Topological Vector Spaces Derived From Topological Hypervector Spaces

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Abstract: We introduce topological hypervector spaces on a topological field, in the sense of Tallini, and study some basic properties of this hyperspaces. In this regards we study the relationship between the topology on a hypervector spaces and its complete part. In particular we show that if every open subset of a topological hypervector space is a complete part then its fundamental vector space induced is a topological vector space. Finally, we study the quotient space of topological hypervector spaces and the derived topological space of a topological hypervector space with respect its fundamental relegation.

Key–Words: Topological hypervector space, Fundamental relation, Complete part, Topological fundamental vector space, Homeomorphism.

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1 Introduction

The concept of hypergroups as a generalization of groups was firstly introduced in 1934 at the 8^{th}Congress of Scandinavian Mathematicians by F. Marty [25]. In the following, it has been studied and extended by many researchers. Indeed the notion of hyperstructures is a generalization of classical algebraic structures. M.S. Tallini introduced the notion of hypervector spaces over a field, and studied the basic properties of this hyperspaces(for more details see [31, 32]. Also, some important properties of (fuzzy) hypervector spaces were studied in [3, 4, 5, 6, 7, 8, 9, 15].

On of the main topic in theory of hyperstructures is the study of regular and strongly regular relations. In particular, the fundamental relations on a hyperstructure, as the smallest strongly relation on the hyperstructure such that its derived quotient spaces with respect this relation become an algebraic structure, for example for special hyperstructures such as semihypergroups, hypergroups, hyperrings, hypervector spaces and etc., their corresponding derived algebraic structures with the fundamental relations is semigroup, group, ring, module, vectors space and etc. The fundamental relations play important role in the study algebraic hyperstructures. In fact, these relation construct a connection between the categories of hyperstructures and categories of algebraic structures(for more details see [2]). The fundamental relation β^* on hypergroups introduced by Koskas[24], and was studied mainly by Corsini [11] and Vougiouklis [33].

Later on, Freni [16] introduced the γ^*-relation on a hypergroup, as a generalization of the β^*-relation. Then B. Davvaz et al. [1], R. Ameri et al. [9] and M. Hamidi et al. [18] introduced the ν^*-relation, ξ^*-relation and τ^*-relation, respectively. In [33] T. Vougiouklis introduced the fundamental relation ε^* of H_v-vector space (a general class of hypervector spaces) and in [4], R. Ameri et al. defined the fundamental relation ε^* for a given hypervector space V, over a classical field K (in the sense of Tallini) as the smallest equivalence relation on V such that V/ε^* is a classical vector space over K.

The notion of topological(transposition) hypergroups introduced and studied by R. Ameri([3]), in this paper notions of a (pseudo, strong pseudo) topological (transposition) hypergroups and introduced and the relationships between pseudo topological polygroups and topological polygroups investigated. Also, Heidari et al.([19]) studied the concept of topological hypergroups as a generalization of topological groups. Since then, many researchers have worked on topological hyperstructures (for more details see [10],[12],[14],[17],[19],[20],[21], [22],[26],[27],[28], [29],[30]).

One of classes of topological hyperstructures is hypervector spaces. The notion of a topological hypervector space introduced in [34]. In this paper, we follow [34] and study more properties of topological hyperstructures. In this regards, we consider various kinds of topologies on power set of a topological hypervector space and use them to introduce the various kinds of topological hypervector spaces. In particular, we consider the upper topology over $P^*(V)$, the family of all nonempty subsets of V, and prove that if $(V, +, \circ, K, \mathcal{T})$ is a topological hypervector space such that every its open subset is a complete part, then the quotient space V/ε^*, the set of all equivalence classes by ε^*, is a topological vector space. Finally, we consider a topological vector space $(V, +, \cdot, K, \mathcal{T})$ and its subhyperspace W, to form the topological hypervector space $(\overline{V}, +, \circ, K, \mathcal{T})$ and prove that two hyperspaces $\overline{V}/\varepsilon^*$ and V/W are homeomorphic.

2 Preliminaries

In this section, we review some definitions and results which we need to development our paper.

A *topological group* is a group $(G, .)$ which is also a topological space such that the multiplication map $(g, h) \to gh$ from $G \times G$ to G, and the inverse map $g \to g^{-1}$ from G to G, are both continuous. Similarly, a *topological ring* or a *topological field* are defined. A *topological vector space* is a vector space X over a topological field K (most often the real or complex numbers with their standard topologies) that is endowed with a topology such that vector addition $+ : X \times X \to X$ and scalar multiplication. $: K \times X \to X$ are both continuous functions with respect to product topologies on $X \times X$, and on $K \times X$ and X, respectively, such that the mapping $x \mapsto -x = (-1)x$, is continuous and the topology on X is compatible with its additive group structure.

Let G be a nonempty set and $P^*(G)$ be the family of all nonempty subsets of G. A mapping $\cdot : G \times G \longrightarrow P^*(G)$, where is called a *hyperoperations*, or a hypercomposition on G, that is for all x, y of G, $\cdot(x, y)$, denoted by $x \circ y$, or simply by xy is a nonempty subset of G, and it called *hyperproduct* of x and y. An algebraic system $(G, \cdot_1, \cdot_2, \ldots, \cdot_n)$ is called a *hyperstructure*, the pair $(G, \cdot)$ endowed with only hyperoperation is called a *hypergroupoid*. For every two nonempty subsets A and B of G by $A \cdot B$ we means $\bigcup_{a \in A, b \in B} a \cdot b$.

Definition 1 *([31]) Let K be a field and $(V, +)$ be an abelian group. A hypervector space over K is a quadruple $(V, +, \circ, K)$, where $\circ$ is a mapping $\circ : K \times V \to P^*(V)$, such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:*

(H_1) $a \circ (x + y) \subseteq a \circ x + a \circ y$;

(H_2) $(a + b) \circ x \subseteq a \circ x + b \circ x$;

(H_3) $a \circ (b \circ x) = (ab) \circ x$;

(H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$;

(H_5) $x \in 1 \circ x$,

where for all $A, B \in P^*(V), A + B = \{a + b \mid a \in A, b \in B\}$.

Remark 2 *If in (H_1) the equality holds, then the hypervector space is called strongly right distributive. If in (H_2) the equality holds, the hypervector space is called strongly left distributive. A hypervector space is called strongly distributive hypervector space, if it is both strongly left and strongly right distributive.*

Clearly, every classical vector space over a field K is also an strongly distributive hypervector space over K, with the operations on V and K, which is called a trivial hypervector space. A nonempty subset W of V is called a subhyperspace, if W is itself a hypervector space with the external hyperoperation on V, i.e. for all $a \in K$ and $x, y \in W, x - y \in W$ and $a \circ x \subseteq W$. Let $\Omega = 0 \times 0_V$, where 0_V is the zero of $(V, +)$. If V is either strongly right distributive, or left distributive, then Ω is a subgroup of $(V, +)$. An strongly right distributive hypervector space is strongly left distributive.

Lemma 3 *Let X and Y be topological spaces and let $f : X \to Y$. Then the following statements are equivalent:*

(1) *f is continuous;*

(2) *for all open subset U of Y, $f^{-1}(U)$ is open in X;*

(3) *for all $x \in X$ and all open subset V of X containing $f(x)$, there exists an open subset U of X containing x such that $f(U) \subseteq V$.*

Lemma 4 *[3] Let $(X, \mathcal{T})$ be a topological space, then the family $\mathcal{B}$ consisting of all*

$$S_U = \{W \in P^*(X) : W \subseteq U,\ U \in \mathcal{T}\},$$

is a base for a topology on $P^*(X)$. This topology is denoted by $\mathcal{T}^*$.

Lemma 5 [3, 13] *Let $(X, \circ)$ be a hypergroupoid and $\mathcal{T}$ be a topology on X. Then the following assertions are equivalent:*

(1) *for any $U \in \mathcal{T}$, the set $\{(x, y) \in X \times X : x \circ y \subseteq U\}$ is open in $X \times X$;*

(2) *for every* $x, y \in X$ *and* $U \in \mathcal{T}$ *such that* $x \circ y \subseteq U$*, there exist* $U_x,\ U_y \in \mathcal{T}$ *containing* x, y *respectively, such that* $U_x \circ U_y \subseteq U$*;*

(3) *for every* $x, y \in X$ *and* $U \in \mathcal{T}$ *such that* $x \circ y \subseteq U$*, there exist* $U_x,\ U_y \in \mathcal{T}$ *containing* x, y *respectively,such that* $a \circ b \subseteq U$ *for any* $a \in U_x$ *and* $b \in U_y$*.*

Let $(V, +, \circ, K)$ be hypervector space over a topological field K and $\mathcal{T}$ be a topology on V. In the following we use the topology $\mathcal{T}^*$ on $P^*(V)$ and the product topology on $V \times V$.

3 Topological Hypervector Spaces

In this section we introduce the concept of topological hypervector spaces and study some their properties.

Definition 6 *Let* $(V, +, \circ, K)$ *be a hypervector space over a topological filed* K *and* $(V, \mathcal{T})$ *be a topological space. Then* $(V, +, \circ, K, \mathcal{T})$ *is said to be a topological hypervector space* $(thvs)$
if the operations $+ : V \times V \to V, (x, y) \mapsto x + y$*,* $i : V \to V, x \mapsto -x$ *and the hyperoperation*
$\circ : K \times V \to P^*(V), (a, x) \mapsto a \circ x$ *are continuous.*

Example 7 *Every topological vector space* $(V, +, \cdot, K, \mathcal{T})$ *with hyperoperation* $a \circ x = \{a \cdot x\}$ *is a topological hypervector space over* K*.*

Example 8 *Every hypervector space* $(V, +, \circ, K)$ *with trivial topology* $\mathcal{T}$ *is a topological hypervector space. Since, if we have* $\mathcal{T} = \{\emptyset, V\}$ *then*
$\mathcal{T}^* = \{\emptyset, S_V\} = \{\emptyset, P^*(V)\}$*.*

Example 9 *Let* $K = V = Z_2 = \{\bar{0}, \bar{1}\}$*. Then* $(V, +, \circ, K)$ *is a hypervector space, where* $a \circ x = \{\bar{0}, \bar{1}\}$ *for any* $a \in K$ *and* $x \in V$*. Let* $\mathcal{T} = \{\emptyset, \{\bar{0}\}, \{\bar{1}\}, V\}$ *be a topology on* $V = K$*. We have* $\mathcal{T}^* = \{\emptyset, \{\{\bar{0}\}\}, \{\{\bar{1}\}\}, \{\{\bar{0}\}, \{\bar{1}\}\}, P^*(V)\}$*. It is clear that* V *is a topological hypervector space.*

Example 10 *By considering the external hyperoperation* $\circ : R \times R^2 \to P^*(R^2), a \circ (x, y) = a \cdot x \times R$ *then* $(R^2, +, \circ, R)$ *is a strongly distributive hypervector space. The family* $\mathcal{B} = \{(x, y) : a < x < b, y \in R\}$ *is a base for a topology on* R*. Then* $(R^2, +, \circ, R, \mathcal{T})$ *is a topological hypervector space.*

Example 11 *Let:*

$$\circ : R \times R \to P^*(R),\ a \circ x = \{a \cdot x, -a \cdot x\}$$

be a external hyperoperation on R.Then $(R, +, \circ, R)$ *is a hypervector space, but it is neither the right distributive nor the left distributive. With standard topology on* R*,* $(R, +, \circ, R, \mathcal{T})$ *is a topological hypervector space.*

Example 12 *Let:*

$$\circ : R \times R \to P^*(R),\ a \circ x = \{a \cdot x, -a \cdot x, 0\}$$

be a external hyperoperation on R. Then $(R, +, \circ, R)$ *is a hypervector space, but it is neither the right distributive nor the left distributive. With standard topology on* $V = R$ *and discrete topology on* $K = R$*,* V *is a topological hypervector space.*

Topological hypervector spaces are a generalization of topological vector spaces but some characteristics of topological vector spaces are not valid in topological hypervector spaces. If V is a $thvs$, $(V, +)$ is a topological group.

Lemma 13 *Let* V *be a* $thvs$*. Then*

(1) *for fixed* $x \in V$*, the map* $y \mapsto x + y$ *is a homeomorphism of* V *onto* V*;*

(2) *if* U *is open and* $x \in V$*, then* $x + U$ *is open; if* U *is open and* A *is any subset of* V*, then* $A + U$ *is open;*

(3) *for fixed* $a \in K$*, the map* $x \mapsto a \circ x$ *is continuous, but not necessarily open. In Example 11,* $U = (2, 3)$ *is open and* $2 \circ (2, 3) = (-6, -4) \cup (4, 6)$ *is also open, but in the Example 12,* $U = (2, 3)$ *is open and* $2 \circ (2, 3) = (-6, -4) \cup \{0\} \cup (4, 6)$ *is not open in R.*

The complete parts were introduced for the first time by Koskas [24]. Then, this concept was studied by many authors. Let $(V, +, \circ, K)$ be a hypervector space over K and A be a nonempty subset of V. We say that A is a *complete part* of V, if for nonzero natural number n, for all $a_1, \ldots, a_n$ of K, and for all $x_1, \ldots, x_n$ of V, the following implication holds:

$$A \cap \sum_{i=1}^{n} a_i \circ x_i \neq \emptyset \Longrightarrow \sum_{i=1}^{n} a_i \circ x_i \subseteq A.$$

Theorem 14 *Let* V *be a* $thvs$*,* $A \subseteq V$ *and* U *be an open subset of* V*, such that* U *is a complete part of* V*. Then* $A \subseteq a^{-1} \circ U$ *if and only if* $a \circ A \subseteq U$ *for all* $a \in K$*.*

Suppose that $A \subseteq a^{-1} \circ U$ and $x \in A$. So $x \in a^{-1} \circ U$, and there exists $u \in U$, such that $x \in a^{-1} \circ u$ thus, $a \circ x \subseteq a \circ (a^{-1} \circ u) = 1 \circ u$. We have $u \in 1 \circ u, u \in U$, which implies that $1 \circ u \subseteq U$ since U is complete part. Therefore $a \circ x \subseteq U$.

Conversely, suppose that $a \circ A \subseteq U$ and $a \in K$. Then, we have $A \subseteq a^{-1} \circ (a \circ A) \subseteq a^{-1} \circ U$.

Theorem 15 *Let* U *be an open subset of a* $thvs$*, such that* U *is a complete part. Then*

(1) *$a \circ U$ is an open subset of V for any $a \in K, a \neq 0$;*

(2) *for any subset A of K and for $0 \neq a \in A$, $A \circ U$ is open.*

(1) The map $P_a : V \to P^*(V),\ P_A : x \mapsto a \circ x$ is continuous. For $a \neq 0$ we have

$$P_{a^{-1}}^{-1}(S_U) = \{x \in V : a^{-1} \circ x \subseteq U\} = a \circ U,$$

thus $a \circ U$ is open. (2) Since the union of open subsets is open, therefore $A \circ U = \bigcup_{a \in A} a \circ U$ is open.

4 Topological Fundamental Vector Spaces

In this section, the concept of a topological fundamental vector space derived of a topological hypervector space is introduced. Let $(V, +, \circ, K)$ be a hypervector space over K. The fundamental relation ε^* of V was introduced by T. Vougiouklis in [33] as the smallest equivalence relation on H_v-vector space, a general class of hypervector spaces, such that the quotient V/ε^* is a vector space over K. In the following, we introduce the fundamental relation on hypervector spaces in the sense of Tallini, and study the relationship between V and V/ε^* in the way of [4].

let U be the set of all finite linear combinations of elements of V with coefficient in K, that follows

$$U = \left\{ \sum_{i=1}^{n} a_i \circ x_i : a_i \in K, x_i \in V, n \in N \right\}.$$

Now, consider the ε-relation over V by

$$x \varepsilon y \Longleftrightarrow \exists u \in U : \{x, y\} \subseteq u,\ \forall x, y \in V.$$

Let ε^* be the transitive closure of ε. We define addition operation and scalar multiplication on V/ε^* by

$$\{\oplus : V/\varepsilon^* \times V/\varepsilon^* \to V/\varepsilon^* \varepsilon^*(x) \oplus \varepsilon^*(y) = \{\varepsilon^*(t) : t \in \varepsilon^*(x) + \varepsilon^*(y)\ \ ,\}$$

and

$$\{\odot : K \times V/\varepsilon^* \to V/\varepsilon^* a \odot \varepsilon^*(x) = \{\varepsilon^*(z) : z \in a \circ \varepsilon^*(x)\}.$$

Theorem 16 *(see [33]) Let $(V, +, \circ, K)$ be a hypervector space over K. Then,*

(1) *$\varepsilon^*(a \circ x) = \varepsilon^*(y)$ for all $y \in a \circ x, \forall a \in K, \forall x \in V$, where $\varepsilon^*(a \circ x) = \bigcup_{b \in a \circ x} \varepsilon^*(b)$.*

(2) *$\varepsilon^*(x) \oplus \varepsilon^*(y) = \varepsilon^*(x + y)$.*

(3) *$\varepsilon^*(0)$ is the identity element of $(V/\varepsilon^*, \oplus)$.*

(4) *$(V/\varepsilon^*, \oplus, \odot, K)$ is a vector space over K.*

The vector space $(V/\varepsilon^*, \oplus, \odot, K)$ is called fundamental vector space of V.

Theorem 17 *Let $(V, +, \circ, K)$ be a hypervector space over K and $(V/\varepsilon^*, \oplus, \odot, K)$ be the fundamental vector space of V. Then the canonical map $\pi : V \to V/\varepsilon^*$, such that $\pi(x) = \varepsilon^*(x)$, is an epimorphism.*

Let $x, y \in V$ and $a \in K$, we see that $\pi(x + y) = \pi(x) \oplus \pi(y)$. Now we show that $\pi(a \circ x) = a \odot \pi(x)$. We have $\pi(a \circ x) = \varepsilon^*(a \circ x) = \varepsilon^*(y)$ for all $y \in a \circ x$. On the other hand, we have $y \in a \circ x, x \in \varepsilon^*(x)$ that implies $y \in a \circ \varepsilon^*(x)$ thus, $a \odot \pi(x) = a \odot \varepsilon^*(x) = \{\varepsilon^*(z) : z \in a \circ \varepsilon^*(x)\} = \varepsilon^*(y)$.

Let X be a topological space and $\sim$ be any equivalence relation on X. The quotient set of all equivalence classes is given by the $X/\sim = \{[x] : x \in X\}$. We have the canonical map or quotient map $\pi : X \to X/\sim,\ x \mapsto [x]$, and we define a topology on $X/\sim$ by setting that: $U \subseteq X/\sim$ is open if and only if $\pi^{-1}(U)$ is open in X. Then it is easy to verify that:

(1) the canonical map π is continuous.

(2) the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ s.t. π is continuous.

(3) the canonical map π is not necessarily open or closed.

Theorem 18 *Let $(V, +, \circ, K)$ be a topological hypervector space over K, such that every open subset of V is a complete part. Then the canonical map $\pi : V \to V/\varepsilon^*$ is open.*

Let W be an open subset of V and $x \in \pi^{-1}(\pi(W))$, we have $\pi(x) \in \pi(W)$ thus there exists $v \in W$ such that $\pi(x) = \pi(v)$ and $x \in \varepsilon^*(v)$.Hence, there exist $a_1, \ldots, a_n \in K$ and $x_1, \ldots, x_n \in V$, such that $\{x, v\} \subseteq \sum_{i=1}^{n} a_i \circ x_i$. Since W is open so there exists an open subset U of V, such that $v \in U \subseteq W$. Hence we have $v \in U \cap \sum_{i=1}^{n} a_i \circ x_i$ and U is complete part, so $x \in \sum_{i=1}^{n} a_i \circ x_i \subseteq U \subseteq W$. Thus, $x \in U \subseteq \pi^{-1}(\pi(W))$. Therefore, $\pi(W)$ is open in V/ε^*.

Theorem 19 *Let $(V, +, \circ, K, \mathcal{T})$ be a topological hypervector space over K such that every open subset of V is a complete part . Then, $(V/\varepsilon^*, \oplus, \odot, \mathcal{T}^*)$ is a topological vector space over K, where $\mathcal{T}^*$ is the quotient topology on V/ε^*.*

By Theorem16, $(V/\varepsilon^*, \oplus, \odot)$ is a vector space. We show that the mappings

$$\oplus : (\pi(x), \pi(y)(\mapsto \pi(x) \oplus \pi(y)$$

and

$$\odot : (a, \pi(x)) \mapsto a \odot \pi(x)$$

are continuous, where $\oplus = \oplus_{\varepsilon^*}$ and $\odot = \odot_{\varepsilon^*}$.

(1) Let U be an open subset of V/ε^* and $x, y \in U$, such that $\pi(x) \oplus \pi(y) \in U$. So, we have $\pi(x + y) \in U$ or $x + y \in \pi^{-1}(U)$. Since $\pi^{-1}(U)$ is open in V and V is $thvs$, it follows that there exist open subsets U_1, U_2 of V such that $x \in U_1, y \in U_2$ and $U_1 + U_2 \subseteq \pi^{-1}(U)$ or $\pi(U_+ U_2) \subseteq U$, thus $\pi(U_1) \oplus \pi(U_2) \subseteq U$.

(2) Let U be an open subset in V/ε^* and $a \in K, x \in V$ such that $a \odot \pi(x) \in U$. There exists $z \in a \circ \pi(x)$ and we have $\pi(z) \in U$ so $z \in \pi^{-1}(U)$. Since $a \circ x \subseteq a \circ \pi(x)$, so $a \circ x \subseteq \pi^{-1}(U)$. Thus there exists open subsets U_1 and U_2 containing a and x from K and V, respectively such that $U_1 \circ U_2 \subseteq \pi^{-1}(U)$ hence $U_1 \odot \pi(U_2) \subseteq U$. Since we have $\pi(U_1 \circ U_2) = \pi(\bigcup_{a \in U_1} a \circ U_2) = \bigcup_{a \in U_1} \pi(a \circ U_2)$
$= \bigcup_{a \in U_1}(a \odot \pi(U_2)) = U_1 \odot \pi(U_2)$.

A *topological vector space* (tvs) is a vector space V over a topological field K equipped with a topology such that the maps $(x, y) \mapsto x + y$ and $(a, x) \mapsto a \cdot x$ are continuous from $X \times X \to X$ and $K \times X \to X$, respectively.

Let X, Y be two vector space over K. A mapping $f : X \to Y$ is called *homomorphism* if for any $x, y \in X$ and for any $\lambda \in K$,

$$f(x + y) = f(x) + f(y), f(\lambda x) = \lambda f(x).$$

A bijective homomorphism between two vector spaces X and Y over K is called *algebraic isomorphism* and we say that X and Y are algebraically isomorphic $X \cong Y$. Let X and Y be two tvs on K. A *topological isomorphism* (*homeomorphism*) from X to Y is a algebraic isomorphism which is also continuous and open.

Let V be a tvs and $W \subseteq V$ be a linear subspace of V. The quotient space V/W consists of cosets $x + W = [x]$ and the quotient map $\pi : V \to V/W$ is defined by $\pi(x) = x + W$.

We construct a topological hypervector space such as $\overline{V}$ using a classical topological vector space V and its linear subspace W and prove that $\overline{V}/\varepsilon^*$ and V/W are homeomorphic.

Theorem 20 *For a linear subspace W of a tvs V, the quotient map $\pi : V \to V/W$ is a continuous and open map, when V/W is equipped with the quotient topology.*

The mapping π is continuous by the definition of the quotient topology. Let U be open in V. Then we have

$$\pi^{-1}(\pi(U)) = U + W = \bigcup_{v \in W} (U + v),$$

since $U + v$ is open for any $v \in W$, hence $\pi^{-1}(\pi(U))$ is open in V as a union of open sets. Therefore, $\pi(U)$ is open in V/W.

Theorem 21 *[23] Let W be a linear subspace of a tvs V. Then the quotient space V/W equipped with the quotient topology is a tvs.*

Let $(V, +, \cdot, K)$ be a classical vector space and W be a linear subspace of V and let $\overline{V} = V$. Then $(\overline{V}, +, \circ, K)$ is a strongly distributive hypervector space where

$$\circ : K \times V \to P^*(V),\ a \circ x = a \cdot x + W,$$

$\overline{V}$ is said to be the associated hypervector space concerning the vector space V.

Theorem 22 *Let $(V, +, \cdot, K)$ be a classical vector space and W be a linear subspace of V.
Then $\overline{V}/\varepsilon^* \cong V/W$.*

We define a mapping $f : \overline{V}/\varepsilon^* \to V/W$ by $f(\varepsilon^*(x)) = x + W$.

(1) the mapping f is well-defined. Let $\varepsilon^*(x) = \varepsilon^*(y)$, it follows that $x\varepsilon^* y$ and we have $x \in 1 \circ x = x + W$, $y \in 1 \circ y = y + W$, since the two sets $x + W$ and $y + W$ are equal or disjoint subset of V/W, thus $x + W = y + W$ and so $f(\varepsilon^*(x)) = f(\varepsilon^*(y))$.

(2) f is linear. Since, f$(\varepsilon^*(x) + \varepsilon^*(y)) = f(\varepsilon^*(x + y)) = x + y + W$
$= x + W + y + W$
$= f(\varepsilon^*(x)) + f(\varepsilon^*(y))$. and $f(a \odot \varepsilon^*(x)) = f(\varepsilon^*(z))$, $z \in a \circ \varepsilon^*(x)$, on the other hand, $a \cdot x \in 1 \circ (a \cdot x) = a \circ x \subseteq a \circ \varepsilon^*(x)$ which implies that f$(\varepsilon^*(z)) = f(\varepsilon^*(a \cdot x)) = a \cdot x + W$
$= a \circ (x + W) = a \circ f(\varepsilon^*(x))$.

(3) The mapping f is surjective. For one-to-one property of f, let $\varepsilon^*(x) \in Ker(f)$. Then $f(\varepsilon^*(x)) = x + W = W$ thus $x \in W$. Therefore, $\varepsilon^*(W) = \varepsilon^*(0) = 0_{\overline{V}/\varepsilon^*}$, which implies that f is one-to-one. Consequently, f is an algebraic isomorphism.

Theorem 23 *Let $(V, +, \cdot, K, \mathcal{T})$ be a tvs. Then $(\overline{V}, +, \circ, K, \mathcal{T})$ is a topological hypervector space.*

It is enough to show that the mapping $\circ : K \times \overline{V} \to P^*(V),\ a \circ x = a \cdot x + W$ is continuous. Let U be an open subset of V. the mapping "$\circ$" is continuous if and only if $\{(a,x) \in K \times \overline{V} : a \circ x \subseteq U\}$ is an open subset of $K \times \overline{V}$ for all $U \in \mathcal{T}$. We have $a \circ x \subseteq U \Rightarrow a \cdot x + W \subseteq U$. Since $a \cdot x \in a \cdot x + W \subseteq U$ and the mapping "$\cdot$" is continuous, there exist U_1 and U_2 containing a and x respectively, such that $U_1 \cdot U_2 \subseteq U$.

Theorem 24 *Let $(V, +, \cdot, K, \mathcal{T})$ be a tvs and W be a linear subspace of V. Then $\overline{V}/\varepsilon^*$ and V/W are topologically isomorphic.*

By Theorem 22, the map

$$f : \overline{V}/\varepsilon^* \to V/W,\ f(\varepsilon^*(x)) = x + W$$

is algebraic isomorphism. It is enough to show that f is continuous and open. Suppose that A is open in V/W. We show that $\pi^{-1}(f^{-1}(A))$ is open in V/W. Let $x \in \pi^{-1}(f^{-1}(A))$, then $\pi(x) \in f^{-1}(A)$, and so $f(\pi(x)) \in A$, thus $x + W \in A$. Since the canonical map $q : V \to V/W$ is continuous, there exists an open subset U_x containing x of V such that $U_x \subseteq q^{-1}(A)$. We show that $U_x \subseteq \pi^{-1}(f^{-1}(A))$. If $t \in U_x$, then $t + W \in A$, and so $t \in \pi^{-1}(f^{-1}(A))$. Therefore, $\pi^{-1}(f^{-1}(A))$ is open in $\overline{V}$, and f is continuous.
Now suppose that A is an open subset of $\overline{V}/\varepsilon^*$. We show that $f(A)$ is an open subset of V/W. Let $x + W \in f(A)$, then $\varepsilon^*(x) \in A$. Since the canonical mapping $\pi : \overline{V} \to \overline{V}/\varepsilon^*$ is continuous, there exists an open subset U_x containing x of V such that $U_x \subseteq \pi^{-1}(A)$. We show that $\{z + W : z \in U_x\} \subseteq f(A)$. If $z \in U_x$, then $z + W = f(\varepsilon^*(x)) \in f(A)$, thus $f(A)$ is open in V/W. Therefore f is open.

5 Conclusion

In this paper the notion of upper topology for topological hypergroups in [3] has been generalized to topological hypervector spaces, for short THVS(in the sense of Tallini) and the topological properties of them was investigated. Also, by considering the fundamental relation ε^*, as the smallest equivalence relation on a THVS space V, the topological behavior of the fundamental vector space V/ε^* was investigated. In particular, it was proved that if in a topological hypervector space V every open sets is a complete part, then the canonical map $\pi : V \to V/\varepsilon^*$ is a open mapping and the fundamental vector space V/ε^* is a topological vector space, too. Finally, for a topological vector space $(V, +, \cdot, K, \mathcal{T})$ and its subhyperspace W of V, it was shown that the quotient space $(V/W, +, \circ, K, \mathcal{T})$ is also a topological hypervector space and $V/W/\varepsilon^*$ is homeomorphic to $\cong V/W$.

We hope that is paper encourage intrusted researchers to work in this topic and develop more results of topological hypervector spaces, in particular extend this results to other classes of hypervector spaces, such as Krasner hypervector spaces, H_V-vector spaces and etc.

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Reza Ameri proposed, carried out the research, and wrote the article. M. H. and A. S., edited the article. They also commented on it. They also selected the appropriate journal and submitted the article. All authors have agreed to the manuscript.

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