Weil-Nachbin Theory for Locally Compact Groups

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Abstract: - Assume H is a normal subgroup of the locally compact group G with measures μ and η the on G and H, respectively, and assume $m: G \to R_+$ is a continuous homomorphism from G to a group R_+ of positive real numbers with the operation of multiplication, then we establish that for the existence of a measure ϑ on the quotient group G/H, it is necessary and sufficient that $\Delta_H^r(h) = m(h)\Delta_G^r(h)$ holds for all $h \in H$ so that each m-relatively invariant measure on G/H is a quotient measure $\mu_\eta = \mu/\eta$; also, we show that the m-relatively invariant measure ϑ on G/H can be presented in the form $\vartheta = \rho(m\mu)$ where $\rho: G \to G/H$ is a projection mapping.

Key-Words: - m-relative invariant measure, C^*-algebra, induced measure, Haar measure, locally compact group.

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1. Introduction, some definitions, and some results

Let G be a locally compact group, let $C_C(G)$ be a space of real-valued continuous functions with compact support.

Definition 1. A *Radon measure on a locally compact group G is called a linear form μ on $C_C(G)$ such that for any compact set $K \subset G$ restriction of the linear form μ to subspace $C_C(K) \subset C_C(G)$ functions of $C_C(G)$ which support contains in K, is continuous in the topology of uniform convergence. The value $\mu(\psi)$ of the Rason measure μ on the continuous function $\psi \in C_C(G)$ with compact support is called a Radon integral of the function ψ*.

As a consequence of the definition, we have that for any compact subset $K \subset G$ there exists a constant $\tilde{c}(K)$ dependent on K such that the equality

$$\left|\mu(\psi)\right| \le \tilde{c}\left\|\psi\right\|_{C_c(G)}$$

holds for all $\psi \in C_C(G)$.

Let $C_C^+(G)$ be s set of all finite positive continuous functions with compact supports. We denote by $\wp_+(G)$ the set of all lower semicontinuous positive functions i.e., all functions ψ such that at every point g_0 of its domain satisfy the following condition

$$\lim_{g \to g_0} \inf_{g \in G} \psi(g) = \psi(g_0).$$

Definition 2. ***Let μ be positive Radon measure on G, then the upper integral $\mu^*(\psi)$ of a function $\psi \in \wp_+(G)$ is defined by***

$$\mu^*(\psi) = \sup_{\varphi \in C_C^+(G),\ \varphi \le \psi} \mu(\varphi).$$

The upper integral of an arbitrary positive function $\psi : G \to R^+$ is defined by

$$\mu^*(\psi) = \inf_{\varphi \in \wp_+(G),\ \varphi \ge \psi} \mu^*(\varphi).$$

Definition 3. ***The outer measure $\mu^*(E)$ of an arbitrary subset $E \subset G$ is an upper integral $\mu^*(1_E)$ of the characteristic function*** 1_E of E.

The set $\mathrm{M}(G)$ of all Radon measures μ on the locally compact space G is the space of all linear forms on the vector space $C_C(G)$ and thus $\mathrm{M}(G)$ is a topological space with the $*$-weak or so-called wide topology of the weak convergence. If G is a compact group then the wide topology coincides with the classical weak topology.

Wide topology in $\mathrm{M}(G)$ can be defined by seminorms $\mu \mapsto \sup_{1 \le i \le k} |\mu(\psi_i)|$, where $\{\psi_i\}_{1 \le i \le k} \subset C_C(G)$ is an arbitrary finite sequence of functions of $C_C(G)$.

The measure theory measure on locally compact topological spaces is a dynamically developing branch of mathematics, a few new articles pertaining to this topic can be found in the list of references [1-12].

In this article, we clarify the definition of L^p - spaces for a locally compact group and describe it properties; also we show that assume H is a normal subgroup of the locally compact group G and assume μ and η are measures on G and H, presume $m : G \to R_+$ is a continuous homomorphism from a group G to a group of positive real numbers with the operation of multiplication, then we establish the existence of a measure ϑ on the quotient group G/H if and only if $\Delta_H^r(h) = m(h)\Delta_G^r(h)$ for all $h \in H$.

2. L^p - spaces

Let $C_C(G,B)$ be a space of real-valued continuous functions with compact support from a locally compact group G to a Banach space B. Let μ be a positive measure on G.

Definition 4. For each number $p \ge 1$ and for every mapping $\psi : G \to B$, we defined a finite or infinite value $\|\psi\|_p$ by

$$\|\psi\|_p = \left(\mu^*\left(\|\psi\|_B^{\ p}\right)\right)^{\frac{1}{p}}.$$

Applying Holder inequality we obtain the following lemma.

Lemma. ***Let $\alpha \ne 0$ be an arbitrary scalar, let $1 \le p < +\infty$ and ψ, φ be arbitrary two mappings $G \to B$, then we have***

$$\|\alpha\psi\|_p = |\alpha| \|\psi\|_p$$

$$\|\psi + \varphi\|_p \le \|\psi\|_p + \|\varphi\|_p.$$

Definition 5.

1. ***For $1 \le p < +\infty$, the set of all mappings ψ from G to B such that $\phi \in C_C(G)$ will be denoted by $\mathcal{L}^p(G,B)$ or simply $\mathcal{L}^p(G)$ or even $\mathcal{L}^p$.***

2. ***Let G be a locally compact group with a positive measure μ, B be a Banach space, we denote $L^p(G)$ the closure in the topology $\mathcal{L}^p(G)$ of the vector space $C_C(G,B)$ of all real-valued continuous functions with compact support, more***

precisely, $L^p(G)$ is the separable Banach space associated with $L^p(G)$, we are employing the usual agreement to use the same symbol for the space of equivalence classes and the space of individual symbols belonging to the class.

For the definition, we have that $\psi \in L^p(G)$ if and only if for any $\varepsilon > 0$ there exists a continuous function $\phi \in C_C(G)$ with a compact support such that $\|\psi - \phi\|_p \le \varepsilon$. Thus, the space $L^p(G)$ consists of limits of sequences of functions of $C_C(G)$ in $\mathscr{L}^p$-topology. Convergence in L^p-topology is called convergence in the p-th mean.

Theorem 1. ***Let sequence $\{\psi_k\} \subset L^p(G)$ be fundamental or sequence Cauchy in $L^p(G)$, then there exists a subsequence $\{\psi_{k(n)}\} \subset \{\psi_k\} \subset L^p(G)$ such that:***

1. The series $\sum_n \left\|\psi_{k(n+1)} - \psi_{k(n)}\right\|$ converges as a positive number series;

2. The series $\sum_n \psi_{k(n+1)}(g) - \psi_{k(n)}(g)$ converges absolutely for μ-almost everywhere;

3. If $\psi(g) \overset{\mu-almost\ \ every\ \ where}{=} \lim_{n\to\infty} \psi_{k(n)}(g)$ and ψ is well defined on G, then $\psi \in L^p(G)$ and

$$\lim_{k\to\infty} \left\|\psi_{k(n)} - \psi\right\|_p = 0;$$

4. There exists a lower semicontinuous positive function $\varphi \in \wp_+(G)$, $\varphi : G \to R_+$ such that $\varphi \in L^p(G, R)$ and $\left\|\psi_{k(n)}(g)\right\| \le \varphi(g)$ for all $g \in G$.

Proof. The fundamentality condition states that for any $\varepsilon > 0$ there exists an integer number k_0 such that $\|\psi_k - \psi_m\|_p \le \varepsilon$ for all $k \ge k_0$ and $m \ge k_0$. So by induction on n, we define a strictly increasing sequence $\{k(n)\}$ of positive integer numbers $k(n)$ that are dependent on n such that $\left\|\psi_{k(n+1)} - \psi_{k(n)}\right\|_p \le 2^{-n}$. Thus, the mapping series $\sum_n \psi_{k(n+1)} - \psi_{k(n)}$ converges in the p-th mean and has a sum $\sum_n \psi_{k(n+1)} - \psi_{k(n)} = \phi \in L^p(G)$, which equals $\psi = \phi + \psi_{k(1)}$. We denote a finite or infinite function $\varphi(g) = \sum_n \left\|\psi_{k(n)}(g)\right\|$ since $\|\varphi\|_p = \sum_n \|\psi_k\|_p < \infty$ we obtain that function $\varphi : G \to R_+$ is μ-almost everywhere finite. The series $\sum_n \psi_{k(n+1)} - \psi_{k(n)}$ converges absolutely μ-almost everywhere and $\|\psi(g)\| \le \sum_n \|\psi_k(g)\| = \varphi(g)$ so $\psi \in L^p(G)$. If we denote $\zeta(g) = \sum_n \left\|\psi_{k(n+1)}(g) - \psi_{k(n)}(g)\right\|$, $g \in G$ then $\|\zeta(g)\|_p < \infty$, therefore, exists a lower semicontinuous function φ such that $\varphi \ge \zeta + \left\|\psi_{k(1)}\right\|$ and $\|\zeta\|_p < \infty$. The theorem is proven.

Corollary. Let Θ be an everywhere dense subset of $\mathscr{L}^p(G)$ then for each function $\psi \in \mathscr{L}^p(G)$ there exists a sequence $\{\phi_k\} \subset \Theta$ such that $\lim_{k\to\infty} \|\phi_k - \psi\|_p = 0$, and $\lim_{k\to\infty} \|\phi_k(g) - \psi(g)\| = 0$ for μ-almost all g. As an everywhere-dense subset of $\mathscr{L}^p(G)$, we can always take the space of real-valued continuous functions with compact support.

Since the L^p-space is metrizable there exists a fundamental sequence $\{\psi_k\} \subset \Theta$ so that

$\lim_{k\to\infty}\left\|\psi_k-\psi\right\|_p=0$ then the corollary follows from the previous theorem.

3. Induce measures

Let G be a locally compact group G with a positive Radon measure μ. Let Γ be a locally compact subgroup of the group G. Since the set Γ is the intersection of open and closed sets of G, the subset Γ is μ-measurable. For any function $\psi\in C_C(\Gamma)$, we denote the function $\tilde{\psi}$ defined on G given by

$$\tilde{\psi}(g)=\begin{cases}\psi(g), & g\in\Gamma,\\ 0, & g\in G\setminus\Gamma.\end{cases}$$

Definition 6. ***If Γ is a locally compact subgroup of a locally compact group G with a positive measure μ, then the measure μ_Γ on Γ induced by the positive measure μ on G is given by***

$$\int\psi d\mu_\Gamma=\int\tilde{\psi}d\mu$$

for all functions $\psi\in C_C(\Gamma)$, where

$$\tilde{\psi}(g)=\begin{cases}\psi(g), & g\in\Gamma,\\ 0, & g\in G\setminus\Gamma.\end{cases}$$

Lemma 1. ***Let K be a compact subset of G then the equality***

$$\mu_K(\tilde{K})=\mu(\tilde{K})$$

holds for any compact subset $\tilde{K}$ of K.

Proof. Let $\chi_{\tilde{K}}$ be the characteristic function of the set $\tilde{K}$ then $\chi_{\tilde{K}}$ is lower semicontinuous function and therefore $\chi_{\tilde{K}}$ is lower envelope of some of the decreasing functional collection $\{\psi_\alpha\}\subset C_C(K)$ so that

$$\mu_K(\tilde{K})=\inf_\alpha\int\psi_\alpha d\mu_K=$$
$$=\inf_\alpha\int\tilde{\psi}_\alpha d\mu_K=\mu(\tilde{K}),$$

where $\tilde{\psi}_\alpha(g)=\begin{cases}\psi_\alpha(g), & g\in K,\\ 0, & g\in G\setminus K\end{cases}$ for all α; since $\tilde{\psi}_\alpha$ is an upper semicontinuous function then the lower envelope of decreasing filer collection $\{\psi_\alpha\}$ equals $\chi_{\tilde{K}}$, and the lemma is proven.

The measure μ on the locally compact group G is called relatively invariant with a left m_ℓ or right m_r multiplier if $\mu(\varphi\psi)=m_\ell(\varphi)\mu(\psi)$ or $\mu(\psi\varphi)=m_r(\varphi)\mu(\psi)$ holds for all $\psi\in C_C(G)$. The multipliers m_ℓ and m_r are continuous homomorphisms $G\to R^+$, where R^+ is a group of positive real numbers with the operation of multiplication.

Let H be a normal subgroup of the locally compact separable group G. The quotient group G/H consists of all left cosets G/H. Let η be a left Haar measure on a normal subgroup H of G. For any function $\psi\in C_C(G)$ with $\operatorname{supp}(\psi)\subset K\subset G$, we define a function ψ_ℓ by

$$\psi_\ell(g)=\int\psi(gh)d\eta(h),$$

so that $\psi_\ell(gh)=\psi_\ell(g)$ for every $h\in H$ since η is left invariant measure. Let mapping $\rho:G\to G/H$ be an orbit projection then we define a continuous function $\psi_\ell=\psi_\eta\circ\rho$, the function ψ_η is continuous on G/H and with $\operatorname{supp}(\psi_\eta)\subset\rho(K)\subset G/H$, and therefore $\psi_\eta\in C_C(G/H)$. Now, we write

$$\int\psi(ghk^{-1})d\eta(h)=\Delta_H^r\int\psi(gh)d\eta(h),$$

Δ_H^r is the right modulus of H.

Definition 7. ***We introduce a morphism*** $\Lambda : C_C(G) \to C_C(G/H)$ ***given by*** $\Lambda(\psi) = \psi_\eta$.

Statement 1. ***The function*** $\Lambda : C_C(G) \to C_C(G/H)$ ***given*** $\Lambda(\psi) = \psi_\eta$ ***maps*** $\Lambda : C_C^+(G) \to C_C^+(G/H)$, ***where*** $C_C^+(G)$ ***and*** $C_C^+(G/H)$ ***are sets of positive continuous functions with compact support.***

Proof. Let $\varphi \in C_C^+(G/H)$ with $\operatorname{supp}(\varphi) = \tilde{K} \subset G/H$ and $\rho : G \to G/H$ be an orbit projection. There exists a compact set $K \in G$ such that $\rho(K) = \tilde{K}$. There exists a function $\phi \in C_C^+(G/H)$ such that $\phi(g) = 1, \quad g \in G$. We define function

$$\phi_\ell(g) = \int \phi(gh)\, d\eta(h)$$

that is positive lower bound on K and so it is positive on the saturation KH of compact K so the continuous function $(\phi \circ \rho)(\phi_\ell)^{-1}$ is nonnegative so that $\psi = \phi(\varphi \circ \rho)(\phi_\ell)^{-1} \in C_C^+(G)$ so that $\psi_\ell = \varphi \circ \rho$.

Definition 8. ***Let*** G ***be a locally compact separable group with a Radon measure*** μ ***and*** H ***be a normal subgroup of*** G, ***and let measure*** μ ***satisfies*** $\mu(\psi h) = \Delta_H^r(h)\mu(\psi)$ ***for all*** $\psi \in C_C(G)$ ***and*** $h \in H$ ***then we define a measure*** μ_η ***by*** $\mu_\eta(\psi_\eta) = \mu(\psi)$ ***for all*** $\psi_\eta \in C_C(G/H)$.

For all $\varphi \in C_C(G/H)$, we define a measure ϑ on G/H by $\vartheta(\varphi) = \mu(\psi)$ for $\psi \in C_C(G)$ such that $\psi_\eta = \varphi$.

We denote $\vartheta^\#(\psi) = \vartheta(\psi_\eta)$ for all $\psi \in C_C(G)$.

Statement 2. ***If*** $\psi_\eta = 0$ ***then*** $\mu(\psi) = 0$ ***for all*** $\psi \in C_C(G)$.

Proof. Assume $\psi, \varphi \in C_C(G)$, we have

$$\begin{aligned}
\mu(\psi\varphi_\ell) &= \int \psi(g)\, d\mu(g) \int \varphi(gh)\, d\eta(h) = \\
&= \int \psi(g)\varphi(gh)\, d\mu(g) \int d\eta(h) = \\
&= \int d\eta(h)\Delta_H^r(h^{-1}) \int \psi(gh^{-1})\varphi(g)\, d\mu(g) = \\
&= \int \varphi(g)\, d\mu(g) \int \psi(gh^{-1})\Delta_H^r(h^{-1})\, d\eta(h) = \\
&= \int \varphi(g)\, d\mu(g) \int \psi(gh)\, d\eta(h) = \mu(\varphi\psi_\ell),
\end{aligned}$$

so that $\mu(\psi(\varphi_\eta \circ \rho)) = \mu(\varphi(\psi_\eta \circ \rho))$ hence from $\psi_\eta = 0$ follows $\mu(\psi(\varphi_\eta \circ \rho)) = 0$ for all $\varphi \in C_C(G)$. Assume $\operatorname{supp}(\psi) = K \subset G$ and assume $\chi \in C_C(G/H)$ so that $\chi(h) = 1, \quad h \in \rho(K)$ then there exists $\varphi \in C_C(G)$ such that $\varphi_\eta = \chi$ on $\rho(K)$ therefore $\varphi_\eta \circ \rho = 1$ on K so that $\psi = \psi(\varphi_\eta \circ \rho)$ and $\mu(\psi) = \mu(\psi(\varphi_\eta \circ \rho)) = 0$.

We have the following lemma.

Lemma 2. ***Let*** ϑ ***be a measure on*** G/H ***then there exists natural measure*** μ ***given by***

$$\mu(\psi) = \vartheta(\psi_\beta)$$

for all $\psi \in C_C(G)$.

Remark. The measure $\mu_\eta = \mu/\eta$ is called the quotient of μ and η.

Lemma 3. ***Let*** G ***be a locally compact group with an invariant Haar measure*** μ ***and let*** ϑ ***be a normalized Haar measure on a normal compact subgroup*** H ***of*** G. ***Let*** $\rho : G \to G/H$ ***be an orbit projection then, the***

quotient measure $\mu_\eta = \mu / \eta$ equivalent to the induced measure $\rho(\mu)$.

Proof. The function $\rho(\mu)$ is a Radon measure on G/H given by

$$\rho(\mu)(\varphi) = \mu(\varphi \circ \rho)$$

for all $\varphi \in C_C(G/H)$. Since $m = \Delta_H^r$ for any $\varphi \in C_C(G/H)$ the support of a function $\varphi \circ \rho$ is a compact set in G so that there exist $\psi_\eta = \varphi$, and $\psi = \varphi \circ \rho \in C_C(G)$, and $\psi_\ell = \varphi \circ \rho$ so $\psi_\eta = \varphi$. Therefore for any $\varphi \in C_C(G/H)$ we have $\mu_\eta(\varphi) = \mu(\varphi \circ \rho)$.

Lemma 4. ***Let ϑ be a measure on the quotient group G/H such that $\vartheta(\psi_\eta) = \mu(\psi)$ for all $\psi \in C_C(G)$ then $\Delta_H^r = m$, the reverse is also true, assume $\Delta_H^r = m$ then $\vartheta(\psi_\eta) = \mu(\psi)$ for all $\psi \in C_C(G)$.***

Proof. Assume $\vartheta(\psi_\eta) = \mu(\psi)$, since $(\psi h)_\eta = \Delta_H^r(h)\psi_\eta$ for all $h \in H$, $\psi \in C_C(G)$ then $\mu(\psi h) = \Delta_H^r(h)\mu(\psi)$ for all $h \in H$, $\psi \in C_C(G)$ so $\Delta_H^r = m$.

Assume $\Delta_H^r = m$ then $\mu_\eta(\varphi) = \mu(\psi)$ for $\varphi \in C_C(G/H)$ thus $\mu(\psi) = \vartheta(\psi_\eta)$ for measure ϑ and for all $\psi \in C_C(G)$.

Theorem (Weil) 2. ***Let H be a normal subgroup of the locally compact group G and let μ and η be the measures on G and H, respectively. Let $m: G \to R_+$ be a continuous homomorphism from a group G to a group of positive real numbers with the operation of multiplication. There exists a measure ϑ on the quotient group G/H if and only if $\Delta_H^r(h) = m(h)\Delta_G^r(h)$ for all $h \in H$ and then every m-relatively invariant measure on G/H is a quotient measure $\mu_\eta = \mu / \eta$.***

Proof. For all $\psi \in C_C(G)$, we have $\vartheta^\#(\psi) = \vartheta(\psi_\eta)$. For a m-relatively invariant measure ϑ on G/H we have $\vartheta(g\psi_\eta) = m(g)\vartheta(\psi_\eta)$ therefore $\vartheta((g\psi)_\eta) = m(g)\vartheta(\psi_\eta)$ and $\vartheta^\#(g\psi) = m(g)\vartheta^\#(\psi)$ thus $\vartheta^\#$ is a relatively invariant measure on the group G.

Now, since for all $h \in H$ and $\psi \in C_C(G)$ we have $\mu(\psi) = \vartheta(\psi_\eta)$ and $\mu(\psi h) = \Delta_H^r(h)\mu(\psi)$ follows $m_r(h) = m(h)\Delta_G^r(h)$ so that $m = \Delta_H^r$ thus $\vartheta = \vartheta^\#{}_\eta = \mu_\eta$. The theorem is proven.

Theorem 3. ***Let H be a normal subgroup of the locally compact group G and let $m: G \to R_+$ be a continuous homomorphism from a group G to a group of positive real numbers with the operation of multiplication. Then, a m-relatively invariant measure ϑ on G/H is given by $\vartheta = \rho(m\mu)$ where $\rho: G \to G/H$ is a projection mapping.***

Proof. Since $m(h) = \Delta_G^r(h) = \Delta_G^r(h) = 1$ for all $h \in H$, there exists a m-relatively invariant measure $\vartheta = \mu_\eta = \rho(\mu) = \rho(m\mu)$.

References:

[1] B. Abadie, Takai duality for crossed products by Hilbert C ∗ -bimodules, J. Operator Theory 64 (2010), 19–34.
[2] V.I. Bogachev, Negligible Sets in Infinite-Dimensional Spaces. Analysis Mathematica 44 (2018), pp. 299–323. DOI: https://doi.org/10.1007/s10476-018-0503-7.
[3] Ch. H. Cashen and A. Martin, Quasi-isometries between groups with two-ended splittings. Math. Proc. Cambridge Philos. Soc., 162(2):249–291, 2017.
[4] D. Cohn, Measure theory, 2nd ed., Springer, 2013.

[5] K. H. Hofmann and F. G. Russo, The probability that x and y commute in a compact group, Math. Proc. Cambridge Philos. Soc. 153 (2012), no. 3, 557-571.
[6] G. Levitt, Generalized Baumslag-Solitar groups: rank and finite index subgroups. Ann. Inst. Fourier (Grenoble), 65(2):725–762, 2015.
[7] Martino, M. C. H. Tointon, M. Valiunas and E. Ventura, Probabilistic Nilpotence in infinite groups, to appear in Israel J. Math.
[8] M. Soleimani Malekan, A. Abdollahi and M. Ebrahimi, Compact groups with many elements of bounded order, J. Group Theory, 23 (2020) no. 6 991–998.
[9] J. Park and Y. Atchade, Markov chain monte Carlo algorithms with sequential proposals. Statistics and Computing, (2020), 30(5):1325–1345.
[10] D. Ravasini, Compactivorous Sets in Banach Spaces. Proceedings of the American Mathematical Society 150.5 (May 2022), pp. 2121–2129. DOI: https://doi.org/10.1090/proc/15851.
[11] N. Memic, Mahler coefficients of locally scaling transformations on Zp. Colloquium Mathematicum. 162 (2020), 53–76.
[12] F. Mukhamedov, O. Khakimov, A. Souissi, Supercyclic and hypercyclic generalized weighted backward shifts over a non-Archimedean c0(N) space, Mathematics, 9(22) (2021), 2986.
[13] Y. Jing, Chieu-Minh Tran, and R. Zhang, A nonabelian Brunn-Minkowski inequality, arXiv:2101.07782 (2021).
[14] M.I. Yaremenko Calderon-Zygmund Operators and Singular Integrals, Applied Mathematics & Information Sciences: Vol. 15: Iss. 1, Article 13, (2021).

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