Analytical Solutions of the Blasius Equation by Perturbation Iteration Method

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Abstract:- The Blasius equation is treated by employing the Perturbation Iteration method. Analytical solutions are derived for different perturbation iteration algorithms. Solutions are contrasted with the numerical solution obtained by an adaptive step size Runge-Kutta algorithm. It is found that the series type perturbation iteration solution better represents the behavior inside the boundary layer whereas the exponentially decaying perturbation iteration solution better represents the real solution outside the boundary layer. A composite expansion uniting both solutions and valid over the whole region is constructed using the gamma interval functions. The composite analytical solution is indistinguishable from the numerical one and can replace the numerical solution in calculations.

Key-Words:- Perturbation Methods, Perturbation-Iteration Algorithm, Boundary Layer Theory, Blasius Equation, Analytical Solutions

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1 Introduction

Due to their complexity, Navier-Stokes equations are hard to solve analytically except for some restricted boundary conditions. An excellent approximation of the Navier-Stokes equation was proposed [1] by introducing the boundary layer concept which led to vast applications in technology especially in the field of aerodynamics. The partial differential equations were transformed into an ordinary differential equation, namely the Blasius equation, via similarity transformations and an approximate series solution to the equation were given [1]. For boundary layer type of problems, in the vicinity of the boundary, a sharp divergence from the global solution exists. The solution inside the boundary layer is usually called the inner solution and the solution outside the boundary layer, the outer solution in the context of perturbation analysis. The outer solution is usually a solution converging to a simple form whereas the boundary layer solution exists in the vicinity of the boundary with a sharp deviation from the outer solution. The two different characteristics of the solutions make it hard to establish an analytical solution throughout the whole domain.

Due to its fundamental importance, Blasius equation, a third order nonlinear ordinary differential equation, attracted the attention of many researchers. A vast number of analytical and numerical techniques were employed in search of solutions. Blasius equation is solved by the Variational Iteration Method (VIM) and its variants [2-4], Adomian Decomposition Method [5, 6], Parameter Iteration method [7], reproducing kernel method [8], semi analytic iterative method [9], Iteration perturbation method [10], an analytical self-consistent method [11], Homotopy Perturbation method [12] and a variant of it [13], Quasi linearization method [14], Adomian Kamal Transform method [15], the Differential Transform Method [16], the Optimal Homotopy Asymptotic Method [17], Sinc-Collocation Method [18]. Bougoffa and Wazwaz [19] assumed an initial exact solution which does not satisfy the boundary conditions and found an analytical iterative solution. Numerical solutions of the equation were presented by applying the Crocco-Wang transformation [20]. For a theoretical mathematical study of the generalized Blasius equation, see [21].

The Perturbation Iteration method (PIM) is applied to Blasius equation for the first time in this work. PIM is a systematic way of producing perturbation iteration algorithms, PIA(n,m) where n represents the number of correction terms in the perturbation expansion and m represents the order of the derivatives in the Taylor expansions. The method is developed originally for nonlinear algebraic equations [22] and the formalism is later applied to differential equations [23,24]. In the last decade, the method has been successfully applied to many mathematical models arising from physical problems [25-48]. A general convergence analysis of PIM as well as an error analysis is given in [49].

In this work, it is found that the PIA(1,1) solution describes well the real solution inside the boundary layer and the PIA(1,2) solution describes the real solution well in the far end of the boundary layer and outside the boundary layer. Hence, a composite analytical solution is constructed using gamma interval functions which is indistinguishable from the numerical solution within the whole domain.

2 Boundary Layer Equations and Reduction

The boundary layer equations for a fluid passing over a flat plate are [1]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \quad (2)$$

with the no-slip boundary conditions at the plate and the condition at infinity being

$$u(x,0)=0, \quad v(x,0)=0, \quad u(x,\infty)=U(x) \quad (3)$$

where $u(x,y)$ and $v(x,y)$ represents the x and y components of the velocity, U is the inviscid incompressible velocity outside the boundary layer and ν is the kinematic viscosity. Blasius transformed the equations into a nonlinear ordinary differential equation via the similarity transformations

$$\eta=y\sqrt{\frac{U}{\nu x}}, \quad u(x,y)=Uf'(\eta),$$

$$v(x,y)=\frac{1}{2}\sqrt{\frac{\nu U}{x}}(\eta f'(\eta)-f(\eta)) \quad (4)$$

which leaded to the well-known Blasius equation

$$f'''+\frac{1}{2}ff''=0 \quad (5)$$

with the transformed boundary conditions for the flow over a flat plate

$$f(0)=0\ ,\ f'(0)=0,\ f'(\infty)=1\ . \quad (6)$$

From the similarity transformations, it is obvious that the x component of the velocity is directly related to $f'(\eta)$ and the y component is related to $f(\eta)$ and $f'(\eta)$. Analytical and numerical solutions will be presented in the following sections.

3. PIA(1,1) Solution

The general functional form of the Blasius equation is

$$F(f,f'',f''';\varepsilon)=f'''+\frac{1}{2}\varepsilon ff''=0 \quad (7)$$

where ε is added in front of the nonlinear term as a book-keeping parameter which will be eliminated in the final iteration equation. In the PIA(1,1) algorithm, only one correction term in the perturbation expansion and first order derivatives in the Taylor expansion are considered. Hence,

$$f_{n+1}=f_n+\varepsilon(f_c)_n \qquad n=0,1,2,\ldots \quad (8)$$

where f_n is the n'th iteration solution, $\varepsilon(f_c)_n$ is the correction term at the n'th iteration. Substituting (8) into (7) and expanding the Taylor series up to first order derivatives in the vicinity of ε=0 yields the iteration algorithm

$$F|_{\varepsilon=0}+F_f\big|_{\varepsilon=0}\varepsilon(f_c)_n+F_{f''}\big|_{\varepsilon=0}\varepsilon(f_c)_n''+F_{f'''}\big|_{\varepsilon=0}\varepsilon(f_c)_n'''+\varepsilon F_\varepsilon|_{\varepsilon=0}\cong 0 \quad (9)$$

Since $F|_{\varepsilon=0}=f_n'''$, $F_f\big|_{\varepsilon=0}=0$, $F_{f''}\big|_{\varepsilon=0}=0$, $F_{f'''}\big|_{\varepsilon=0}=1$, $F_\varepsilon|_{\varepsilon=0}=\frac{1}{2}f_nf_n''$, and $\varepsilon(f_c)_n=f_{n+1}-f_n$, substituting all into (9) yields the iteration algorithm

$$f_{n+1}'''=-\frac{1}{2}f_nf_n'' \qquad n=0,1,2,\ldots\ . \quad (10)$$

The boundary value problem is transformed into an initial value problem

$$f(0)=0\ ,\ f'(0)=0,\ f''(0)=\beta \quad (11)$$

for practical purposes. Starting from an initial trivial solution

$$f_0=0 \quad (12)$$

the consecutive iteration solutions which satisfy the conditions (11) are

$$f_1=\frac{\beta}{2}\eta^2 \quad (13)$$

$$f_2=\frac{\beta}{2}\eta^2-\frac{\beta^2}{240}\eta^5 \quad (14)$$

$$f_3=\frac{\beta}{2}\eta^2-\frac{\beta^2}{240}\eta^5+\frac{11\beta^3}{161280}\eta^8-\frac{\beta^4}{5702400}\eta^{11} \quad (15)$$

and the derivative of the third iteration is

$$f_3'=\beta\eta-\frac{\beta^2}{48}\eta^4+\frac{11\beta^3}{20160}\eta^7-\frac{\beta^4}{518400}\eta^{10} \quad (16)$$

Higher iterations are not considered, as the denominators of the coefficients become extensively large and the aim is to seek a simple solution.

The PIA(1,1) in fact generates a series type solution which can be deduced by substituting

$$f=\sum_{i=0}^{\infty}a_i\eta^i \quad (17)$$

into the original equation and equating to zero the coefficients of like powers

$$i(i-1)(i-2)a_i+\frac{1}{2}\sum_{j=0}^{i-1}(i-j-1)(i-j-2)a_{i-j-1}a_j=0, \quad i=3,4,5,\ldots \quad (18)$$

The initial conditions require $a_0=0$, $a_1=0$ and $a_2=\frac{\beta}{2}$. Other coefficients can be calculated up to any arbitrary order of η. The series solution up to η^{11} is

$$f = \frac{\beta}{2}\eta^2 - \frac{\beta^2}{240}\eta^5 + \frac{11\beta^3}{161280}\eta^8 - \frac{5\beta^4}{4257792}\eta^{11} \quad (19)$$

which is also the solution given by Blasius [2]. Comparing (15) and (19), one realizes that there is a discrepancy in the coefficient of the last term. This stems from the termination of the iteration at n=3. However, since the terms higher than η^{11} are totally ignored in the series expansion, with the previous experience for such solutions, PIA(1,1) solution may include the effects of such terms in the last term and may slightly perform better than the series solution. By using Variational Iteration Method, Wazwaz [3] calculated higher order terms also.

To test the analytical solutions, an adaptive step size numerical solution employing Runge-Kutta method is used. The Blasius equation is expressed as a system of first order equations by defining $f_1 = f, f_2 = f', f_3 = f''$,

$$f_1' = f_2 \quad (20)$$

$$f_2' = f_3 \quad (21)$$

$$f_3' = -\frac{1}{2}f_1 f_3 \quad (22)$$

and the boundary value problem is converted to an initial value problem

$$f_1(0) = 0, f_2(0) = 0, f_3(0) = \beta \quad (23)$$

The specific value of β which makes the solution satisfy the condition $f_2(\infty) = 1$ is determined by the shooting technique

$$\beta = 0.3320573 \quad (24)$$

Within the tolerances of our numerical algorithm, there is no need to take a more precise value. As mentioned earlier, f' determines the x component of the velocity whereas f' and f determines the y component. Hence, comparisons of these quantities are given in Figures 1 and 2.

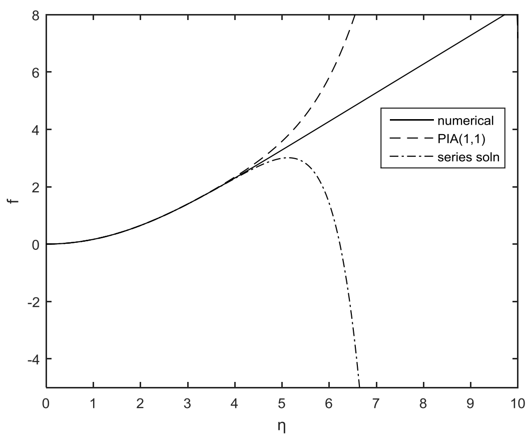

Fig. 1: Comparisons of PIA(1,1) and series solutions with the numerical solution for f

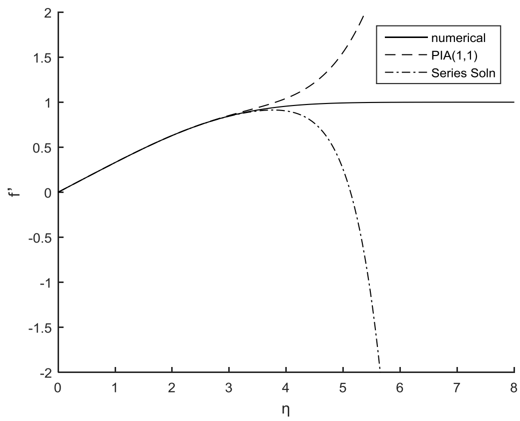

Fig. 2: Comparisons of PIA(1,1) and series solutions with the numerical solution for f'

From both figures, the analytical solutions start diverging after $\eta = 3$. The series solution is slightly more divergent than the PIA(1,1) solution as expected. From Figure 2, the boundary layer reaches the fully developed state after $\eta = 4$. Excellent match is observed in the interval [0, 3] but the solutions are not reliable as one proceeds to higher η values.

For a general convergence and error analysis, refer to [49] for details.

4 PIA(1,2) Solution

For constructing the PIA(1,2) algorithm, one takes one correction term in the perturbation expansion similar to the previous case, i.e. Eq. (8), but expands the Taylor series up to second order derivatives

$$F|_{\varepsilon=0} + F_f|_{\varepsilon=0}\varepsilon(f_c)_n + F_{f''}|_{\varepsilon=0}\varepsilon(f_c)_n'' + F_{f'''}|_{\varepsilon=0}\varepsilon(f_c)_n''' + F_\varepsilon|_{\varepsilon=0}\varepsilon + \frac{1}{2}F_{ff}|_{\varepsilon=0}(\varepsilon(f_c)_n)^2 + \frac{1}{2}F_{f''f''}|_{\varepsilon=0}\left(\varepsilon(f_c)_n''\right)^2 + \frac{1}{2}F_{f'''f'''}|_{\varepsilon=0}\left(\varepsilon(f_c)_n'''\right)^2 + \frac{1}{2}F_{\varepsilon\varepsilon}|_{\varepsilon=0}\varepsilon^2 + F_{ff''}|_{\varepsilon=0}\varepsilon(f_c)_n\varepsilon(f_c)_n'' + F_{ff'''}|_{\varepsilon=0}\varepsilon(f_c)_n\varepsilon(f_c)_n''' + F_{f\varepsilon}|_{\varepsilon=0}\varepsilon^2(f_c)_n + F_{f''f'''}|_{\varepsilon=0}\varepsilon(f_c)_n''\varepsilon(f_c)_n''' + F_{f''\varepsilon}|_{\varepsilon=0}\varepsilon^2(f_c)_n'' + F_{f'''\varepsilon}|_{\varepsilon=0}\varepsilon^2(f_c)_n''' \cong 0 \quad (25)$$

After evaluating the derivatives at $\varepsilon = 0$ and using (8), one has the iteration equation

$$f_{n+1}''' + \frac{1}{2}f_n f_{n+1}'' + \frac{1}{2}f_n'' f_{n+1} = \frac{1}{2}f_n f_n'', n = 0,1,2, \ldots \quad (26)$$

The boundary value problem need not be transformed as in the PIA(1,1) case, that is

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1\ . \quad (27)$$

Starting with an initial constant solution

$$f_0 = 2\alpha \quad (28)$$

the first iteration solution is

$$f_1 = -\frac{1}{\alpha}(1 - e^{-\alpha\eta}) + \eta \quad (29)$$

which satisfies the conditions (27) for any α. The first derivative is

$$f_1' = 1 - e^{-\alpha\eta} \quad (30)$$

For the second iteration,

$$f_2''' + \frac{1}{2}f_1 f_2'' + \frac{1}{2}f_1'' f_2 = \frac{1}{2}f_1 f_1'' \quad . \quad (31)$$

Upon substituting f_1, the equation is a variable coefficient linear third order equation, hard to solve. Instead, just for the coefficients at the left-hand side of the equation, one may assume a simplification by taking f_0 instead of f_1. The final solution satisfying the boundary conditions is

$$f_2 = -\frac{1}{\alpha}\left(1 + \frac{5}{8\alpha^2}\right) + \eta + \left[\frac{1}{\alpha}\left(1 + \frac{3}{4\alpha^2}\right) + \frac{1}{2\alpha^2}\eta + \frac{1}{4\alpha^2}\eta^2\right]e^{-\alpha\eta} - \frac{1}{8\alpha^3}e^{-2\alpha\eta} \quad (32)$$

with its derivative

$$f_2' = 1 - \left(1 + \frac{1}{4\alpha^2} + \frac{1}{4}\eta^2\right)e^{-\alpha\eta} + \frac{1}{4\alpha^2}e^{-2\alpha\eta} \quad (33)$$

Note that α remains arbitrary in the first and second iteration solutions. It can be selected so that the match between the numerical results is better. In Figures 3 and 4, the first and second iteration solutions are contrasted with the numerical ones.

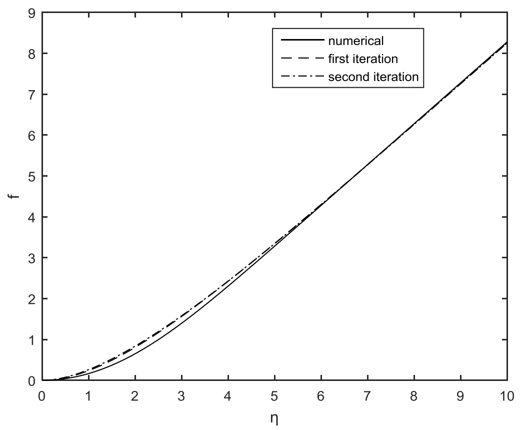

Fig. 3: Comparisons of first and second iteration PIA(1,2) with the numerical solution for f

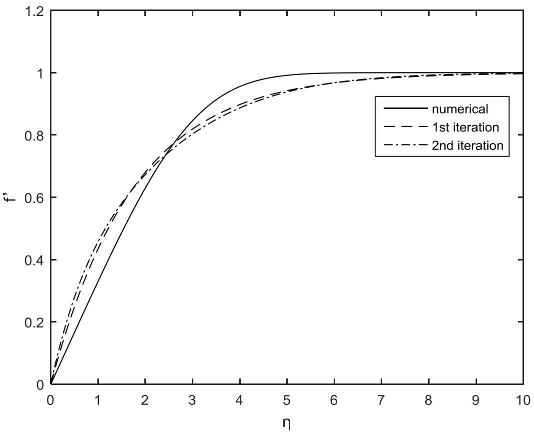

Fig. 4: Comparisons of first and second iteration PIA(1,2) with the numerical solution for f'

For the first iteration $\alpha = 0.57$, and for the second iteration $\alpha = 0.96$. From both figures, it is evident that the second iteration solution does not perform better than the first iteration. One reason might be that, in obtaining the second iteration, a simplifying assumption about the coefficients of the equation is made. On the contrary, there are no blow ups in the whole domain of interest as in the PIA(1,1) and series solution cases and the solution is admissible albeit with some small fractional errors. The errors introduced are mainly inside the boundary layer. Outside the boundary layer, the solution matches better with the numerical solution.

The convergence and error analysis of PIM is an important issue and the problem is addressed in detail in [49].

5 A Composite Solution

The aim is to construct an analytical solution which can replace the numerical solution and is valid throughout the whole domain of interest. Based on the results presented in the previous sections, PIA(1,1) solution will be taken in the interval [0 η_i] and PIA(1,2) solution will be taken in the interval [η_i ∞]. The reason for this choice is that PIA(1,1) is more successful in representing the behavior inside the boundary layer and PIA(1,2) is more successful outside it. For a smooth connection at the intermediate junction point η_i, the functions, the first and second derivatives should be equated. At η_i, from (15), they are

$$f_i = \frac{\beta}{2}\eta_i^2 - \frac{\beta^2}{240}\eta_i^5 + \frac{11\beta^3}{161280}\eta_i^8 - \frac{\beta^4}{5702400}\eta_i^{11} \quad (34)$$

$$f_{ip} = \beta\eta_i - \frac{\beta^2}{48}\eta_i^4 + \frac{11\beta^3}{20160}\eta_i^7 - \frac{\beta^4}{518400}\eta_i^{10} \quad (35)$$

$$f_{ipp} = \beta - \frac{\beta^2}{12}\eta_i^3 + \frac{11\beta^3}{2880}\eta_i^6 - \frac{\beta^4}{51840}\eta_i^9 \quad (36)$$

The first iteration PIA(1,2) is sufficient for calculations and the form of the solution is

$$f^0 = c_1 + c_2(\eta - \eta_i) + c_3 e^{-\alpha(\eta-\eta_i)} \quad (37)$$

which is called the outer solution valid as η approaches infinity. Equating the function, first and second derivatives of the outer solution to the inner solution at the intermediate point, i.e. Eqs. (34)-(36), the coefficients are calculated

$$c_1 = f_i - \frac{f_{ipp}}{\alpha^2}, \; c_2 = f_{ip} + \frac{f_{ipp}}{\alpha}, \; c_3 = \frac{f_{ipp}}{\alpha^2}. \quad (38)$$

To write the composite solution as a single expression, one may define a new gamma interval function with properties

$$\gamma[a,b)(x) = \begin{cases} 1 & a \le x < b \\ 0 & x < a, x \ge b \end{cases} \quad (39)$$

The composite solution and its first derivative are expressed as single analytical expressions

$$f(\eta) = \left\{\frac{\beta}{2}\eta^2 - \frac{\beta^2}{240}\eta^5 + \frac{11\beta^3}{161280}\eta^8 - \frac{\beta^4}{5702400}\eta^{11}\right\}\gamma[0,\eta_i)(\eta) + \left\{f_i - \frac{f_{ipp}}{\alpha^2} + \left(f_{ip} + \frac{f_{ipp}}{\alpha}\right)(\eta - \eta_i) + \frac{f_{ipp}}{\alpha^2}e^{-\alpha(\eta-\eta_i)}\right\}\gamma[\eta_i, \infty)(\eta) \quad (40)$$

$$f'(\eta) = \left\{\beta\eta - \frac{\beta^2}{48}\eta^4 + \frac{11\beta^3}{20160}\eta^7 - \frac{\beta^4}{518400}\eta^{10}\right\}\gamma[0,\eta_i)(\eta) + \left\{f_{ip} + \frac{f_{ipp}}{\alpha}\left(1 - e^{-\alpha(\eta-\eta_i)}\right)\right\}\gamma[\eta_i, \infty)(\eta) \quad (41)$$

The composite expansions (40) and (41) are contrasted with the numerical solutions in Figures 5 and 6. In the calculations, η_i=3, $\alpha = 1.23$, β =0.3320573 are selected.

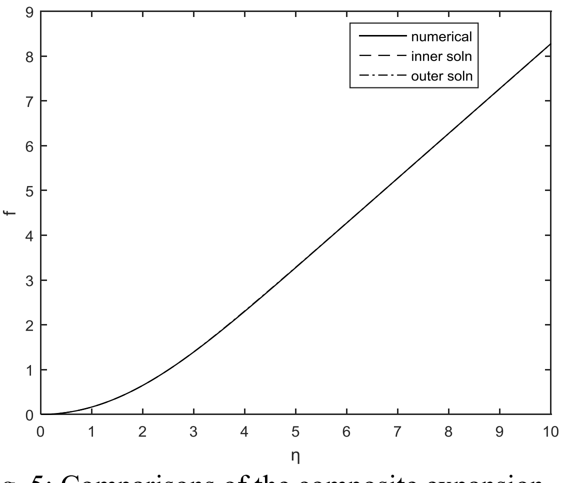

Fig. 5: Comparisons of the composite expansion with the numerical solution for f

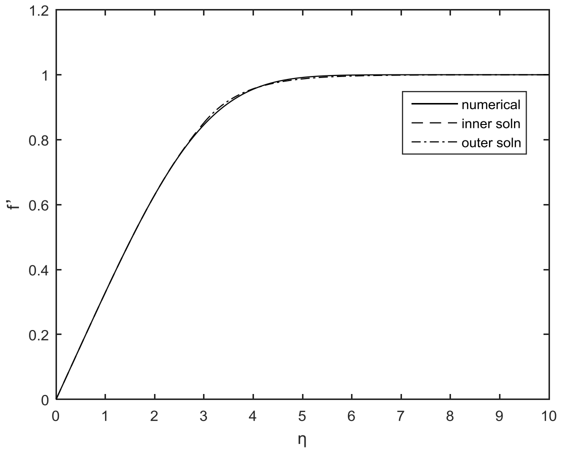

Fig. 6: Comparisons of the composite expansion with the numerical solution for f'

From the figures, the composite solution is almost indistinguishable from the numerical solution. Hence, instead of the discrete numerical solution, the composite solution can be used safely for a continuous expression of the solution.

6 Concluding Remarks

Perturbation iteration method is used for solving the Blasius equation for the first time. Iteration algorithms PIA(1,1) and PIA(1,2) are employed in search of approximate analytical solutions. It is found that PIA(1,1) better represents the numerical solutions within the boundary layer adjacent to the plate whereas PIA(1,2) better represents the numerical solutions at the far edge of the boundary layer and outside the boundary layer. Hence a composite expansion valid throughout the whole domain is constructed by combining both solutions. Most of the series solutions presented in the literature diverge at the edge and outside the boundary layer because the nature of the solutions is different inside and outside of the boundary layer. The composite expansion has the advantage of representing the solutions precisely in both domains. As a final conclusion, the continuous composite expansion can safely replace the discrete numerical solutions.

References:

[1] Schilichting H., *Boundary Layer Theory*, McGraw Hill, New York, 2004.

[2] He J., Approximate analytical solution of Blasius equation, *Communications in Nonlinear Science and Numerical Simulation*, Vol. 4, No. 1, 1999, pp. 75-78.

[3] Wazwaz A. M., The variational iteration method for solving two forms of Blasius equation on a half-infinite domain, *Applied Mathematics and Computation*, Vol. 188, 2007, pp. 485-491.
[4] Sajid M., Abbas Z., Ali N. and Javed T., A hybrid variational iteration method for Blasius equation, *Applications and Applied Mathematics: An International Journal*, Vol. 10, No. 1, 2015, 223-229.
[5] Wang L., A new algorithm for solving classical Blasius equation, *Applied Mathematics and Computation*, Vol. 157, 2004, No. 1-9.
[6] Abbasbandy S., A numerical solution of Blasius equation by Adomian's decomposition method and comparison with homotopy perturbation method, *Chaos, Solitons and Fractals*, Vol. 31, 2007, pp. 257-260.
[7] Lin J., A new approximate iteration solution of Blasius equation, *Communications in Nonlinear Science and Numerical Simulation*, Vol. 4, No. 2, 1999, pp. 91-94.
[8] Akgül A., A novel method for the solution of Blasius equation in semi-infinite domains, *An International Journal of Optimization and Control: Theories and Applications*, Vol. 7, No. 2, 2017, 225-233.
[9] Selamat M. S., Halmi N. A. and Ayob N. A., A semi analytic iterative method for solving two forms of Blasius equation, *Journal of Academia*, Vol. 7, No. 2, 2019, pp. 76-85.
[10] He J. H., A simple perturbation approach to Blasius equation, *Applied Mathematics and Computation*, Vol. 140, 2003, pp. 217-222.
[11] Yasuri A. K., An analytical self-consistent method for differential forms of the Blasius equation, *Mathematical Methods in the Applied Sciences*, Vol. 46, 2023, pp. 5836-5849.
[12] Ganji D. D., Babazadeh H., Noori F., Pirouz M. M.and Janipour M., An application of homotopy perturbation method for nonlinear Blasius equation to boundary layer flow over a flat plate, *International Journal of Nonlinear Science*, Vol. 7, No. 4, 2009, pp. 399-404.
[13]Aminikhah H., An analytical approximation for solving nonlinear Blasius equation by NHPM, Numerical Methods for Partial Differential Equations, Vol. 26, No. 6, 2010, pp. 1291-1299.
[14] Delkhosh M. and Cheraghian H., An efficient hybrid method to solve nonlinear differential equations in applied sciences, *Computational and Applied Mathematics*, Vol. 41, Article No. 322, 2022.
[15] Khandelwal R., Kumawat P. and Khandelwal Y., Solution of the Blasius equation by using Adomian Kamal Transform, *International Journal of Applied and Computational Mathematics*, Vol. 5, Article No. 20, 2019.
[16] Peker H. A., Karaoğlu O. and Oturanç G., The differential transformation method and pade approximant for a form of Blasius equation, *Mathematical and Computational Applications*, Vol. 16, No. 2, 2011, pp. 507-513.
[17] Marinca V. and Herişanu N., The optimal homotopy asymptotic method for solving Blasius equation, *Applied Mathematics and Computation*, Vol. 231, 2014, pp. 134-139.
[18] Parand K., Dehghan M. and Pirkhedri A., Sinc-collocation method for solving the Blasius equation, *Physics Letters A*, Vol. 373, 2009, pp. 4060-4065.
[19] Bougoffa L and Wazwaz A. M., New approximate solutions of the Blasius equation, *International Journal of Numerical Methods for Heat and Fluid Flow*, Vol. 25, No. 7, 2015, pp. 1590-1599.
[20] Asaithambi A., Numerical solution of the Blasius equation with Crocco-Wang transformation, *Journal of Applied Fluid Mechanics*, Vol. 9, No. 5, 2016, pp. 2595-2603.
[21] Makhfi A. and Bebbouchi R., On the generalized Blasius equation, *Afrika Matematika*, Vol. 31, 2020, pp. 803-811.
[22] Pakdemirli M., Boyacı H., Generation of root finding algorithms via perturbation theory and some formulas, *Applied Mathematics and Computation*, Vol. 184, No. 2, 2007, pp. 783-788.
[23] Aksoy Y. and Pakdemirli M., New Perturbation-Iteration Solutions for Bratu-type Equations, *Computers & Mathematics with Applications*, Vol. 59, 2010, pp. 2802-2808.
[24] Aksoy Y., Pakdemirli M., Abbasbandy, S, Boyacı, H., New Perturbation-Iteration Solutions for Nonlinear Heat Transfer Equations. *International Journal of Numerical Methods for Heat & Fluid Flow*, Vol. 22, 2012, pp. 814-828.
[25] Al Saif A. J. and Harfash A. J., A comparison between the reduced differential transform method and perturbation-iteration algorithm for solving two dimensional unsteady incompressible Navier-Stokes equations, *Journal of Applied Mathematics and Physics*, Vol. 6, 2018, pp. 2518-2543.
[26] Al-Saif A. J. and Harfash A. J., Perturbation Iteration Algorithm for solving heat and mass transfer in the unsteady squeezing flow between parallel plates, *Journal of Applied Computational Mechanics*, Vol. 5, No. 4, 2019, pp. 804-815.
[27] Bahşi M. M. and Çevik M., Numerical solution of Pantograph type delay differential equations using Perturbation Iteration Algorithms, *Journal of Applied Mathematics*, Vol. 2015, 2015.

[28] Bildik N. Optimal Perturbation-Iteration method for solving Telegraph equations, *International Journal of Applied Physics and Mathematics*, Vol. 7, No. 3, 2017, pp. 165-172.
[29] Bildik N. and Deniz S., A new efficient method for solving delay differential equations and a comparison with other methods, *The European Physical Journal Plus*, Vol. 132, Article No. 51, 2017.
[30] Bildik N. and Deniz S., New analytic approximate solutions to the generalized regularized long wave equations, *Bulletin of Korean Mathematical Society*, Vol. 55, No. 3, 2018, pp. 749-762.
[31] Bildik N. and Deniz S., Solving the Burgers and long wave equations using the new perturbation iteration technique, *Numerical Methods for Partial Differential Equations*, Vol. 34, 2018, pp. 1489-1501.
[32] Bildik N. and Deniz S., Comparative study between optimal homotopy asymptotic method and perturbation iteration technique for different types of Nonlinear equations, *Iranian Journal of Science and Technology, Transactions A-Science*, Vol. 42, 2018, 647-654.
[33] Deniz S. and Bildik N., A new analytical technique for solving Lane-Emden type equations arising in astrophysics, *Bulletin of Belgium Mathematical Society-Simon Stevin*, Vol. 24, No. 2, 2017, pp. 305-320.
[34] Deniz S. and Bildik N., Optimal Perturbation-Iteration method for Bratu type problems, *Journal of King Saud University-Science*, Vol. 30, 2018, pp. 91-99.
[35] Gahamanyi M., Ntganda J. M. and Hapgar M. S. D., Perturbation Iteration Method for solving mathematical model of glucose and insulin in diabetic human during physical activity. *Open Journal of Applied Sciences*, Vol. 6, 2016, pp. 826-838.
[36] Harfash A. J. and Al-Saif A. J., MHD flow of fourth grade fluid solve by PIA algorithm, *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences*, Vol. 59, No. 2, 2019, pp. 220-231.
[37] Khalid M., Sultana M., Zaidi F. and Khan J., A solution of a water quality model in a uniform stream channel using new iterative method, *International Journal of Computer Applications*, Vol. 115, No. 6, 2015, pp. 1-4.
[38] Khalid M., Sultana M., Zaidi F. and Arshaul U., An effective perturbation-iteration algorithm for solving Riccati differential equations, *International Journal of Computer Applications*, Vol. 111, No. 10, 2015, pp. 1-5.
[39] Khalid M., Sultana M., Zaidi F. and Uroosa A., Solving linear and non-linear Klein-Gordon equations by new perturbation iteration transform method, *TWMS Journal of Applied Engineering Mathematics*, Vol. 6, No. 1, 2016, pp. 115-125.
[40] Pakdemirli M., Perturbation–iteration method for strongly nonlinear vibrations, *Journal of Vibration and Control*, Vol. 23, No. 6, 2017, pp. 959-969.
[41] Singh R. P. and Reddy Y. N., Perturbation Iteration method for solving differential difference equations having boundary layer, *Communications in Mathematics and Applications* Vol. 11, No. 4, 2020, pp. 617-633.
[42] Srivastava H. M., Deniz S. and Saad K. M., An efficient semi-analytical method for solving the generalized regularized long wave equations with a new fractional derivative operator, *Journal of King Saud University-Science*, Vol. 33, Article No. 101345, 2021.
[43] Şenol M., Alquran M. and Kasmaci H. D., On the comparison of perturbation-iteration algorithms and residual power series method to solve fractional Zakharov-Kuznetsov equation, *Results in Physics*, Vol. 9, 2018, pp. 321-327.
[44] Şenol M., Atpinar S., Zararsız Z., Salahshour S. and Ahmadian A., Approximate solution of time fractional fuzzy partial differential equations, *Computational and Applied Mathematics*, Vol. 38, Article No. 18, 2019.
[45] Şenol M. and Dolapçı I. T., On the Perturbation Iteration Algorithm for fractional differential equations. *Journal of King Saud University-Science*, Vol. 28, 2016, pp. 69-74.
[46] Şenol M. and Kasmaci H. D., Perturbation-Iteration Algorithm for systems of fractional differential equations and convergence analysis, *Progress in Fractional Differentiation and Applications*, Vol. 3, No. 4, 2017, pp. 271-279.
[47] Tasbozan O., Şenol M., Kurt A. and Özkan O., New solutions of fractional Drinfeld-Sokolov-Wilson system in shallow water waves, *Ocean Engineering*, Vol. 161, 2018, pp. 62-68.
[48] Yıldız V., Pakdemirli M. and Aksoy Y., Parallel plate flow of a third- grade fluid and a Newtonian fluid with variable viscosity. *Zeitschrift fur Naturforschung A*, Vol. 71, No. 7, 2016, pp. 595-606.
[49] Bildik N., General Convergence Analysis for the Perturbation Iteration Technique, *Turkish Journal of Mathematics and Computer Science*, Vol. 6, 2017, pp. 1-9.

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