) - g(\nabla_Z^\perp NTW, V) \quad (60)$$

for any $Z, W \in \Gamma(D^\theta), X \in \Gamma(D \oplus D^\perp)$ and $V \in \Gamma(T^\perp M)$.

Proof. For every $Z, W \in \Gamma(D^\theta)$, $X = PX + RX \in \Gamma(D \oplus D^\perp)$ and utilizing equations (11), (14) and (26), we obtain:

$$g(\bar{\nabla}_Z W, X) = g(\bar{\nabla}_Z TW, \phi X) + g(\bar{\nabla}_Z NW, \phi X)$$
$$g(\bar{\nabla}_Z W, X) = -g(\bar{\nabla}_Z \phi TW, X) + g(\bar{\nabla}_Z NW, \phi X)$$

Applying (22) and (26) in above, we have:

$$g(\bar{\nabla}_Z W, X) = -g(\bar{\nabla}_Z (T^2 W + NTW), X) + g(-A_{NW}Z + \nabla_Z^\perp NW, \phi X)$$
$$g(\bar{\nabla}_Z W, X) = -g(\bar{\nabla}_Z T^2 W, X) - g(\bar{\nabla}_Z NTW, X) - g(A_{NW}Z, \phi X) + g(\nabla_Z^\perp NW, \phi X)$$

Utilizing (22) and Lemma 3.5 in above equation, we have:

$$g(\bar{\nabla}_Z W, X) = -cos^2\theta g(\bar{\nabla}_Z W, X) - g(-A_{NTW}Z + \nabla_Z^\perp NTW, X) - g(A_{NW}Z, \phi X) + g(\nabla_Z^\perp NW, \phi X)$$
$$(1 + cos^2\theta) g(\bar{\nabla}_Z W, X) = g(A_{NTW}Z, X) - g(\nabla_Z^\perp NTW, X) - g(A_{NW}Z, \phi X) + g(\nabla_Z^\perp NW, \phi X)$$
$$(1 + cos^2\theta) g(\bar{\nabla}_Z W, X) = g(A_{NTW}Z, X) - g(A_{NW}Z, \phi X) + g(\nabla_Z^\perp NW, \phi X)$$

Using (31) in above, we have:

$$(1 + cos^2\theta) g(\bar{\nabla}_Z W, X) = g(A_{NTW}Z, X) - g(A_{NW}Z, TPX + TRX + NPX + NRX) + g(\nabla_Z^\perp NW, TPX + TRX + NPX + NRX)$$

Applying the fact that $NPX = 0$ in above, we have:

$$(1 + cos^2\theta) g(\bar{\nabla}_Z W, X) = g(A_{NTW}Z, X) - g(A_{NW}Z, TPX) + g(\nabla_Z^\perp NW, NRX) \quad (61)$$

Now, for any $Z, W \in \Gamma(D^\theta), V \in \Gamma(T^\perp M)$ and using (11), (14) and (26), we have:

$$g(\bar{\nabla}_Z W, V) = -g(\bar{\nabla}_Z \phi TW, V) + g(\bar{\nabla}_Z NW, \phi V)$$

Using (22) and (26), we have:

$$g(\bar{\nabla}_Z W, V) = -g(\bar{\nabla}_Z (T^2 W + NTW), V) + g(-A_{NW}Z + \nabla_Z^\perp NW, \phi V)$$
$$g(\bar{\nabla}_Z W, V) = -g(\bar{\nabla}_Z T^2 W, V) - g(\bar{\nabla}_Z NTW, V) + g(-A_{NW}Z + \nabla_Z^\perp NW, \phi V)$$

Using (22), (27) and Lemma 3.5 in above, we have:

$$g(\bar{\nabla}_Z W, V) = -cos^2\theta g(\bar{\nabla}_Z W, V) - g(-A_{NTW}Z + \nabla_Z^\perp NTW, V) - g(A_{NW}Z, tV + nV) + g(\nabla_Z^\perp NW, tV + nV)$$

$$(1 + cos^2\theta) g(\bar{\nabla}_Z W, V) = -g(\nabla_Z^\perp NTW, V) - g(A_{NW}Z, tV) + g(\nabla_Z^\perp NW, nV) \quad (62)$$

Since the distribution D^θ defines a totally geodesic foliation on M, therefore from (61) and (62), we have desired results.

6 Conclusion

New researchers or audiences can use the above to find some interesting results using Golden Structure or Golden Riemannian Manifolds, 3-dimensional Lorentzian Concircular Structures. These intersections of mathematical concepts offer fertile ground for research and can unveil fascinating properties and relationships. Researchers and audiences interested in these topics may discover compelling results by investigating the interplay between these areas and exploring the unique characteristics they bring to each other.

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- Sheeba Rizvi was responsible for the development and design of the methodology
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