Directly indecomposible multialgebras

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Abstract: The aim of this paper is the study directly indecomposible multialgebras. In this regards, first the isomorphism theorems and correspondence theorem for multialgebras. Then by applying congruences relation on multialgebras factor multialgebras are constructed and some important properties of them are obtained. In particular, it is shown that every finite multialgebra is isomorphic to a direct products of directly indecomposable of multialgebras. Finally, subdirect products and subdirect irreducible of multialgebras are investigated and Birkoff's theorem is extended to multialgebras.

Key–Words: Multialgebra, Fundamental relation, Isomorphism, Congruence relation, Factor congruence, Directly indecomposible, Subdirectly irreducible.

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1 Introduction

A multialgebra can be considered as a relational systems which generalize the universal algebras. In [17] Schweigert studied the congruence of multialgebras. R. Ameri et al. introduced and studied hyperalgebraic system in [2]; some more properties of multialgebras such as identities, fundamental relation and direct limit and etc. has been studied by C. Pelea (for more details see [12], [13], [14]). In this paper we follow [16] to study isomorphism theorems, directly indecomposible and subdirect products of multialgebras. This paper is organized in 5 sections. In Section 2, we gather the definition and basic properties of multialgebras which we need to development our paper. In Section 3 the isomorphism theorems and correspondence theorem for multialgebras has been proved. In Section 4, by using the notions of congruence, factor congruence and direct product of multialgebras it is shown that every finite multialgebra is isomorphic to a direct product of directly indecomposable multialgebras. Finally, in Section 5 subdirectly irreducible of multialgebras are introduced and a necessary and sufficient condition that a multialgebra is subdirectly irreducible is obtained. Finally, the Birkoff's theorem has been extended to multialgebras.

2 Preliminaries

In this section we gather all definitions and results of multialgebras, which we need to development our paper. In the sequel H is a fixed nonvoid set, $P^*(H)$ is the family of all nonvoid subsets of H, and for a positive integer n H^n denotes the set of all $n-$tuples elements of H.

For a positive integer n a $n-$ary *hyperoperation* β on H is a function $\beta : H^n \rightarrow P^*(H)$. We say that n the *arity* of β. A subset S of H is *closed* under the n-ary hyperoperation β if $(x_1, \ldots, x_n) \in S^n$ implies that $\beta(x_1, \ldots, x_n) \subseteq S$. A nullary hyperoperation on H is just an element of $P^*(H)$; i.e. a nonvoid subset of H.

An n-ary *relation* ρ on H is a subset of H^n. We also say that the arity of ρ is n. Orders and equivalence relations on H are the best examples of binary (i.e. 2-array) relations on H. Henceforth sometimes we use hyperoperation instead of the n-ary hyperoperation. A *hyperalgebraic system* or a *multialgebra* $\langle H, (\beta_i, \mid i \in I)\rangle$ is the set H with together a collection $(\beta_i, \mid i \in I)$ of hyperoperations on H.

A subset S of a multialgebra $H = \langle H, (\beta_i, \mid i \in I)\rangle$ is a *submultialgebra of H if S is closed under each hyperoperation β_i, for all $i \in I$, that is $\beta_i(a_1, ..., a_n) \subseteq S$, whenever $(a_1, ..., a_n) \in S^n$. The type of H is the map from I into the set $\mathbb{N}^*$ of nonnegative integers assigning to each $i \in I$ the arity of β_i.*

A binary relation ρ on a set M is called compatible (resp. strong compatible) with an n-ary hyperoperation β if $x_1 \rho y_1, ..., x_n \rho y_n$ implies that

$$\beta(x_1, ..., x_n)\overline{\rho}\beta(y_1, ..., y_n),$$

$$(\beta(x_1,...,x_n)\overline{\rho_S}\beta(y_1,...,y_n)),$$

where for nonempty subsets A and B of M,

$$A\overline{\rho}B \iff (\forall a\in A\ \exists b\in B : a\rho b \text{ and } \forall b\in B\ ,\ \exists a\in A : b\rho a),$$

and

$$A\overline{\rho_S}B \iff \forall a\in A, \forall b\in B\ \ a\rho b.$$

Let $\langle H,(\beta_i,\mid i\in I)\rangle$ be a multialgebra. A binary relation ρ on M is called (resp. strong) congruence if ρ is an equivalence relation and (resp. strongly) compatible with every $\beta_i, i\in I$.

For $n>0$ we extend an $n-$ary hyperoperation β on H to an $n-$ary operation $\overline{\beta}$ on $P^*(H)$ by setting for all $A_1,...,A_n\in P^*(H)$

$$\overline{\beta}(A_1,...,A_n)=\bigcup\{\beta(a_1,...,a_n)\mid a_i\in A_i(i=1,...,n)\} \quad (1)$$

It is easy to see that $\langle P^*(H),(\overline{\beta}_i,\mid i\in I)\rangle$ is an algebra. whenever possible we write a instead of the the singleton $\{a\}$; e.g. for a binary hyperoperation $\circ$ and $a,b,c\in H$ we write $a\circ(b\circ c)$ for

$$\{a\}\circ(\{b\}\circ\{c\})=\bigcup\{a\circ u|u\in b\circ c\}.$$

An equivalence relation on A compatible (resp. strongly compatible) with a multialgebra H on A is congruence (resp. strong congruence) of H. Denote by $Con(H)(resp.Cons(H))$ the set of all congruences (resp. strong congruences) of H.

Let $H=\langle A,(\beta_i,\mid i\in I)\rangle$ be a multialgebra and let $\theta\in Con(H)$. Let $A/\theta=\{B_j|j\in J\}$ be the set of blocks of θ. For every $i\in I$ define $\overline{\beta}_i$ on A/θ as follows:

Let $j_1,...,j_{m_i}\in J$ be arbitrary and let $a_l\in B_{j_l}$ for $l=1,...,m_i$. Define

$$\overline{\beta}_i(B_{j_1},...,B_{j_{m_i}})=\{B_j|j\in J,\ B_j \text{ meets } \beta_i(a_1,...,a_{m_i})\} \quad (2)$$

Since $\theta\in Con(H)$, it can be verified that $\overline{\beta}_i$ is well defined m_i-ary hyperoperation on A/θ. Call $H/\theta=\langle A/\theta,(\overline{\beta}|j\in J)\rangle$ a factor multialgebra of H. If, moreover, $\theta\in Con(H)$, then every $\overline{\beta}_i$ is singleton valued, i.e. an operation on A/θ, and H/θ is an algebra. For semihypergroups this fact are in [1] the general case is in [11].

We view binary relation on A as subsets of A^2 and so for a multialgebra H on A the sets $Con(H)$ and $Cons(H)$ are naturally ordered by set inclusion. First we characterize the poset $(Con(H,\subseteq)$. Recall that for a binary relations ρ and σ on A the relation product (also called de Morgan product) is

$$\rho\circ\sigma=\{(x,y)\in A^2|(x,u)\in\rho,(u,y)\in\sigma \text{ for some } u\in A\}.$$

It is well known and easy to show that the relation product is associative with the unital element $\omega=\{(a,a)|a\in A\}$.

***Example 1**. (i) A hypergroupoid is a multialgebra of type (2), that is a set H together with a (binary) hyperoperation $\circ$. A hypergroupoid $(H,\circ)$, which is associative, that is $x\circ(y\circ z)=(x\circ y)\circ z$ for all $x,y,z\in H$ is called a semihypergroup.*

(ii) A hypergroup is a semihypergroup such that for all $x\in H$ we have $x\circ H=H=H\circ x$ (called the reproduction axiom).

An element e in a hypergroup $H=(H,\circ)$ is called an identity of H if for all $x\in H$, on has

$$x\in(e\circ x)\cap(x\circ e).$$

(iii) A polygroup (or multigroup) is a semihypergroup $H=(H,\circ)$ with $e\in H$ such that for all $x,y\in H$

(i) $e\circ x=x=x\circ e$;

(ii) there exists a unique element, $x^{-1}\in H$ such that

$$e\in(x\circ x^{-1})\cap(x^{-1}\circ x),\ \ x\in\bigcap_{z\in x\circ y}(z\circ y^{-1}),\ \ y\in\bigcap_{z\in x\circ y}(x^{-1}\circ z).$$

In fact, a polygroup is a multialgebra of type $(2,1,0)$.

***Definition 2**. Let $H=\langle H,(\beta_i,\mid i\in I)\rangle$ and $\overline{H}=\langle\overline{H},(\overline{\beta}_i,\mid i\in I)\rangle$ be two similar multialgebras. A map h from H into $\overline{H}$ is called a*

(i) A homomorphism if for every $i\in I$ and all $(a_1,...,a_{n_i})\in H^{n_i}$ we have that
$h(\beta_i((a_1,...,a_{n_i}))\subseteq\beta_i(h(a_1),...,h(a_{n_i}))$;

(ii) a good homomorphism if for every $i\in I$ and all $(a_1,...,a_{n_i})\in H^{n_i}$ we have
$h(\beta_i((a_1,...,a_{n_i}))=\beta_i(h(a_1),...,h(a_{n_i}))$.

For a map $h: H\longrightarrow\overline{H}$ set

$$\ker h=\{(a,a')\mid a,a'\in H, \text{ and } h(a)=h(a')\}.$$

It is well known and it can be easily seen that $kerh$ is an equivalence relation on H. If h is a good homomorphism, then it can be easily seen that θ is a strong congruence on H. Setting for all $a \in H$, $\phi(a) = a/\theta$.

Definition 3. *A universal algebra is a pair $< A, (f_i : i \in I) >$ where A is a nonempty set and $(f_i : i \in I)$ is a family of finitary operations on A indexed by I. A finitary operation is an $n-$ary operation for some n, and $n-$ary operation on A is any function f from A^n to A, n is the rank of f. In above we assume for every $i \in I$,n_i is the rank of f_i, and $< n_i, i \in I >$ is called tape of A.*

Definition 4. *Let $\tau =< n_i : i \in I >$ be a sequence over N=$\{1,2,...\}$. By a multialgebra of tape τ, we understand a pair $< H, (f_i : i \in I) >$, where H is a nonempty set and f_i is an n_i-ary hyper operation on H,i.e, a map $f_i : H^{n_i} \to P^*(H)$, for each $i \in I$.*

Remark 5. *Let $< A, (f_i : i \in I) >$ be a universal multialgebra . A induces an algebra $< P^*(A), (f_i : i \in I) >$ with the operations:*
$f_i(A_0, ..., A_{n_i}) = \bigcup\{f_i(a_0, ..., a_{n_i-1})|a_i \in A_i, \forall i \in \{0, ..., n_i - 1\}$
for $A_0, ..., A_{n_i-1} \in P^*(A)$. We denote this algebra by $P^*(A)$.

Definition 6. Let A be a multialgebra. The fundamental relation α^* on A is the smallest equivalence relation on A such that A/α^* is a universal algebra.

***Lemma 7** [1]*. *If ρ and σ are binary relations on A compatible with H, then $\tau = \rho \circ \sigma$ is compatible with H.*

***Lemma 8** [1]*. (i) The relation $\omega = \{(a, a) \mid a \in A\}$ is compatible with H and

(ii) the relation A^2 is strongly compatible with H.

***Lemma 9** [1]*. Let H be a multialgebra on H. Let $h > 0$ and let $\{\sigma_j | j \in J\}$ be a set of h-ary relations on H strongly compatible with H. Then $\sigma = \bigcap_{j\in J} \sigma_j$ is strongly compatible with H.

3 Isomorphism theorems of multialgebras

***Theorem 10** [1]*. Let $H = \langle H, (\beta_i, | \ i \in I)\rangle$ and $H' = \langle H', (\beta'_i, | \ i \in I)\rangle$ be similar multialgebras, let h be a good homomorphism from H onto H', and let ϕ be the quotient map corresponding $\theta = \ker h$. Then

(i) θ is a congruence relation on H;

(ii) ϕ is a good homomorphism from H onto H/θ;

(iii) the unique function f from H/θ onto H' satisfying $\phi \circ f = h$ is a good isomorphism from H/θ onto H'.

Proposition 11. Let H be a multialgebra and let θ be the least element of $Cons(H)$. Then $(Cons(H), \subseteq)$ is lattice isomorphic to the congruence lattice of the algebra H/θ.

Definition 12. Suppose H is an multialgebra and $\phi, \theta \in Con(H)$ with $\theta \subseteq \phi$. Then let

$$\phi/\theta = \Big\{\langle a/\theta, b/\theta\rangle \in (H/\theta)^2 : \ \langle a, b\rangle \in \phi\Big\}.$$

Lemma 13. If $\phi, \theta \in Con(H)$ and $\theta \subseteq \phi$, then ϕ/θ is congruence on H/θ [6].

Theorem 14 (Second Isomorphism Theorem). *If $\phi, \theta \in Con(H)$ and $\theta \subseteq \phi$, then the map*

$$\alpha : (H/\theta)\big/(\phi/\theta) \to H/\phi$$
$$\alpha\big((a/\theta)\big/(\phi/\theta)\big) = a/\phi$$

is an isomorphism from $\big(H/\theta\big)\big/(\phi/\theta\big)$ to H/ϕ.

Proof. *Let $a, b \in A$.*
Then if $(a/\theta)/(\phi/\theta) = (b/\theta)/(\phi/\theta)$ then it is equal to $\big(a/\theta, b/\theta\big) \in (\phi/\theta)$ i.e $(a, b) \in \phi$ then $a/\phi = b/\phi$ i.e α is well-defined.

Now for β an n-ary function symbol and $a_1, \ldots, a_n \in H$ we have

$$\alpha\beta^{(H/\theta)/(\phi/\theta)}\Big((a_1/\theta)/(\phi/\theta), \ldots, (a_n/\theta)/(\phi/\theta)\Big)$$
$$= \alpha\Big(\beta^{H/\theta}(a_1/\theta, \ldots, a_n/\theta)/(\phi/\theta)\Big)$$

(by definition of factor multialgebra.)

then

$$= \alpha\Big(\big(\beta^{H}(a_1, \ldots, a_n)/\theta\big)\Big/(\phi/\theta)\Big)$$
$$= \bigcup \beta^{H}(a_1, \ldots, a_n)/\phi = \beta^{H/\phi}\big(a_1/\phi, \ldots, a_n/\phi\big)$$
$$= \beta^{H/\phi}\big(\alpha(a_1/\theta)/(\phi/\theta)\big), \ldots, \alpha\big((a_n/\theta)/(\phi/\theta)\big).$$

□

Definition 15. Suppose H' is subset of H and θ is a congruence on H. Let $H'^{\theta} = \{a \in H : \ H' \cap a/\theta \neq \emptyset\}$. Let H'^{θ} be the submultialgebra of H generated by H'^{θ}. Also, define $\theta|_{H'}$ be $\theta \cap H'^2$, the restriction of θ on H'.

Lemma 16. If H' is a submultialgebra of H and $\theta \in Con(H)$ (resp. $cons(H)$), then

(i) The universe of H'^{θ} is H'^{θ}.

(ii) $\theta|_{H'}$ is a congruence (resp. strong congruence) on H'.

Proof.

(i) Let β be any $n-$ary hyperoperation and $a_1, \ldots, a_n \in H'^{\theta}$. By definition, there exist $b_1, \ldots, b_n \in H'$ such that $\langle a_i, b_i \rangle \in \theta$, $i = 1, \ldots, n$. Because of congruency of θ on H and H' is submultialgebra of H, we have $\beta(a_1, \ldots, a_n)\bar{\theta}\beta(b_1, \ldots, b_n)$.

Let $a \in \beta(a_1, \ldots, a_n)$.

So there exist $b \in \beta(b_1, \ldots, b_n)$ s.t $a/\theta = b/\theta$. Then $H' \cap (a/\theta) \neq \emptyset$ or $a \in H'^{\theta}$. Therefore $\beta(a_1, \ldots, a_n) \subseteq H'^{\theta}$.

(ii) Proof is straightforward.

□

Theorem 17 (Third Isomorphism Theorem). If H' is a submultialgebra of H and $\theta \in Con(H)$, then

$$H' \Big/ \theta|'_H \cong H'^{\theta} \Big/ \theta|_{H'^{\theta}}.$$

Proof. *We prove that the mapping $\alpha : H' \Big/ \theta|_{H'} \to H'^{\theta} \Big/ \theta|_{H'^{\theta}}$ is defined by $\alpha\big(b/\theta|_{H'}\big) = b/\theta|_{H'^{\theta}}$ is Isomorphism. First we prove α is one-to-one:*

Let $b_1 \Big/ \theta|_{H'^{\theta}} = b_2 \Big/ \theta|_{H'^{\theta}}$. Then $\langle b_1, b_2 \rangle \in \theta|_{H'^{\theta}} = \theta \cap (H'^{\theta})^2$.

So $\langle b_1, b_2 \rangle \in \theta$ and $\langle b_1, b_2 \rangle \in (H'^{\theta})^2$. Therefore, $b_1/\theta \cap H' \neq \emptyset$ and $b_2/\theta \cap H' \neq \emptyset$. Consequently there exist $b'_1, b'_2 \in H'$, such that $b_1/\theta = b'_1/\theta$ and $b_2/\theta = b'_2/\theta$. Thus

$$\langle b'_1, b'_2 \rangle \in \theta$$
$$\langle b'_1, b'_2 \rangle \in (H')^2,$$

Consequently $\langle b'_1, b'_2 \rangle \in \theta \cap H'^2 = \theta|_{H'}$, consequently $\langle b_1, b_2 \rangle \in \theta \cap H'^2 = \theta|_{H'}$. then $b_1 \big/ \theta|_{H'} = b_2 \big/ \theta|_{H'}$. α is onto because, if $b/\theta|_{H'^{\theta}} \in H'^{\theta}/\theta|_{H'^{\theta}}$, such that $b \in H'^{\theta} \setminus H'$ then $H' \cap b/\theta \neq \emptyset$ i.e there exist $b_1 \in H'$ such that

$$\alpha\big(b_1/\theta|_{H'}\big) = b/\theta|_{H'^{\theta}}.$$

At last we prove α is a homomorphism.

$$\begin{aligned}
&\alpha\beta\big(b_1/\theta|_{H'}, \ldots, b_n/\theta|_{H'}\big) \\
&= \alpha\Big(\frac{\beta(b_1, \ldots, b_n)}{\theta|_{H'}}\Big) \\
&= \beta(b_1, \ldots, b_n)/\theta|_{H'^{\theta}} \\
&= \beta(\frac{b_1}{\theta|_{H'^{\theta}}}, \ldots, \frac{b_n}{\theta|_{H'^{\theta}}}) \\
&= \beta\big(\alpha(\frac{b_1}{\theta|_{H'}}), \ldots, \alpha(\frac{b_n}{\theta|_{H'}})\big) \\
&= \beta\Big(\alpha(\frac{b_1}{\theta|_{H'}}), \ldots, \alpha(\frac{b_n}{\theta|_{H'}})\Big).
\end{aligned}$$

□

Note that if L is a lattice and $a, b \in L$ with $a \leq b$, then the interval $[a, b]$ is subuniverse interval of a lattice L, where $a \leq b$, by $[a, b]$, we mean the corresponding sublattice of L.

Theorem 18 (Correspondence Theorem). Let H be an multialgebra and let $\theta \in con(H)$ (res. $cons(H)$). Then the mapping ψ on $[\theta, \nabla_H]$ defined by $\psi(\phi) = \phi/\theta$ is a lattice isomorphism from $[\theta, \nabla_H]$ to $con(H/\theta)$ (resp. $cons(H/\theta)$), where $[\theta, \nabla_H]$ is a sublattice of $con(H)$ (resp $cons(H)$)

Proof. *ψ is one to one. Because: for $\phi, \phi' \in [\theta, \nabla_H]$ with $\phi \neq \phi'$. Then without loss of generality, we can assume that there are $a, b \in H$ with $(a, b) \in \phi - \phi'$. Then $(a/\theta, b/\theta) \in (\phi/\theta) \setminus (\phi'/\theta)$ so $\psi(\phi) \neq \psi(\phi')$.*

To show that ψ is onto, suppose $\rho \in con(H/\theta)$ (resp. $cons(H/\theta)$) and define φ to be $\ker(\pi_\rho\pi_\theta)$, where π_ρ, π_θ are canonical projections. Then for $a, b \in H$, $\langle a/\theta, b/\theta \rangle \in \varphi/\theta$ if and only if $\langle a, b \rangle \in \varphi = \ker(\pi_\rho\pi_\theta)$ if and only if $\pi_\rho\pi_\theta(a) = \pi_\rho\pi_\theta(b)$ if and only if and only if $(a/\theta)/\rho = (b/a)/\rho)$ if and only if $\langle a/\theta, b/\theta \rangle \in \rho$. So $\varphi/\theta = \rho$. let f is n-ary hyperoperation on H and $a_1, \ldots, a_n \in H$.

we have

$$\begin{aligned}
\psi f^H(a_1, \ldots, a_n) &= f^H(a_1, \ldots, a_n)/\theta \\
&= f^{H/\theta}(a_1/\theta, \ldots, a_n/\theta) \\
&= f^{H/\theta}(\psi a_1, \ldots, \psi a_n).
\end{aligned}$$

so ψ is strong homomorphism. □

4 Directly indecomposible multialgebras

Definition 19. Let H_1 and H_2 be two mulialgebras of the same type $\boldsymbol{\mathcal{F}}$.

Define the (direct) product of multialgebra $H_1 \times H_2$ to be the multialgebra whose universe is the set $H_1 \times H_2$ and for $f \in \mathcal{F}_n$ and $a_i \in H_1$ and $a'_i \in H_2$; $1 \le i \le n$:

$$f^{H_1 \times H_2}\big((a_1,b_1),\dots,(a_n,b_n)\big)$$
$$= \big(f^{H_1}(a_1,\dots,a_n), f^{H_2}(b_1,\dots,b_n)\big)$$
$$\{(a,b) | a \in f^{H_1}(a_1,\dots,a_n),\ b \in f^{H_2}(b_1,\dots,b_n)\}.$$

Note that in general neither H_1 nor H_2 is embedable in $H_1 \times H_2$, although in special case like hypergroups, it is possible because there is always a trivial subhyperalgebra.
However, both H_1 and H_2 are homomorphic image of $H_1 \times H_2$.

Definition 20. The mapping $\pi_i : H_1 \times H_2 \to H_i$, $i \in \{1,2\}$;
defined by

$$\pi_i\big((a_1,a_2)\big) = a_i,$$

is called the projection map on the $i'th$ coordinate of $H_1 \times H_2$.

Theorem 21. For $i = 1,2$ the mapping $\pi_i : H_1 \times H_2 \to H_i$ is a surjective strong homomorphism from $H = H_1 \times H_2$ to H_i. Furthermore, in $con(H_1 \times H_2)$ we have

$$\ker \pi_1 \cap \ker \pi_2 = \Delta,$$
$$\ker \pi_1 \ \ and \ \ \ker \pi_2 \ \ premute,$$

and

$$\ker \pi_1 \vee \ker \pi_2 = \nabla.$$

Proof. *It is easy to check that π_i is a strong surjective homomorphism (epimorphism). Now*

$$\big((a_1,a_2),(b_1,b_2)\big) \in \ker \pi_i$$
$$iff\ \pi_i\big((a_1,a_2)\big) = \pi_i\big(b_1,b_2)\big)$$
$$iff \qquad a_i = b_i$$

Thus

$$\ker \pi_1 \cap \ker \pi_2 = \Delta.$$

Now, consider $(a_1,b_1),(b_1,b_2)$ are any two element of $H_1 \times H_2$, then

$$(a_1,a_2) \in \ker \pi_1(a_1,b_2) \ker \pi_2(b_1,b_2),$$

and

$$(a_1,a_2) \in \ker \pi_2(b_1,a_2) \ker \pi_1(b_1,b_2),$$

hence

$$\nabla = \ker \pi_1 \circ \ker \pi_2.$$

But then $\ker \pi_1$ and $\ker \pi_2$ permute, and their join is ∇. □

Definition 22. A congruence ρ on H is a factor congruence if there is a congruence σ on H such that

$$\rho \wedge \sigma = \Delta,$$

and

$$\rho \vee \sigma = \nabla.$$

and ρ permute with σ.

The pair ρ and σ is called a pair of factor congruence on H.

Theorem 23. If ρ and σ is a pair of factor congruence on H, then

$$H \simeq H/\rho \times H/\sigma$$

under the map

$$\alpha(a) = (a/\rho, a/\sigma).$$

Proof. *It is straight forward to see α is injective.*
For every $(a/\rho, b/\sigma) \in H_1/\rho \times H_2/\sigma$, $\langle a,b\rangle \in H$. So, there exist $c \in H_1$ such that $a\rho c\sigma b$. Then

$$\alpha(c) = \langle c/\rho, c/\sigma\rangle \in \langle a/\rho, b/\sigma\rangle.$$

Thus α is onto. Finally for $\beta \in \mathcal{F}_n$ and $a_1,\dots,a_n \in H$,

$$\alpha\beta^H(a_1,\dots,a_n)$$
$$= \big(\beta^H(a_1,\dots,a_n)/\rho, \beta^H(a_1,\dots,a_n)/\sigma\big)$$
$$= \beta^{H/\rho}(a_1/\rho,\dots,a_n/\rho), \beta^{H/\sigma}(a_1/\sigma,\dots,a_n/\sigma)$$
$$= \beta^{H/\rho \times H/\sigma}\big((a_1/\rho,a_1/\sigma),\dots,(a_n/\rho,a_n/\sigma)\big)$$
$$= \beta^{H/\rho \times H/\sigma}\big(\alpha a_1,\dots,\alpha a_n\big);$$

hence α is indeed an isomorphism. □

Definition 24. A multialgebra H is (directly) indecomposable if H is not isomorphic to a direct product of two nontrivial multialgebras.

Example 25. Any finite multialgebra H with $|H|$ a prime number must be directly indecomposable.
By theorem 4.3 and 4.5 we have:

Corollary 26. H is directly indecomposable if and only if the only factor congruences on H are Δ and ∇.

Proof. *Suppose Δ and ∇ be the only factor congruences on H. Then*

$$H/\Delta = \{a/\Delta | a \in H\} = \{a\}$$

and

$$H/\nabla = \{a/\nabla | a \in H\} = H.$$

So $|a/\Delta| = 1$ *i.e* H/Δ *is trivial. So by 4.5*

$$H \simeq H/\Delta \times H/\nabla,$$

i.e H *is directly indecomposible.*
Now suppose H *be directly indecomposible. So* H *is not isomorphic to a direct product of two nontrivial multialgebras. If* H *have any factor congruence except* Δ, ∇*, then by 4.5*

$$H \cong H/\rho \times H/\sigma$$

that is conflict.
We can easily generalized the binary product $H_1 \times H_2$ *as follows.* □

Definition 27. Let $(H_i)_{i\in I}$ be a family of multialgebras of type $\boldsymbol{\mathcal{F}}$. The (direct) product $H = \prod H_i$ is a multialgebra with universe $\prod_{i\in I} H_i$ and such that for $\beta \in \boldsymbol{\mathcal{F}}_n$ and $a_1, \dots, a_n \in \prod_{i\in I} H_i$,

$$\beta^H(a_1, \dots, a_n)(i) = \beta^{H_i}\big(a_1(i), \dots, a_n(i)\big)$$

for $i \in I$, i.e β^H is defined coordinate-wise.
The empty product $\prod \phi$ is the trivial multialgebra with universe $\{\phi\}$. As before we have projection maps

$$\pi_j : \prod_{i\in I} H_i \to H_j$$

for $j \in I$ defined by

$$\pi_j(a) = a(j).$$

which give surjective strong homomorphisms

$$\pi_j : \prod_{i\in I} H_i \to H_j.$$

If $i = \{1, 2, \dots, n\}$, we also write $H_1 \times \dots \times H_n$. If I is arbitrary but $H_i = H$ for all $i \in I$, then we usually write H^I for the direct product, and call it a (direct) power of H. H^ϕ is a trivial multialgebra.

Theorem 28. If H_1, H_2 and H_3 are of type $\boldsymbol{\mathcal{F}}$, then

(a) $H_1 \times H_2 \simeq H_2 \times H_1$ under $\alpha\big((a_1, a_2)\big) = (a_2, a_1)$.

(b) $H_1 \times (H_2 \times H_3) \simeq H_1 \times H_2 \times H_3$ under

$$\alpha\big((a_1, (a_2, a_3)\big) = (a_1, a_2, a_3).$$

Proof. *(a) For any* $\beta \in \mathcal{F}$ *and* $a_1 \in H_1$*,* $a_2 \in H_2$ *define*

$$\alpha : H_1 \times H_2 \to H_2 \times H_1$$
$$\alpha(a_1, b_1) = (b_1, a_1)$$

α *is well-defined and one-to-one because:*

$$\alpha(a_1, b_1) = \alpha(a_1', b_1')$$
$$iff \quad (b_1, a_1) = (b_1', a_1')$$

So $b_1 = b_1'$*,* $a_1 = a_1'$*. Thus* $(a_1, b_1) = (a_1', a_2')$*.*
Now, we prove α *is homomorphism. For any* $a_1, b_1 \in H_1$ *and* $a_2, b_2 \in H_2$*,*

$$\begin{aligned}
&\alpha\big(\beta^{H_1\times H_2}(a_1, a_2), (b_1, b_2)\big)\\
&= \alpha\big(\beta^{H_1}(a_1, b_1), \beta^{H_2}(a_2, b_2)\big)\\
&= \beta^{H_2}(a_2, b_2), \beta^{H_1}(a_1, b_1)\\
&= \beta^{H_2\times H_1}\big((a_2, a_1), (b_2, b_1)\big)\\
&= \beta^{H_2\times H_1}\big(\alpha(a_1, a_2), \alpha(b_1, b_2)\big).
\end{aligned}$$

(b) For H_1, H_2, H_3 *of type* $\mathcal{F}$*, defined*

$$\alpha : H_1 \times (H_2 \times H_3) \to H_1 \times H_2 \times H_3$$
$$\alpha\big(a_1, (a_2, a_3)\big) = (a_1, a_2, a_3)$$

α *is homomorphism because:*

$$\begin{aligned}
&\beta^{H_1\times H_2\times H_3}\big(\alpha(a_1, (a_2, a_3), \alpha(a_1', (a_2', a_3'))\big)\\
&= \beta^{H_1\times H_2\times H_3}\big((a_1, a_2, a_3), (a_1', a_2', a_3')\big)\\
&= \big(\beta(a_1, a_1'), \beta(a_2, a_2'), \beta(a_3, a_3')\big)\\
&= \alpha\big(\beta(a_1, a_1'), (\beta(a_2, a_2'), \beta(a_3, a_3'))\big)\\
&= \alpha\big(\beta(a_1, a_1'), \beta\big((a_2, a_3), (a_2', a_3')\big)\big)\\
&= \alpha\big(\beta((a_1, (a_2, a_3)), (a_1', (a_2', a_3'))\big).
\end{aligned}$$

□

Definition 29. (i) If $\alpha_i : H \to H_i$, $i \in I$ are maps, then the natural map $\alpha : H \to \prod_{i\in I} H_i$ is defined by

$$(\alpha a)(i) = \alpha_i a.$$

(ii) If we are given maps $\alpha_i : H_i \to H_i'$, $i \in I$, then the natural map

$$\alpha : \prod_{i\in I} H_i \to \prod_{i\in I} H_i'$$

is defined by

$$(\alpha a)(i) = \alpha_i(a(i)).$$

Theorem 30. (i) If $\alpha_i : H \to H_i$, $i \in I$ is a family of homomorphism, then the natural map α is a homomorphism from H to $H^* = \prod_{i\in I} H_i$.
(ii) If $\alpha_i : H_i \to H_i$, $i \in I$, is an indexed family of (resp. strong) homomorphism, then the natural map α is a (resp. strong) homomorphism from $H^* = \prod_{i\in I} H_i$ to $H'^* = \prod_{i\in I} H'_i$.

Proof. *Suppose $\alpha_i : H \to H_i$ is a homomorphism for $i \in I$. Then for $a_1, \dots, a_n \in H$ and $\beta \in \mathcal{F}_n$ we have*

$$\begin{aligned}&\big(\alpha\beta^H(a_1,\dots,a_n)\big)(i)\\ &= \alpha_i\beta^H(a_1,\dots,a_n)\\ &= \beta^{H_i}(\alpha_i a_1,\dots,\alpha_i a_n)\\ &= \beta^{H_i}\big((\alpha a_1)(i),\dots,(\alpha a_n)(i)\big)\\ &= \beta^{\prod H_i}\big(\alpha a_1,\dots,\alpha a_n\big)(i);\end{aligned}$$

Hence,

$$\alpha\beta^H(a_1,\dots,a_n) = \beta^{\prod H_i}(\alpha a_1,\dots,\alpha a_n),$$

so α is a homomorphism in (a).
Now (b) is consequence of (a) because the maps

$$\prod H_i \xrightarrow{\pi} H_i \xrightarrow{\alpha_i} H'_i \xrightarrow{\beta_i} \prod H'_i$$

indeed that $\beta_i \circ (\alpha_i \circ \pi_i)$ is homomorphism. □

Definition 31. If $a_1, a_2 \in H$ and $\alpha : H \to H'$ is a map we say α separate a_1 and a_2 if

$$\alpha a_1 \neq \alpha a_2.$$

The maps $\alpha_i : H \to H_i$, $i \in I$, separate points if for each $a_1, a_2 \in H$ with $a_1 \neq a_2$ there is an α_i such that

$$\alpha_i(a_1) \neq \alpha_i(a_2).$$

Lemma 32. For an indexed family of maps $\alpha_i : H \to H_i$, the following are equivalent:

(i) The map α_i sperate points.

(ii) α is injective.

(iii) $\bigcap_{i\in I} \ker \alpha_i = \Delta$.

Proof. *Routine.* □

Theorem 33. Let $\alpha_i : H \to H_i$, $i \in I$, be a family of (resp. strong) homomorphisms, $i \in I$, then natural (resp.strong) homomorphism $\alpha : H \to \prod_{i\in I} H_i$ is an (resp. strong) embedding if and only if $\bigcap_{i\in I} \ker \alpha_i = \Delta$ if and only if the maps α_i separate points.

Proof. *Immediate from 4.15.* □

5 Subdirect products of multialgebras

Definition 34. A multialgebra H is a subdirect product of an indexed family $(H_i)_{i\in I}$ of multialgebras if

(i) $H \leq \prod_{i\in I} H_i$, and

(ii) $\pi_i(H) = H_i$ for each $i \in I$.

An (resp.strong) embedding $\alpha : H \to \prod_{i\in I} H_i$ is (resp.strong) subdirect if $\alpha(H)$ is a (resp.strong) subdirect product of the H_i.
Note that if $I = \phi$, the H is a subdirect of ϕ if and only if $H = \prod \phi$, is trivial multialgebra.

Lemma 35 *[16]*. If $\theta_i \in con(H)$ for $i \in I$ and $\bigcap_{i\in I} \theta_i = \Delta$, then the natural homomorphism

$$\varphi : H \to \prod_{i\in I} H/\theta_i$$

defined by

$$\varphi(a)(i) = a/\theta_i$$

is a subdirect (strong) embedding.

Proof. *Proof is similar to the proof for algebras and omitted.* □

Definition 36. A multialgebra H is subdirectly irreducible if for every subdirect embedding

$$\alpha : H \to \prod_{i\in I} H_i$$

there is an $i \in I$ such that

$$\pi_i \circ \alpha : H \to H_i$$

is an isomorphism.
The following result give a characterization of subdirectly irreducible multialgebras is most useful in practice.

Theorem 37. A multialgebra H is subdirectly irreducible if and only if H is trivial or there is a minimum congruence in $conH - \{\Delta\}$.

Proof. *($\Longrightarrow$) If H is not trivial and $conH - \{\Delta\}$ has no minimum element then $\cap(con(H - \{\Delta\}) = \Delta$. Let $I = conH - \{\Delta\}$. Then the natural map $\nu : H \to \prod_{\theta\in I} H/\theta$ is a subdirect embedding by Lemma* 5.2*, and as the natural map $H \to H/\theta$ is not*

injective for $\theta \in I$, it follows that H is not subdirectly irreducible.
($\Longleftarrow$) If H is trivial and $\nu : H \to \prod_{i\in I} H_i$ is a subdirect embedding then each H_i is trivial; hence each $\pi_i \circ \nu$ is an isomorphism. So suppose H is not trivial, and let $\theta = \cap\big(conH - \{\Delta\}\big) \neq \Delta$. Choose $\langle a, b\rangle \in \theta$, $a \neq b$. If $\nu : H \to \prod_{i\in I} H_i$ is a subdirect embedding then for some i, $(\nu a)(i) \neq (\nu b)(i)$, hence $(\pi_i \circ \nu)(a) \neq (\pi_i \circ \nu)(b)$.
Thus, $\langle a, b\rangle \notin \ker(\pi_i \circ \nu)$ so $\theta \nsubseteq \ker(\pi_i \circ \nu)$. But this implies $\ker(\pi_i \circ \nu) = \Delta$. So $\pi_i \circ \nu : H \to H_i$ is an isomorphism. □

In the latter case the minimum element in $\cap(con(H) - \{\Delta\})$, a principal congruence and the congruence lattice of H looks like the following diagram

Theorem 38 (Birkhoff). Every multialbera H is isomorphism to a subdirect product of subdirectly irreducible.

Proof. *By previous theorem we know trivial multialgebras are subdirectly irreducible. Then we only need to consider the case of nontrivial H, that proof is easily by Zorn's Lemma.* □

Some important question about multialgebras:

1. *Let $\mathcal{A}$ be a multialgebra and $\mathcal{A}^*$ be its fundamental algebra. Under what conditions the corresponding fundamental algebra and $\mathcal{A}$ are satisfying the same identities?.*
2. *Let V be a variety of multialgebras, what identities hold in the variety generated by $(\{V(\mathcal{A}^*)|\mathcal{A} \in V\}$?.*

It is convenient to introduce the following notation:
If V is variety of multialgebras, by V^* we denote the variety generated by $\{V^*(\mathcal{A})|\mathcal{A} \in V\}$, where V^* is denoted the corresponding elements of variety V in V^*.(See theorem 2 in Ivica Bosnjak et. all, 2003)
There is another subject that has attracted attention of algebraists; If K is a class of multialgebras, and $\mathcal{A}$ and $\mathcal{B}$ are isomorphic multialgebras from K, then $\mathcal{A}^* \cong \mathcal{B}^*$. It is natural to ask whether the converse it is true, i.e. is it true that for any $\mathcal{A}$, $\mathcal{B}$ from K it holds:

$$\mathcal{A}^* \cong \mathcal{B}^* \Rightarrow A \cong B?$$

Clearly the implication is not true always (for example let $\mathcal{A}$ and $\mathcal{B}$ be two total hypergroups)
($x \circ y = A \ \forall x, y \in A$). Clearly, $\mathcal{A}^* \cong \mathcal{B}^* = (e)$. Now if we choose sets A and B such that $|A| \neq |B|$, then $A \ncong B$

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