New exact traveling wave solutions of the non-linear (2+1)-dimensional Klein-Gordon equation

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Abstract: In this article, we discuss the nonlinear (2+1)-dimensional Klein-Gordon equation with an additional term. The functional variable method is used to construct exact solutions of the quadratic and cubic nonlinear (2+1)-dimensional Klein-Gordon equation. The exact solutions of these equations including soliton and periodic wave solutions are obtained. The advantage of the used method beyond other existing methods is that it provides more new exact solutions. Some selected solutions of the equations are presented graphically by Matlab program. This method is efficient and it can be successfully used to obtain another nonlinear wave equations in mathematical physics and engineering.

Key-Words: the quadratic non-linear Klein-Gordon equation, periodic solutions, the cubic non-linear Klein-Gordon equation functional, variable method, trigonometric function, soliton solutions, hyperbolic function.

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1 Introduction

The investigation of travelling wave solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. Nonlinear evolution equations are widely used to describe complex phenomena in various fields of sciences, especially in physics such as, plasma physics, fluid mechanics, optical fibers, solid state physics, nonlinear optics and so on.

The Klein-Gordon equation appeared studying solitons condensed matter physics, investigating the interaction of solitons in a collisionless plasma and the recurrence of initial states[1, 2, 3].

This equation is expressed in the following basic form

$$u_{tt} - u_{xx} - u_{yy} = a(u).$$

The KG equation most probably first arose in a mathematical context with $a(u) = e^u$ in the theory of constant surfaces in Liouville's work. The KG equation with cubic non-linearity $a(u) = u^3 - u$ has been used in [4]. In application, the nonlinear term is a nonlinear force. For $a(u) = sinu$ equation becomes sine-Gordon equation, which could also be sine-Gordon equation, double sinh-Gordon equation depending on the form of $a(u)$.

In order to obtain the exact solutions of nonlinear evolution equations, a number of methods have been proposed, such as the exp-function method [5, 6], the homotopy perturbation method [7], the tanh method [8], the Jacobi elliptic function method [9], functional variable method[10, 11, 12, 13, 14], Hirota method [15], Backlund transform method[16], exp-function method[17] and G/G' expansion method[18, 19].

The Klein-Gordon equation is a relativistic version of the Schrodinger equation describing free particles, which was proposed by Oskar Klein and Walter Gordon in 1926. Various methods have been developed to get approximate and numerical solutions of the nonlinear Klein-Gordon equations, such as Adomain decomposition method[20], the variational iteration method[21] and the modified Laplace decomposition method[22].

In this article, we study the non-linear (2+1)-dimensional KG equation with additional term

$$u_{tt} - u_{xx} - u_{yy} + b(u) = 0, \qquad (1)$$

where $b(u) = \alpha u + \beta u^n + \gamma u$, $u(x, y, t)$ is unknown functions, $n = 2, 3$, $x \in R$, $y \in R$, $t \geq 0$, α, β and γ are any constants.

We construct the nonlinear (2+1)-dimensional Klein-Gordon equation with an additional term. The functional variable method is used to construct exact solutions of the quadratic and cubic nonlinear (2+1)-dimensional Klein-Gordon equation. The exact solutions of these equations including soliton and periodic wave solutions are obtained. The advantage of the used method beyond other existing methods is that it provides more new exact solutions. Some selected solutions of the equations are presented graphically by Matlab program. This method is efficient and it can be successfully used to obtain another nonlinear wave equations in mathematical physics and engineering.

2 Algorithm for finding solutions

In this section the functional variable method [23] is used to find the exact solitary and periodic wave solutions of the quadratic non-linear (2+1)-dimensional Klein-Gordon equation:

$$u_{tt} - u_{xx} - u_{yy} + \alpha u + \beta u^2 + \gamma u = 0. \tag{2}$$

The wave variable

$$u(x,y,t) = u(\xi),\ \xi = px + qy - kt \tag{3}$$

will convert Eq.2 to the following ordinary differential equation

$$u'' = \frac{\beta u^2 + \mu(t)u}{p^2 + q^2 - k^2}, \tag{4}$$

where $u' = \frac{du}{d\xi}$, $\mu = \alpha + \gamma$, $p = const,\ q = const$ and k is the speed of the traveling wave.

We make transformation in which the unknown function u is considered as a functional variable in the form

$$u' = F(u), \tag{5}$$

then, the solution can be found by the relation

$$\int \frac{du}{F(u)} = \xi + C, \tag{6}$$

here C is a constant of integration which is set equal to zero for convenience.

Some successive differentiations of u in terms of F are given as

$$u'' = \frac{dF(u)}{du}\frac{du}{d\xi} = \frac{dF(u)}{du}F(u) = \frac{1}{2}\frac{d(F^2(u))}{du}. \tag{7}$$

It follows that

$$\frac{1}{2}\frac{d\left(F^2(u)\right)}{du} = \frac{\beta u^2 + \mu u}{p^2 + q^2 - k^2}. \tag{8}$$

Integrating Eq.8 and after simple simplification, we get

$$F(u) = u\sqrt{\frac{2\beta u + 3\mu}{3\left(p^2 + q^2 - k^2\right)}}. \tag{9}$$

From Eq.5 and Eq.9, we deduce that

$$\frac{du}{u\sqrt{2\beta u + 3\mu}} = \frac{d\xi}{\sqrt{3\left(p^2 + q^2 - k^2\right)}}. \tag{10}$$

Integrating equation Eq.10 and taking into account Eq.3, we obtain the solution of Eq.2:

$$u(x,y,t) = \frac{3(\alpha+\gamma)}{2\beta}\times$$

$$\times\left(cth^2\left(\frac{1}{2}\sqrt{\frac{\alpha+\gamma}{p^2+q^2-k^2}}\,(px+qy-kt)\right) - 1\right). \tag{11}$$

To obtain solitary and periodic wave solutions of Eq.2, the following conditions must be added to the free parameters in Eq.11:

(i) if $\frac{\alpha+\gamma}{p^2+q^2-k^2} > 0, p^2 + q^2 - k^2 \neq 0$, then we get the following solitary solution to Eq.2:

$$u(x,y,t) = \frac{3(\alpha+\gamma)}{2\beta}\times$$

$$\times\left(cth^2\left(\frac{1}{2}\sqrt{\frac{\alpha+\gamma}{p^2+q^2-k^2}}\,(px+qy-kt)\right) - 1\right). \tag{12}$$

(ii) if $\frac{\alpha+\gamma}{p^2+q^2-k^2} < 0, p^2 + q^2 - k^2 \neq 0$, then we get the following periodic solution to Eq.2:

$$u(x,y,t) = -\frac{3(\alpha+\gamma)}{2\beta}\times$$

$$\times\left(ctg^2\left(\frac{1}{2}\sqrt{\frac{\alpha+\gamma}{k^2-p^2-q^2}}\,(px+qy-kt)\right) + 1\right). \tag{13}$$

3 Solutions of the cubic non-Linear (2+1)-dimensional Klein-Gordon equation

In this section, we find the exact solitary and periodic wave solutions of the cubic non-linear (2+1)-dimensional Klein-Gordon equation:

$$u_{tt} - u_{xx} - u_{yy} + \alpha u + \beta u^3 + \gamma u = 0. \tag{14}$$

Using the wave variable

$$u(x,y,t) = u(\xi),\ \xi = px + qy - kt, \tag{15}$$

in Eq.14, we obtain the following ordinary differential equation

$$u'' = \frac{\beta u^3 + \mu u}{p^2 + q^2 - k^2}, \tag{16}$$

where $\mu = \alpha + \gamma$.

Taking into account Eq.7 and Eq.16 we get the expression for the function $F(w)$:

$$F(u) = u\sqrt{\frac{\beta u^2 + 2\mu}{2\left(p^2 + q^2 - k^2\right)}}. \tag{17}$$

From Eq.7 and Eq.17, we deduce that

$$\frac{du}{u\sqrt{\beta u^2 + 2\mu}} = \frac{d\xi}{\sqrt{2\left(p^2 + q^2 - k^2\right)}}. \tag{18}$$

Integrating Eq.18, we obtain

$$u(x,y,t) = \sqrt{\frac{2(\alpha+\gamma)}{\beta}} \times$$

$$\times\sqrt{cth^2\left(\sqrt{\frac{\alpha+\gamma}{p^2+q^2-k^2}}\,(px+qy-kt)\right) - 1}. \tag{19}$$

To obtain solitary and periodic wave solutions of Eq.2, the following conditions must be added to the free parameters in Eq.19:

(i) if $\alpha+\gamma > 0, \beta > 0, p^2+q^2-k^2 > 0$ or $\alpha+\gamma < 0, \beta < 0, p^2+q^2-k^2 < 0$ then we deduce the following solitary wave solution

$$u(x,y,t) = \sqrt{\frac{2(\alpha+\gamma)}{\beta}} \times$$

$$\times\sqrt{cth^2\left(\sqrt{\frac{\alpha+\gamma}{p^2+q^2-k^2}}\,(px+qy-kt)\right) - 1}. \tag{20}$$

(ii) if $\alpha+\gamma > 0, \beta > 0, p^2+q^2-k^2 < 0$ or $\alpha+\gamma < 0, \beta < 0, p^2+q^2-k^2 > 0$ then we deduce the following periodic wave solution

$$u(x,y,t) = \sqrt{\frac{2(\alpha+\gamma)}{\beta}} \times$$

$$\sqrt{ctg^2\left(\sqrt{\frac{\alpha+\gamma}{k^2-p^2-q^2}}\,(px+qy-kt)\right) + 1}. \tag{21}$$

4 Physical explanation of the solutions

We shown graphs of the obtained solitary and the periodic wave solutions. The graphical illustrations of the solutions are demonstrated in the Figure 1-4.

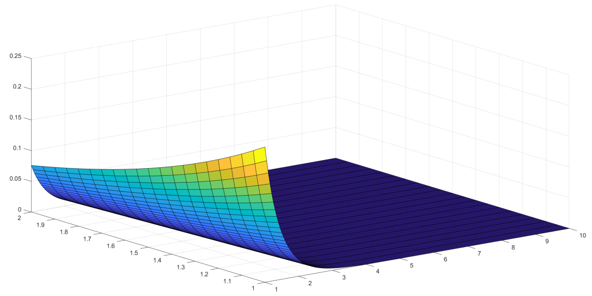

Figure 1: Solitary wave solution of the quadratic non-linear (2+1)-dimensional Klein-Gordon equation for $y = 0$, $\alpha = 2$, $\beta = 6$, $p = 2$, $q = 1$, $k = -1$ and $\gamma = 2$.

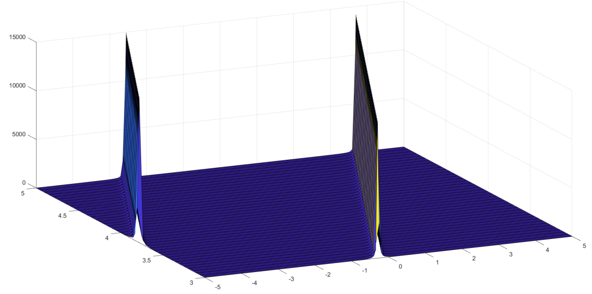

Figure 2: Periodic wave solution of the quadratic non-linear (2+1)-dimensional Klein-Gordon equation for $y = 0$, $\alpha = 1$, $\beta = -3$, $p = 1$, $q = 1$, $k = 2$ and $\gamma = 1$.

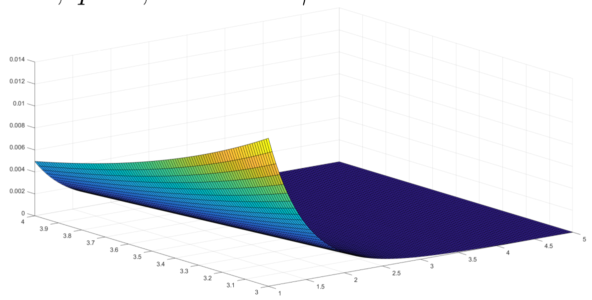

Figure 3: Solitary wave solution of the cubic non-linear (2+1)-dimensional Klein-Gordon equation for $y = 0$, $\alpha = 2$, $\beta = 8$, $p = 2$, $q = 1$, $k = -1$ and $\gamma = 2$.

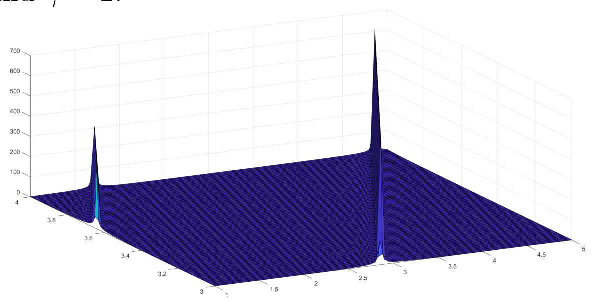

Figure 4: Periodic wave solution of the cubic non-linear (2+1)-dimensional Klein-Gordon equation for $y = 0$, $\alpha = 1$, $\beta = 2$, $p = 1$, $q = 1$, $k = 2$ and $\gamma = 1$.

5 Conclusions

We have shown the functional variable method to solve the quadratic and cubic nonlinear (2+1)-dimensional Klein-Gordon equation successfully. The obtained solutions may be significant and important for analysing the nonlinear phenomena arising in applied physical sciences. Some graphs were drawn with the aid of Matlab software to illustrate the behavior of these solutions. The results show that the proposed method is effective and can be applied to many other nonlinear evolution equations.

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