Modeling and Analysis of the Monotonic Stability of the Solutions of a Dynamical System

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Abstract: - This study aims to develop an approach for the qualitative analysis of the monotonic stability of specific solutions in a dynamical system. This system models the motion of a point along a conical surface, specifically a straight and truncated circular cone. It consists of two nonlinear ordinary differential equations of the first order, each in a unique form and dependent on a particular parameter. Our proposed method utilizes traditional mathematical analysis of a function with a single independent variable, integrated with combinatorial elements. This methodology enables the precise determination of various qualitative cases where the chosen function's value monotonically decreases as a point moves along the conical surface from a specified starting point to a designated point within a final circular region. We assume that the system's partial solutions include a finite number of inflection points and multiple linear intervals.

Key-Words: - stability, mathematical analysis, monotonic functions, conical surface, nonlinear differential equations, partial solution

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1 Introduction

Analyzing the stability of solutions in nonlinear dynamic systems is a crucial challenge in contemporary science and technology. The conventional method for examining the stability of dynamic systems, represented by ordinary differential equations, involves the second Lyapunov method, which is widely used to assess solution stability [1-4]. However, this traditional approach has a notable limitation: it presumes the knowledge of the Lyapunov function.

Therefore, developing alternative methods to evaluate the stability of partial solutions in systems of ordinary differential equations is both scientifically and practically significant. One such approach is a method based on the concept of monotonic stability in stable partial solutions of nonlinear differential equations. This methodology is explored in several articles [5-7], with the distinct aspect of these studies being their focus on mathematical models framed as systems of first-order ordinary differential equations.

The objective of this work is to perform a qualitative analysis of the monotonic stability of solutions within a dynamic system that describes the motion of a point on a conical surface. We assume that these partial solutions encompass a finite number of inflection points and multiple linear intervals. Our aim is to establish conditions for the monotonic stability of these partial solutions. In achieving the primary results, we propose employing classical mathematical analysis techniques for functions with a single independent variable, combined with combinatorial elements. This method allows for the precise identification of various qualitative scenarios where the value of a selected function decreases monotonically as a point traverses a conical surface from a specified initial point to a designated point within a finite circular area.

2 Preliminaries

Let us assume that all spatial curves depicting the behavior of specific solutions to the dynamical system under consideration are confined to the surface of a right, truncated circular cone. The upper boundary of this cone's surface is a circle with the largest radius ρ_0, while the lower The cone's vertex is situated at the origin O of the rectangular Cartesian coordinate system *OXYZ*. We will refer to this surface as the stability cone, which is illustrated in Fig. 1.

In the mathematical model, we employ a spherical coordinate system. Assume that the initial position of a point moving along the trajectory of the solution to the dynamical system is situated on the upper circle, with its coordinates being $(\rho_0, \theta_0, \varphi_0)$.

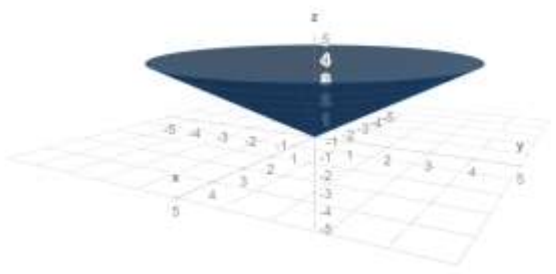

Fig. 1. Stability cone

Additionally, the final position of the point on the trajectory is located on the lower circle, with coordinates $(\rho_1, \theta_1, \varphi_1)$. It is important to note that the condition $\theta_0 = \theta_1 = const$ is satisfied. Figure 2 depicts the trajectory of a point demonstrating monotone stability of the solution on the stability cone.

3 Equations of motion of a point along a conical surface

Let us consider a system of continuous differential equations modeling the motion of a point (R, θ, φ) along a conical surface within a spherical coordinate system

$$\begin{aligned} &\frac{d\rho}{dt} = f_1(\theta_0, \varphi), \\ &\theta = \theta_0 = const, \\ &\frac{d\varphi}{dt} = f_2(\theta_0, \varphi). \end{aligned} \qquad (1)$$

In this context, t is an independent real variable, and $R(t), \varphi(t)$ are non-negative, twice continuously differentiable functions that serve as specific solutions to the dynamical system (1), defined over the interval $t \in [t_0, t_1]$. Additionally, $f_1(\theta_0, \varphi)$ is a known nonpositive, continuously differentiable function, defined over the same interval, while $f_2(\theta_0, \varphi)$ is a nonnegative, continuously differentiable function,also defined within $t \in [t_0, t_1]$. Furthermore, the function $R(t) = \rho(t) / \cos\theta_0$ represents the distance from the origin (0,0,0) to the point (R, θ, φ). Note that the constant value θ_0 acts as a parameter in the system of equations (1).

4 Analysis of the Monotonic Stability of the Solutions of a Dynamical System on a Conical Surface

Let us define the concept of monotone stability for a specific solution $R = R(t)$ within the dynamical system (1)–(2). Consider a non-negative solution $R = R(t)$ of system (1) that meets the following conditions on interval $t \in [t_0, t_1]$ on the conical surface at $\theta = \theta_0 = const$:

(i) the function $R = R(t)$ is defined and twice continuously differentiable;

(ii) the derivative $\frac{d^2R}{dt^2}$ consistently maintains its sign between inflection points $k = 0, 1, 2, \ldots, m, m < \infty$ or (and) within intervals where the derivative $\frac{d^2R}{dt^2} = 0$ ($\forall t \in [t_0, t_1]$).

Definition 1. A non-negative solution $R = R(t)$ of system (1) is considered monotone stable on interval $t \in [t_0, t_1]$ if it satisfies conditions (i)–(ii), and the solution decreases monotonically on the stability cone at $\theta = \theta_0 = const$ on this interval.

Theorem 1. *(Sufficient condition for monotonic stability). If a non-negative solution $R = R(t)$ of system (1) fulfills conditions (i)–(ii), and the derivative $\frac{dR(t)}{dt}$ is negative on interval $t \in [t_0, t_1]$, then this solution is monotone stable in the interval.*

Note. The proof of Theorem 1 is straightforward. It relies on the fundamental sufficient condition for a strictly decreasing function of one variable, while also considering the fulfillment of the inequality $\frac{dR(t)}{dt} = \frac{1}{\cos\theta_0}\frac{d\rho}{dt} < 0$ and conditions (i)–(ii).

Note. The term “qualitatively different cases of monotonic stability of solutions” refers to monotone stable solutions $R = R(t)$ of system (1) that demonstrate a unique type of convexity compared to other solutions exhibiting monotonic stability.

For further analysis of monotone stability, we will consider the first and second derivatives of the solution $R = R(t)$. Regarding the enumeration of qualitatively different cases of monotonic stability of solution $R = R(t)$, the following theorem applies:

Theorem 2. *If a solution $R = R(t)$ of the system equation (1)–(2) satisfies Def. 1, and the number of inflection points in this solution is 0, 1, 2, ..., m, or the number of linear intervals in this solution is 1,2,3, ..., n, then the number of qualitatively* *different cases of monotonic stability of this solution is* ${}_{2m+3}C_1 + {}_{8n}C_1$.

Proof. To prove Theorem 2, let us divide the argument into two parts. First, we determine the number of qualitatively different cases of monotonic stability for particular solutions $R = R(t)$ that conform to the conditions of Def. 1 and have n and *m* inflection points in the interval $t \in [t_0, t_1]$**.** Initially, we identify the count of distinct monotonic stability cases in the absence of inflection points for the function $R = R(t)$ over the interval $[t_0, t_1]$. There are three such cases. In the first linear case $\forall t \in [t_0, t_1]$, the derivatives of the particular solution have the signs: $\frac{dR(t)}{dt} < 0$ and $\frac{d^2R(t)}{dt^2} = 0$. In the second case, the derivatives of the solution exhibit the signs: $\frac{dR(t)}{dt} < 0$ and $\frac{d^2R(t)}{dt^2} > 0$. In the third case, the derivatives of the solution have the signs: $\frac{dR(t)}{dt} < 0$ and $\frac{d^2R(t)}{dt^2} < 0$. Next, consider the scenario where the function $R = R(t)$ has one inflection point within this interval $[t_0, t_1]$. The function $R = R(t)$ is continuously differentiable in the interval $[t_0, t_1]$. Consequently, as a point moves through an inflection point, there is a noticeable change in the convexity of the function's graph. The introduction of one inflection point creates a new interval with a consistent convexity of two possible types of the function $R = R(t)$, yielding two additional qualitatively distinct cases of monotonic stability, distinguished by the sign of the second derivative over this new interval. Each subsequent inflection point similarly contributes the potential for two more qualitatively distinct cases. Thus, for *m* inflection points, the number of qualitatively distinct cases of monotonic stability is given by the equation: ${}_{2m+3}C_1 = 2m + 3$. Secondly, consider the number of qualitatively different cases of monotonic stability for particular solutions $R = R(t)$ that, in addition to non-linear sections with constant convexity, also have *n* linear sections over the interval $t \in [t_0, t_1]$. It is demonstrated that the number of such cases equals ${}_{8n}C_1$, as proven using mathematical induction. For *n*=1, t there are ${}_{8}C_1 = 8$ qualitatively different cases of monotonic stability. This result can be explained as follows: The formation of a single linear section at the beginning of the interval $t \in [t_0, t_1]$ results in two distinct cases of monotonic stability. These cases are characterized by distinctly different types of protrusions in the final nonlinear section of the interval $t \in [t_0, t_1]$. Similarly, it is demonstrated that the emergence of a single linear section in the interval's final part $t \in [t_0, t_1]$ results in two distinct cases of monotonic stability. Specifically, these two cases are differentiated by the nature of the protrusion in the initial nonlinear section of the interval $t \in [t_0, t_1]$. When a linear section is formed in the inner part of the interval $t \in [t_0, t_1]$, it can result in four qualitatively different cases of monotonic stability. These cases are distinguished by their unique constant convexity patterns, which differ in the two nonlinear sections immediately adjacent to the linear section on both the right and left sides. As a consequence, the emergence of a single linear section in the solution $R = R(t)$ can give rise to eight distinct cases of monotonic stability. The foundation of the method of mathematical induction has thus been established. The proof for the second part of this method relies on the fact that each distinct case with *k* linear sections in a specific solution $R = R(t)$ corresponds to two distinct cases in a solution with *k*+1 linear sections. This relationship is straightforward and does not necessitate further elaboration. Therefore, if a formula ${}_{8k}C_1$ defines the number of distinct cases of monotonic stability for *k* linear sections in a particular solution $R = R(t)$, then for *k*+1 linear sections in the solution, the number of different cases of monotonic stability equals ${}_{8k}C_1 \cdot 2$. In this scenario, the equality ${}_{8k}C_1 \cdot 2 = {}_{8(k+1)}C_1$ holds true. Consequently, the second part of the method of mathematical induction, the induction step, is also verified. Therefore, when *n* linear sections are formed in a solution $R = R(t)$, the number of qualitatively different cases of monotonic stability is equal to ${}_{8n}C_1$. Summarizing, the total number of distinct cases of monotonic stability arising from the formation of 0, 1, 2, ...,*m* inflection points or 1,2,3, ...,*n* linear sections in specific solutions $R = R(t)$ is equal to ${}_{2m+3}C_1 + {}_{8n}C_1$. Thus, the theorem is proven.

Note. Theorem 2 does not undertake a qualitative analysis of the cases of monotonic stability in solutions $R = R(t)$ that simultaneously contain inflection points and linear sections.

Let us illustrate the application of the established equality ${}_{2m+3}C_1 + {}_{8n}C_1$ with an example.

Example. Calculate the number of qualitatively different cases of monotonic stability when there are 4 inflection points or 3 linear sections on the solution curve within a given interval.

Solution. Referring to Theorem 2, the number of distinct cases of monotonic stability can be determined as ${}_{2m+3}C_1 + {}_{8n}C_1 = 11 + 24 = 35$ cases, considering m = 4 and n=3.

Definition 2. *The qualitative analysis of the monotonic stability of the partial solution $R = R(t)$ of dynamic system (1) in the interval $t \in [t_0, t_1]$ refers to examining the convexity of a given strictly monotonically decreasing solution in the interval.*

The theorem is established [7].

Theorem 3. *To conduct a qualitative analysis of the monotonic stability of an unknown non-negative solution $R = R(t)$ of system (1), the following conditions must be met:*

(i) the particular solution $R = R(t)$ adheres to Def.1 in the interval $t \in [t_0, t_1]$ and this solution is not linear;

(ii) the first derivative $\frac{d\varphi(t)}{dt}$ of the known continuously differentiable function $\varphi(t)$ is defined in the interval $t \in [t_0, t_1]$ and retains a consistent sign throughout this interval;

(iii) the initial conditions $R(0) > 0$, $\varphi(0)$, and the final value $\varphi(t_1)$ are known.

Note. The proof of Theorem 3 closely aligns with the proof presented in Theorem 3 of the article [7]. Essentially, this proof outlines a method for the qualitative analysis of the monotonic stability of an unknown particular solution $R = R(t)$. Let us examine this method.

Proof (*Method for qualitative analysis of monotonic stability of an unknown nonlinear particular solution $R = R(t)$*).

Assume that the conditions of Def. 1 are satisfied and that the solution $R = R(t)$ is not linear. It's evident that the function $R = R(t)$ decreases monotonically within the interval $t \in [t_0, t_1]$. The second derivative of this function, $\frac{d^2R}{dt^2}$, either maintains its positive or negative sign, or it changes sign at a finite number of the inflection points of the function $R = R(t)$ in the interval $t \in [t_0, t_1]$. It is important to note that the function $R = R(t)$ may be unknown. Let us outline a method for analyzing the sign of the second derivative $\frac{d^2R}{dt^2}$ of this function $R = R(t)$ in the interval $t \in [t_0, t_1]$ for the aforementioned case. To do this, we need to express the second derivative $\frac{d^2R}{dt^2}$ of this function $R = R(t)$. Given that function $R = R(t)$ is a twice-differentiable function, we proceed as follows:

$$\frac{d^2R}{dt^2} = \frac{1}{\cos\theta_0}\frac{df_1}{d\varphi}\frac{d\varphi}{dt}. \qquad (2)$$

According to system (1), the sign of the first derivative $\frac{d\varphi}{dt}$ is constant and known $\forall t \in [t_0, t_1]$. Therefore, to ascertain the sign of the second derivative $\frac{d^2R}{dt^2}$ of the function $R = R(t)$ for all *t* in the interval $[t_0, t_1]$, it is essential to determine the sign of the derivative $\frac{df_1}{d\varphi}$ for all *t* over the interval. Notably, the first derivative $\frac{df_1}{d\varphi}$ is a known smooth function $\frac{df_1}{d\varphi} = F(\varphi)$. Given that the argument φ of the function $F(\varphi)$ changes strictly monotonically $\forall t \in [t_0, t_1]$, if the initial value $\varphi(0)$ is known, then the sign of the derivative $\frac{df_1}{d\varphi} = F(\varphi)$ can be deduced by directly calculating the values of $F(\varphi)$ for all φ within the specified interval $[\varphi(0), \varphi_1]$. This outlines a method for analyzing the sign of the second derivative (2) of this function $R = R(t)$ in the interval $t \in [t_0, t_1]$. Once the sign of this second derivative of the function $R = R(t)$ is determined for the entire interval $t \in [t_0, t_1]$, it enables us to ascertain the convexity type of the strictly decreasing solution $R(t)$ at each point $t \in [t_0, t_1]$. Consequently, in line with Def. 2, we have effectively conducted a qualitative analysis of the monotonic stability of the solution $R = R(t)$ of system (1), which is strictly decreasing within interval $t \in [t_0, t_1]$. With this, the proof of Theorem 3 is concluded.

5 Examples of mathematical models with monotonic stability

Firstly, let us consider an example where the partial solution to the system of ordinary differential equations (1) is represented by a function whose curve forms a conical helix [8]:

$$x = 0.2t\cos t,$$
$$y = 0.2t\sin t, \qquad (3)$$
$$z = 0.25t.$$

This curve is expressed in a spherical coordinate system as follows:

$$\rho = 0.2t,$$

$$\theta_0 = acrtg(1.25) = const, \qquad (4)$$

$$\varphi = t.$$

In this scenario, the function $R(t)$ is linear. Indeed, since $R(t) = \rho(t) / \cos\theta_0$, where $\rho = 0.2t$, $\cos\theta_0 = \cos(acrtg(1.25))$. Consequently, we obtain:

$$R(t) = 0.2t / \cos(acrtg(1.25)). \qquad (5)$$

Letting the variable t vary from -6.5π to 0 and differentiating function (5) with respect to t twice , we find the first and second derivatives: $\dot{R}(t) = 0.2 / \cos\theta_0 < 0$, $\ddot{R}(t) = 0$. According to Theorem 1, this implies that the particular solution (4) is monotonically stable.

Figure 3 illustrates this conical curve, depicting the monotonically stable solution (3) in a Cartesian coordinate system.

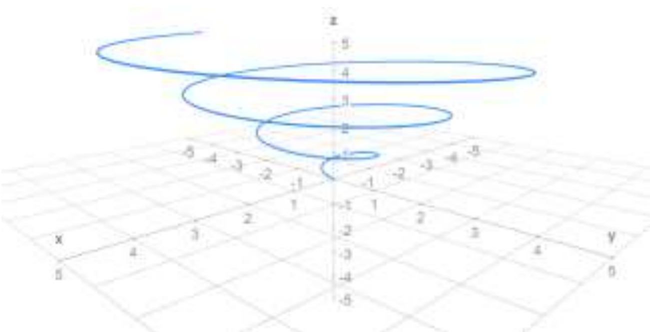

Fig.3. Conical curve of a monotonically stable solution (3)

Figure 4 shows a conical curve describing a monotonically stable solution (3), located on the stability cone.

Let us consider another example where the partial solution to the system of ordinary differential equations (1) is represented by a function that forms a cylindrical conical helix, described as follows [8]:

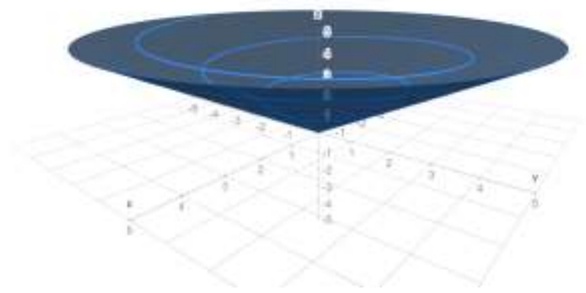

Fig.4. Conical curve of a monotonically stable solution (3) on the stability cone

$$x = 2.5e^{-0.6t}\cos t,$$

$$y = 2.5e^{-0.6t}\sin t, \qquad (6)$$

$$z = 3.5e^{-0.6t}.$$

This curve, when expressed in a spherical coordinate system, is represented by:

$$\rho = 2.5e^{-0.6t},$$

$$\theta_0 = acrtg(1.4) = const, \qquad (7)$$

$$\varphi = t.$$

In this case, the function $R(t)$ is nonlinear. This is evident from the relationship $R(t) = \rho(t) / \cos\theta_0$, where $\rho = 2.5e^{-0.6t}$, $\cos\theta_0 = \cos(acrtg(1.4))$. Consequently, we derive:

$$R(t) = 2.5e^{-0.6t} / \cos(acrtg(1.4)). \qquad (8)$$

When we let the variable t vary from 0 to 10 and differentiate function (8) with respect to t twice, we obtain the first and second derivatives: $\dot{R}(t) = -1.5e^{-0.6t} / \cos(acrtg(1.4)) < 0$, $\ddot{R}(t) = 0.9e^{-0.6t} / \cos(acrtg(1.4)) > 0$.

Based on Theorem 1, this suggests that the particular solution (4) is monotonically stable. It is apparent that the curve (4) is convex downward throughout the specified interval for the variable t.

Figure 5 illustrates the cylindrical conical curve, depicting the monotonically stable solution in a Cartesian coordinate system.

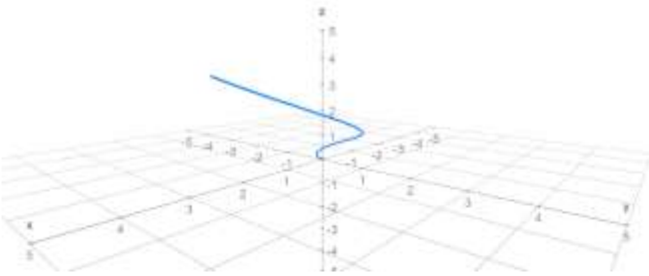

Fig.5. Cylindrical conical curve of a monotonically stable solution (6)

Figure 6 depicts the cylindrical conical curve, which represents a monotonically stable solution as per equation (6). This curve is situated on the stability cone, visually illustrating the stability characteristics of the solution within the context of the system.

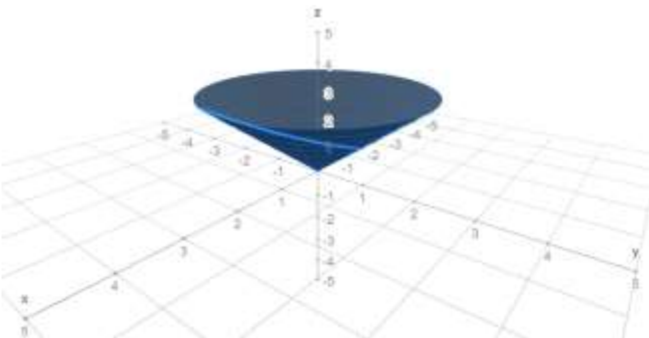

Fig.6. Cylindrical conical curve of a monotonically stable solution (3) on the stability cone

6 Conclusion

The mathematical model describing the motion of a point on a conical surface was formulated as a system of two nonlinear ordinary differential equations, dependent on a specific parameter. These equations were defined using a spherical coordinate system. Mathematical analysis was employed to establish conditions for the monotonic stability of the point's motion on the conical surface. Utilizing combinatorial methods, an expression was derived to calculate the number of monotonic stability cases, incorporating the presence of inflection points and linear sections on the curve being analyzed. Furthermore, the article presents a method for the qualitative analysis of the monotonic stability of the solution, focusing on examining its convexity. Two illustrative examples are provided, demonstrating monotonically stable motion of a point on conical surfaces.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.