The Stability of the Functional Equation f(xy)=f(x)f(y)+g(x)g(y)+f(x)g(y) on Semigroup

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Abstract: - In this work, we establish the stability properties of the functional equation f(xy)=f(x)f(y)+g(x)g(y)+f(x)g(y), x,y in G on semigroup by using the invariant properties of a linear space **T**.

Key-Words: - Stability, semigroup, functional equation, two-sided invariant, linear space, multiplicative function, bounded function.

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1 Introduction

The problem of the stability of functional equations goes back to, [1], who first asked the question concerning the stability of group homomorphisms as follows: Let $(G_1,*)$ be a group and let $(G_2,\bullet,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon> 0$, does there exist $\delta(\varepsilon) > 0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x*y), h(x)h(y))\leq\varepsilon$ for all $x, y \in G_1$, then there is a homomorphism $H: G_1\to G_2$ with $d(h(x), H(x))\leq \delta(\varepsilon)$ for all $x\in G_1$?

And then, [2], took the case of an approximately additive map f of E into E', (where E and E' are Banach spaces) that satisfies Hyers' inequality $|f(x+y)-f(x)-f(y)|\leq \varepsilon$ for all $x, y \in E$. Moreover, he proved that there exists a unique additive map l of E into E' satisfying $|f(x) - l(x)| \leq \varepsilon$ for all $x \in E$. [3], generalized Hyers' theorem for additive mappings and, [4], for linear mappings by considering an unbounded Cauchy difference.

Several researchers have widely studied the stability of functional equations. The progress and developments of this discipline can be found in, [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24].

This paper aims to study the stability properties of the following functional equation:

$$f(xy)=f(x)f(y)+g(x)g(y)+f(x)g(y), \quad x,y \text{ in } G \qquad (1)$$

where G denotes a semigroup, **C** is the set of complex numbers, f, and g are **C**-valued functions on G with g being a nonzero function.

In the case where g is a zero-function, equation (1) will be written as:

$$f(xy)=f(x)f(y), \quad x,y \text{ in } G \qquad (2)$$

So f is a multiplicative function.

The stability of the Cosine-Sine functional equation:

$$f(xy)=f(x)f(y)+g(x)g(y)+h(x)h(y), \quad x,y \text{ in } G \qquad (3)$$

Obtained by, [25], on an amenable group, and the general solution acquired by, [26], on groups.

The stability of the functional equations

$$f(x\sigma(y))=f(x)f(y)-g(x)g(y), \quad x,y \text{ in } G \qquad (4)$$

and

$$f(x\sigma(y))=f(x)g(y)+g(x)f(y), \quad x,y \text{ in } G \qquad (5)$$

where G is an amenable group and $\sigma: G\to G$ is an involutive automorphism (σ is an involutive automorphism meaning that: $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all x, y in G) was established by, [27]. For $\sigma = I$, where I designates the identity map of G, the functional equation (4) and (5) becomes the cosine addition law

$$f(xy)=f(x)f(y)-g(x)g(y), \quad x,y \text{ in } G \qquad (6)$$

and sine addition law

$$f(xy)=f(x)g(y)+g(x)f(y), \quad x,y \text{ in } G \qquad (7)$$

that, [28], proved the stability properties.

In this work, we extend the results of, [28], to the functional equation (1) from amenable group to semigroup.

To begin we need some definitions.

Definition 1.
Let G be a semigroup and **T** the linear space of a complex-valued function on G.

Then we say that the functions f,g: G→**C** are linearly independent modulo **T**, if $\lambda f + \mu g \in \mathbf{T}$ imply that $\lambda = \mu = 0$ for all λ and μ in **C**.

We say that the linear space **T** is two-sided invariant if the $f \in \mathbf{T}$ implies that the functions x→f(xy) and x→f(yx) belongs to **T** for all y in G.

Definition 2.
Let G be a semigroup and **m**:G→**C** a function.

We say that **m** is a multiplicative function if $\mathbf{m}(xy) = \mathbf{m}(x)\mathbf{m}(y)$ for all x, y in G.

2 The Main Result

Our main result is, by lemma 1 we prove that equation (1) holds on in the case that f and g are linearly independent, second by lemma 2 we get some properties of the solutions of equation (1), third by the theorem we prove the stability of the functional equation (1).

Lemma 1.
Let G be a semigroup, f,g: G → **C** tow functions with g a nonzero function and **T** the linear space of **C** valued functions tow sided invariant on G.
We suppose that f and g are linearly independent modulo **T** if the functions

$$x \to f(xy)-f(x)f(y)-g(x)g(y)-f(x)g(y)$$

and

$$x \to f(xy)-f(yx)$$

belongs to **T** for all y in G, then
f(xy)=f(x)f(y)+g(x)g(y)+f(x)g(y), for all x,y in G

Proof.
We use a similar calculation to that of the proof of [28], Lemma 3.1
Let F(x,y)=f(xy)-f(x)f(y)-g(x)g(y)-f(x)g(y) for all x, y in G

As $g \neq 0$, then there exist y_0 in G such as $g(y_0) \neq 0$

Therefore
$F(x, y_0)= f(xy_0)-f(x)f(y) - g(x)\, g(y_0)-f(x)g(y_0)$
So,
$g(x)=\frac{1}{g(y_0)} f(xy_0) - (\frac{f(y_0)}{g(y_0)} +1)f(x)- \frac{1}{g(y_0)} F(x, y_0)$

Let $\alpha_0 = \frac{1}{g(y_0)}$ and $\alpha_1 = \frac{f(y_0)}{g(y_0)}$
We get,
$g(x)= \alpha_0 f(xy_0)-(\alpha_1+1) f(x)- \alpha_0 F(x, y_0)$
We have,

$$\begin{aligned}
f[(xy)z]&=f(xy)f(z)+g(xy)g(z)+f(xy)g(z)+F(x y, z)\\
&=[f(x)f(y)+g(x)g(y)+f(x)g(y)+F(x,y)]f(z)\\
&\quad+[\alpha_0 f(xyy_0)-(\alpha_1+1) f(xyy_0)\\
&\quad-\alpha_0 F(xy, y_0)]\, g(z)\\
&\quad+[f(x)f(y)+g(x)g(y)+f(x)g(y)\\
&\quad+F(x,y)]g(z) +F(xy, z)\\
&=f(x)f(y)f(z)+g(x)g(y)f(z)+f(x)g(y)f(z)\\
&\quad+ F(x, y)f(z)+\alpha_0 f(x y)g(z)+\alpha_1 f(xyy_0)g(z)\\
&\quad-\alpha_1 F(xy,y_0)g(z)+f(x)f(y)g(z)\\
&\quad+g(x)g(y)g(z)\\
&\quad+f(x)g(y)g(z)+F(x,y)g(z)+F(xy,z)
\end{aligned}$$

and,

$$\begin{aligned}
f[(xy)z]&=f(x)f(y)f(z)+g(x)g(y)f(z)+f(x)g(y)f(z)\\
&\quad+F(x, y)f(z)+ \alpha_0 f(x)\, f(yy_0)\, g(z)\\
&\quad+ \alpha_0 g(x)g(yy_0)g(z)+ \alpha_0 f(x)\, g(yy_0)g(z)\\
&\quad+ \alpha_0 F(x,yy_0)g(z)-(\alpha_1+1)f(x)f(y)g(z)\\
&\quad-(\alpha_1+1)g(x)g(y)g(z)-(\alpha_1+1)f(x)g(y)g(z)\\
&\quad-(\alpha_1+1)F(x, y)g(z)- \alpha_0 F(x y, y_0)g(z)\\
&\quad+f(x)f(y)g(z)+g(x)g(y)g(z)+f(x)g(y)g(z)\\
&\quad+F(x, y)g(z)+F(xy, z)
\end{aligned}$$

on the other hand,

$$\begin{aligned}
f[(xy)z]&=f[x(yz)]\\
&=f(x)f(yz)+g(x)g(yz)+f(x)g(yz)+F(x,yz)\\
&=f(x)[f(yz)+g(yz)]+g(x)g(yz)+F(x,yz)
\end{aligned}$$

By using the fact that f and g are linearly independent modulo **T** and also **T** is a tow sided invariant linear space, we get:

$$g(yz)=g(y)f(z)+g(y)g(z)+\alpha_0 g(y)g(z)+\alpha_1 g(yy_0)g(z)$$

and

$$\begin{aligned}
f(yz)+g(yz) &=f(y)f(z)+g(y)f(z)+f(y)g(z)+g(y)g(z)\\
&\quad+\alpha_0 f(y)g(z)+\alpha_0 g(y)g(z)\\
&\quad+\alpha_1 f(yy_0)g(z)+\alpha_1 g(yy\alpha_0)\, g(z)
\end{aligned}$$

then

$$\begin{aligned}
F(xy,z)-F(x,yz)=&[\alpha_0 F(xy, y_0)+ \alpha_1 F(x,y)\\
&- \alpha_0 F(x,yy_0)]g(z) - F(x,y)f(z)
\end{aligned}$$

Again, using f and g are linearly independent modulo **T** and the fact that **T** is a tow sided invariant linear space, we obtain: F=0
The lemma is proved.

Lemma 2.
Let G be a semi-group, f,g:G → **C** tow functions with g a nonzero function and **T** be a tow-sided invariant linear space of **C** valued functions on G.
If the functions

$$x \to f(xy)-f(x)f(y)-g(x)g(y)-f(x)g(y)$$

and

$$x \to f(xy)-f(yx)$$

Belongs to **T** for all y in G, then we have one of the following possibilities:

1) f, g in **T**
2) f+g multiplicative, g in **T**
3) $f=\frac{\lambda^2}{\lambda^2+\lambda+1}m+\frac{\lambda+1}{\lambda^2+\lambda+1}b$,
$g=\frac{\lambda}{\lambda^2+\lambda+1}m-\frac{\lambda}{\lambda^2+\lambda+1}b$,
where b:G→ **C** is a function belonging to **T** and m:G → C is a multiplicative function and λ in **C***-$\{e^{i\frac{2\pi}{3}}, e^{-i\frac{2\pi}{3}}\}$ is a constant.
4) f(xy)=f(x)f(y)+g(x)g(y)+f(x)g(y)

Proof.
We use a similar calculation to that of the proof of, [28], Lemma 3.2.

If g in **T**

By using a similar demonstration of the theorem in, [3], we suppose that f is not in **T** and f+g is not multiplicative.

Then there exists y_1, z_1 in G that
$f(y_1z_1)+g(y_1z_1)\neq(f(y_1)+g(y_1))(f(z_1)+g(z_1))$

We have,
$$f(xyz)-f(xy)(f(z)+g(z))=f(xyz)-f(x)(f(yz)+g(yz)) - [f(xy)-f(x)(f(y)+g(y))](f(z)+g(z))+f(x)[f(yz)+g(yz)-(f(y)+g(y))(f(z)+g(z))]$$
For $y=y_1$ and $z=z_1$ we get,
$$f(x)=[f(xy_1z_1)-f(xy_1)(f(z_1)+g(z_1)) - (f(xy_1z_1)-f(x)(f(y_1z_1)+g(y_1z_1))) +(f(xy_1)-f(x)(f(y_1)+g(y_1)))(f(z_1)+g(z_1))] \times[f(y_1z_1)+g(y_1z_1) -(f(y_1)+g(y_1))(f(z_1)+g(z_1))]^{-1}$$

Since the function $x \to f(xy_1)-f(x)(f(y_1)+g(y_1))$ belong to **T**,

Then f in **T** that contradicts the supposition.
So f in **T** or f+g is multiplicative, then 1 and 2 of Lemma 2 is proved

If f, g in **T** and f, g are linearly dependent.
There exist λ in C* and b in **T** such that
$f=\lambda g+b$

Substituting f by $\lambda g+b$ in (1) we get,

$$\begin{aligned}&f(x y)-f(x)f(y)-g(x)g(y)-f(x)g(y)\\ &=\lambda g(x y)+b(xy)-(\lambda g(x)+b(x))(\lambda g(y)+b(y))\\ &-g(x)g(y)-(\lambda g(x)+b(x))g(y)\\ &=\lambda g(xy)-[\lambda^2 g(y)+\lambda b(y)+\lambda g(y)\\ &+g(y)]g(x)+b(xy)-b(x)b(y)-\lambda g(y)b(x)-b(x)g(y)\\ &=\lambda[g(x y)-\tfrac{1}{\lambda}((\lambda^2+\lambda+1)g(y)+\lambda b(y))g(x)]\\ &+b(xy)-b(x)b(y)-(\lambda+1)g(y)b(x)\end{aligned}$$

And by the hypothesis, the function $x \to f(xy)-f(x)f(y)-g(x)g(y)-f(x)g(y)$ belongs to **T**.

We deduce that the function $x \to g(xy)-\frac{1}{\lambda}((\lambda^2+\lambda+1)g(y)+\lambda b(y))g(x)$ belongs to **T**,
and by using the theorem in, [29], we obtain,

$$\frac{\lambda^2+\lambda+1}{\lambda}g+b=m$$

Where m is a multiplicative function.

For $\lambda \neq e^{i\frac{2\pi}{3}}$ and $\lambda \neq e^{-i\frac{2\pi}{3}}$ we obtain

$f=\frac{\lambda^2}{\lambda^2+\lambda+1}m+\frac{\lambda+1}{\lambda^2+\lambda+1}b$ and $g=\frac{\lambda}{\lambda^2+\lambda+1}m-\frac{\lambda}{\lambda^2+\lambda+1}b$
then 3 of Lemma 2 is proved.

If f and g are linearly independent modulo **T**, applying lemma 1 we get 4 of Lemma 2.
So lemma 2 is proved.

Theorem
Let G be a semigroup and f,g: G → **C** tow functions with g a nonzero function.
The function

$$(x,y) \to f(xy)-f(x)f(y)-g(x)g(y)-f(x)g(y)$$

is bounded if only and if one of the following assertions hold on:

1) f and g are bounded functions

2) f+g is a bounded multiplicative function, g is a bounded function
3) f=$\frac{\lambda^2}{\lambda^2+\lambda+1}$m+ $\frac{\lambda+1}{\lambda^2+\lambda+1}$b, and

g =$\frac{\lambda}{\lambda^2+\lambda+1}$ m- $\frac{\lambda}{\lambda^2+\lambda+1}$b, where b:G → **C** is a bounded function and m:G → **C** is a bounded multiplicative function, λ in **C***-{$e^{i\frac{2\pi}{3}}$, $e^{-i\frac{2\pi}{3}}$} is a constant.
4) f(xy)=f(x)f(y)+g(x)g(y)+f(x)g(y)

Proof.
Let **T** the space of **C** valued bounded functions on G. Applying the Lemma 2 we prove the necessity.
If g is a bounded function then we get the assertions 1 and 2.
And the assertions 3 and 4 are followed directly by Lemma 2

3 Conclusion
The results of this paper show that equation (1) has notable stability property, that the difference between the right-hand side and the left-hand side of the equation is bounded if and only if we add a bounded function to the exact solutions.

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