PROOF
DOI: 10.37394/232020.2022.2.22

Preliminary Group Classification of nonlinear wave equation $u_{tt} + u_t = f(x, u_x)u_{xx} + g(x, u_x)$

TSHIDISO MASEBE
Tshwane University of Technology, Pretoria, 0001, SOUTH AFRICA

Abstract: The paper discusses the non-linear wave equations whose coefficients are dependent on first order spatial derivatives. We construct the principal Lie algebra, the equivalence Lie al- gebra, and the extensions by one of the princi- pal Lie algebra. We further construct the op- timal system of one-dimensional subalgebras for rst three extended five-dimensional Lie alge- bras. These are finally used to determine invari- ant solutions of some examples.

Keywords: Principal Lie Algebra, Equivalence Lie algebra, Invariant solution, One-dimensional optimal systems.

Received: June 19, 2021. Revised: August 9, 2022. Accepted: September 5, 2022. Published: October 10, 2022.

1. Introduction

THE Lie group analysis of differential equations is the area of mathematics pioneered by Sophus Lie in the 19th century (1849-1899). The first general solution of the problem of classification was given by Sophus Lie for an extensive class of partial differential equations [4]. Since then many researchers have done work on various families of differential equations. The results of their work have been captured in several outstanding literary works [1, 4].The preliminary group classification by Ibragimov, Torrisi and Valenti [4], gave us up to thirty three equivalence classes of submodels of the wave model of the form

$$u_{tt} = f(x, u_x)u_{xx} + g(x, u_x). \tag{1}$$

The present work examines a model which represents families of the nonlinear wave with dissipation, namely

$$u_{tt} + u_t = f(u_x)u_{xx} + g(u_x). \tag{2}$$

In this work we use the results of one-dimensional optimal systems

(i) of the equivalence Lie algebra to obtain X_5 and hence the classification of the family of equations (2) above ,

(ii) of the extended principal Lie algebra of equation (2) to calculate the invariant solutions of some examples.

The method followed in the construction of the one-dimensional optimal systems is found in the paper by Ibragimov, Torrisi and Valenti [2]. In this paper while constructing the principal Lie algebra, we also show how to determine the Lie point symmetries of (2). We proceed to construct the equivalence Lie algebra, and give the extensions by o ne of the principal algebra of equation (2). We also show the method of determining invariant solutions. The paper also illustrates the construction of one-dimensional optimal systems of extended principal Lie algebras L_5. We conclude by calculating invariant solutions of some one-dimensional subalgebras of each extended algebra L_5.

1.1. Principal Lie Algebra

The principal Lie algebra L_p of the non-linear wave equation with dissipation namely

$$u_{tt} + u_t = f(u_x)u_{xx} + g(u_x),$$

is determined as follows:

Let the generator of equation(2) be given by

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{3}$$

The second prolongation of (3) is given by

$$\widetilde{X}^2 = X + \zeta^t\frac{\partial}{\partial u_t} + \zeta^x\frac{\partial}{\partial u_x} + \zeta^{tt}\frac{\partial}{\partial u_{tt}} + \zeta^{xx}\frac{\partial}{\partial u_{xx}}, \tag{4}$$

where

$$\begin{aligned}\zeta^t &= D_t(\eta) - u_tD_t(\xi^1) - u_xD_t(\xi^2),\\ \zeta^x &= D_x(\eta) - u_tD_x(\xi^1) - u_xD_x(\xi^2),\\ \zeta^{tt} &= D_t(\zeta^t) - u_{tt}D_t(\xi^1) - u_{tx}D_t(\xi^2),\\ \zeta^{xx} &= D_x(\zeta^x) - u_{tx}D_x(\xi^1) - u_{xx}D_x(\xi^2).\end{aligned} \tag{5}$$

The operators D_t and D_x denote the total derivatives with respect to t and x respectively as follows:

$$\begin{aligned}D_t &= \tfrac{\partial}{\partial t} + u_t\tfrac{\partial}{\partial u} + u_{tx}\tfrac{\partial}{\partial u_x} + u_{tt}\tfrac{\partial}{\partial u_t} + ...\\ D_x &= \tfrac{\partial}{\partial x} + u_x\tfrac{\partial}{\partial u} + u_{tx}\tfrac{\partial}{\partial u_t} + u_{xx}\tfrac{\partial}{\partial u_x} + ...\end{aligned} \tag{6}$$

The determining equation of (2) is given by

$$\widetilde{X}^2\left(u_{tt}+u_t-f(u_x)u_{xx}-g(u_x)\right)|_{(2)} = (\zeta^{tt}+\zeta^t-f\zeta^{xx}-f^{u_x}\zeta^x u_{xx}-g\zeta^x)|_{(2)} = 0. \quad (7)$$

In cases of arbitrary f and g it follows that

$$\zeta^{xx}=\zeta^x=0, \text{ and } \zeta^{tt}+\zeta^t=0. \quad (8)$$

From the equation (8) we have that

$$\begin{array}{l}\zeta^{tt}+\zeta^t=\eta_{tt}+u_t\left(2\eta_{tu}-\xi^1_{tt}-2u_x\xi^2_{tu}\right)\\ +u_t^2\left(\eta_{uu}-2\xi^1_{tu}-u_x\xi^2_{uu}\right)-u_t^3\xi^1_{uu}\\ -u_{tx}\left(2\xi^1_t+2u_x\xi^2_u+u_t\xi^2_u\right)\\ +\left(-u_t-f(u_x)u_{xx}-g(u_x)\right)\\ \left(\eta_u-2\xi^1_t-3u_t\xi^1_u\right)+\eta_t+u_t\left(\eta_u-\xi^1_t\right)\\ -u_t^2\xi^1_u-u_x\xi^2_t-u_tu_x\xi^2_u=0.\end{array} \quad (9)$$

From equation (9) we obtain

$$\begin{array}{l}\xi^2_u=\xi^1_t=0.\\ \xi^1_u=\eta_u=0.\\ \xi^2_t=0.\\ \eta_{tt}+\eta_t=0 \quad \Rightarrow \quad \eta=c_1+c_2e^{-t}.\end{array} \quad (10)$$

Thus we have that

$$\xi^1=c, \qquad \xi^2=c, \qquad \eta=c_1+c_2e^{-t}.$$

Thus the principal Lie algebra L_p of the non-linear wave equation with dissipation namely

$$u_{tt}+u_t=f(u_x)u_{xx}+g(u_x),$$

is spanned by the following generators

$$X_1=\frac{\partial}{\partial x}, \quad X_2=\frac{\partial}{\partial t}, \quad X_3=\frac{\partial}{\partial u}, \quad X_4=e^{-t}\frac{\partial}{\partial u}. \quad (11)$$

1.2 Equivalence Lie Algebra and Extensions of the Principal Lie Algebra

The equivalence Lie Algebra, is the non-degerate changes in the variables, x,t and u which carries equation (2) into an equation of the same form. The family of non-linear waves $u_{tt}+u_t=f(u_x)u_{xx}+g(u_x)$, can be written as a system of differential equations

$$\begin{array}{l}u_{tt}+u_t=f^1u_{xx}+f^2\\ f^k_x=f^k_t=f^k_u=f^k_{u_t}=0\end{array} \quad (12)$$

k = 1, 2. The equivalence Lie algebra element for the system (4) is given by the generators

$$E=\xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial u}+\mu^k\frac{\partial}{\partial f^k}$$

where $\xi=\xi(x,t,u)$, $\tau=\tau(x,t,u)$, $\eta=\eta(x,t,u)$, $\mu^k=\mu^k(x,t,u,u_x,u_t,f^1,f^2)$.

We now introduce the following total derivatives

$$\begin{array}{l}\widetilde{D_\alpha}=\frac{\partial}{\partial\alpha}+f^k_\alpha\frac{\partial}{\partial f^k}+f^k_{\alpha t}\frac{\partial}{\partial f^k_t}+\\ f^k_{\alpha x}\frac{\partial}{\partial f^k_x}+f^k_{\alpha u}\frac{\partial}{\partial f^k_u}+f^k_{\alpha u_t}\frac{\partial}{\partial f^k_{u_t}}+...\end{array}$$

for $\alpha\in\{x,t,u,u_t\}$.

The extension of the equivalence algebra element E, takes the form

$$\begin{array}{l}\widetilde{E}=E+\zeta^t\frac{\partial}{\partial u_t}+\zeta^x\frac{\partial}{\partial u_x}+\zeta^{xx}\frac{\partial}{\partial u_{xx}}\\ +\varpi^k_t\frac{\partial}{\partial f^k_t}+\varpi^k_x\frac{\partial}{\partial f^k_x}+\varpi^k_u\frac{\partial}{\partial f^k_u}+\varpi^k_{u_t}\frac{\partial}{\partial f^k_{u_t}},\end{array}$$

where

$$\begin{array}{l}\zeta^i=D_i(\eta)-u_tD_i(\tau)-u_xD_i(\xi)\\ \zeta^{ij}=D_i(\zeta^i)-u_{jt}D_i(\tau)-u_{jx}D_i(\xi)\end{array}$$

for $i,j\in\{x,t\}$ and

$$\begin{array}{l}\varpi^k_\alpha=\widetilde{D_\alpha}(\mu^k)-f^k_t\widetilde{D_\alpha}(\tau)-f^k_x\widetilde{D_\alpha}(\xi)\\ -f^k_u\widetilde{D_\alpha}(\eta)-f^k_{u_t}\widetilde{D_\alpha}(\zeta^t)-f^k_{u_x}\widetilde{D_\alpha}(\zeta^x)\end{array}$$

where $\alpha\in\{x,t,u,u_t\}$, k = 1, 2.

The invariance condition for the system of equations (12)
is given by

$$\widetilde{E}(u_{tt}+u_t-f^1u_{xx}-f^2)|_{(12)}=0 \quad (13)$$

$$\widetilde{E}(f^k_\alpha)=0 \text{ for } \alpha\in\{x,t,u,u_t\}. \quad (14)$$

We thus obtain

$$\zeta^{tt}+\zeta^t-\mu^1u_{xx}-f^{'}\zeta^{xx}-\mu^2=0$$

and

$$\varpi^k_\alpha=0 \text{ for } \alpha\in\{x,t,u,u_t\}.$$

From the equations (13) we have

$$(\mu^k)_\alpha=(\zeta^x)_\alpha=0, \alpha\in\{x,t,u,u_t\}$$

and $k=1,2$, which implies that the μ^k are independent of x,t,u,u_t and hence

$$\mu^k=\mu^k(u_x,f^1,f^2), \qquad k=1,2.$$

Furthermore $(\zeta^x)_\alpha=0$ yields

$$\begin{array}{l}\xi=a_1x+a_2u+p(t)\\ \tau=\tau(t)\\ \eta=b_1u+b_2x+q(t)\end{array} \quad (15)$$

where $a_1,a_2;b_1,b_2$ are constants. The equations (15), together with the invariance condition yield

$$\begin{array}{l}\xi=a_1x+a_2\\ \tau=a_3\\ \eta=a_4u+a_5t+a_6x+a_7\\ \mu^1=2a_1f^1\\ \mu^2=a_5+a_4f^2.\end{array} \quad (16)$$

For the model $u_{tt}+u_t=f(u_x)u_{xx}+g(u_x)$, we have

$$\begin{array}{l}\mu^1=2a_1f\\ \mu^2=a_5+a_4g.\end{array}$$

Therefore we obtain a 7-dimensional equivalence algebra for the non-linear wave equation (2), which is spanned by the following operators

$$E_1=\frac{\partial}{\partial x},\quad E_2=\frac{\partial}{\partial t},\quad E_3=\frac{\partial}{\partial u},\quad E_4=x\frac{\partial}{\partial u}$$
$$E_5=u\frac{\partial}{\partial u}+g\frac{\partial}{\partial g},\quad E_6=t\frac{\partial}{\partial u}+\frac{\partial}{\partial g},\quad E_7=x\frac{\partial}{\partial x}+2f\frac{\partial}{\partial f}. \tag{17}$$

The classification of the equation (2) is obtained by extending the principal Lie algebra $X_1=\frac{\partial}{\partial x}$, $X_2=\frac{\partial}{\partial t}$, $X_3=\frac{\partial}{\partial u}$, $X_4=e^{-t}\frac{\partial}{\partial u}$ by X_5 as follows:

1.3 One-dimensional Optimal System

In order to determine X_5 and hence the classification of equation (2) we will give details of the determination of the one-dimensional optimal systems L_4 below. Since f and g depend on u_x, we prolong the equivalence operators E_i (17), to the following operators

$$\widetilde{E}_i=E_i+\zeta^x\frac{\partial}{\partial u_x},\ \text{for } i=1,2,......,7.$$

Therefore we have

$$\widetilde{E}_i=E_i,\ \text{for } i=1,2,3$$

$$\widetilde{E}_4=x\frac{\partial}{\partial u}+\frac{\partial}{\partial u_x},\quad \widetilde{E}_5=u\frac{\partial}{\partial u}+g\frac{\partial}{\partial g}+u_x\frac{\partial}{\partial u_x} \tag{18}$$

$$\widetilde{E}_6=E_6,\qquad E_7=x\frac{\partial}{\partial x}+2f\frac{\partial}{\partial f}-u_x\frac{\partial}{\partial u_x},$$

We form new operators Z_i by projecting each $\widetilde{E}_i$ (18), onto the (u_x,f,g)-subspace of the $(x,t,u,u_t,u_x,f,g)-$space. We have

$$pr(\widetilde{E}_i)=0,\ \text{for } i=1,2,3$$

$$Z_i=pr(\widetilde{E}_{i+3}),\ \text{for } i=1,2,3,4.$$

$$Z_1=pr(\widetilde{E}_5)=\frac{\partial}{\partial u_x}$$

$$Z_2=g\frac{\partial}{\partial g}+u_x\frac{\partial}{\partial u_x},Z_3=\frac{\partial}{\partial g},$$

$$Z_4=2f\frac{\partial}{\partial f}-u_x\frac{\partial}{\partial u_x},$$

We now consider the algebra L_4, which is spanned by Z_1,Z_2,Z_3,Z_4. We wish to determine the optimal system of one-dimensional subalgebras of the algebra L_4. The non-zero structure constants of L_4 are as follows:

$$[Z_1,Z_2]=Z_1\quad ,[Z_1,Z_4]=-Z_1,\quad [Z_2,Z_3]=-Z_3,$$

The generators of the adjoint algebra L_4^A are given by

$$A_1=Z_1\frac{\partial}{\partial Z_2}-Z_1\frac{\partial}{\partial Z_4}$$

$$A_2=-Z_1\frac{\partial}{\partial Z_1}-Z_3\frac{\partial}{\partial Z_3}$$
$$A_3=Z_3\frac{\partial}{\partial Z_3} \tag{19}$$
$$A_4=Z_1\frac{\partial}{\partial Z_1}$$

In order to obtain the elements of the adjoint group G^A or the group of inner automorphisms of the algebra L_4, we integrate the equations (19) to obtain a four parameter Lie group:

$$A_1:\overline{Z}_2=Z_2+a_1Z_1,\qquad \overline{Z}_4=Z_4-a_1Z_1$$
$$A_2:\overline{Z}_1=a_2^{-1}Z_1,\qquad \overline{Z}_3=a_2^{-1}Z_3$$
$$A_3:\overline{Z}_2=Z_2+a_3Z_3,$$
$$A_4:\overline{Z}_1=a_4Z_1$$

A matrix representation of an arbitrary element of the adjoint group G^A is of the form

$$M=\begin{bmatrix} a_2^{-1}a_4 & a_1 & 0 & -a_1\\ 0 & 1 & 0 & 0\\ 0 & a_2^{-1}a_3 & a_2^{-1} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

If we let $Z\in L_4$ be given by

$$Z=e^1Z_1+e^2Z_2+e^3Z_3+e^4Z_4$$
$$Z\equiv e=(e^1,e^2,e^3,e^4),$$

then $\overline{e}=Me$ defines an equivalence relation in L_4 and hence subdivides this algebra into equivalence classes. The components of Z map as follows under M :

$$\begin{aligned}\overline{e}^1&=a_2^{-1}a_4e^1+a_1(e^2-e^4)\\ \overline{e}^2&=e^2\\ \overline{e}^3&=a_2^{-1}a_3e^2+a_2^{-1}e^3\\ \overline{e}^4&=e^4\end{aligned}$$

Therefore the optimal system of one-dimensional subspaces of L_4, obtained through the adjoint group G^A, are as follows:

Z	**Generator**	**Restrictions**
$Z^{(1)}$	αZ_2+Z_4	$\alpha\neq 1$
$Z^{(2)}$	$\alpha Z_2+\beta Z_3+Z_4$	$\alpha\neq\beta$
$Z^{(3)}$	$Z_1+Z_2+Z_4$	
$Z^{(4)}$	$Z_1+Z_2+\alpha Z_3+Z_4$	
$Z^{(5)}$	Z_3	
$Z^{(6)}$	Z_3+Z_4	
$Z^{(7)}$	Z_1+Z_3	

Consider

$$Z^{(1)}=\alpha Z_2+Z_4,$$

with $\alpha\neq 1$.

$$Z^{(1)}=\alpha(g\frac{\partial}{\partial g}+u_x\frac{\partial}{\partial u_x})+2f\frac{\partial}{\partial f}-u_x\frac{\partial}{\partial u_x}$$

$$= \alpha g \frac{\partial}{\partial g} + 2f \frac{\partial}{\partial f} + (\alpha - 1)u_x \frac{\partial}{\partial u_x}.$$

From the characteristic equation

$$\frac{dg}{\alpha g} = \frac{df}{2f} = \frac{du_x}{(\alpha - 1)u_x},$$

we obtain

$$f = u_x^{\frac{2}{\alpha-1}} \qquad \text{and} \qquad g = u_x^{\frac{\alpha}{\alpha-1}}.$$

To obtain the extending vector X_5, we let

$$\widetilde{Z} = \alpha E_5 + E_7$$

$$= \alpha(u \frac{\partial}{\partial u} + g \frac{\partial}{\partial g}) + x \frac{\partial}{\partial x} + 2f \frac{\partial}{\partial f}.$$

Let X_5 be the projection of $\widetilde{Z}$ onto the $(x, t, u)-$ space, i.e

$$X_5 = x \frac{\partial}{\partial x} + \alpha u \frac{\partial}{\partial u}.$$

For the vectors $Z^{(i)}$, $i = 2, 3, \cdots, 7$, we proceed in a similar manner in order to determine the functions f, g and the extension vector X_5. The classification for equation (2) is given in the following table:

$Z^{(i)}$	$f(u_x)$	$g(u_x)$	X_5	$Restriction$
$Z^{(1)}$	$u_x^{\frac{2}{\alpha-1}}$	$u_x^{\frac{2}{\alpha-1}}$	$x\frac{\partial}{\partial x} + \alpha u \frac{\partial}{\partial u}$	$\alpha \neq 1$
$Z^{(2)}$	$u_x^{\frac{2}{\alpha-1}}$	$\alpha^{-1} u_x^{\frac{2}{\alpha-1} - \beta}$	$x\frac{\partial}{\partial x} + (\alpha u + \beta t)\frac{\partial}{\partial u}$	$\alpha \neq \beta$
$Z^{(3)}$	e^{2u_x}	C	$x\frac{\partial}{\partial x} + (u + x)\frac{\partial}{\partial u}$	
$Z^{(4)}$	e^{2u_x}	αu_x	$x\frac{\partial}{\partial x} + (u + x + \alpha t)\frac{\partial}{\partial u}$	
$Z^{(5)}$				
$Z^{(6)}$	u_x^{-2}	$-\ln x$	$x\frac{\partial}{\partial x} + ut\frac{\partial}{\partial u}$	$\alpha \neq 1$
$Z^{(7)}$	C	u_x	$(t + x)\frac{\partial}{\partial u}$	

(20)

In what follows we will give the classification for equation (2) for the listed generators X_5.

1. If $X_5 = x\frac{\partial}{\partial x} + (x + u)\frac{\partial}{\partial u}$ then $f = e^{2u_x}$, and $g = c$
2. If $X_5 = x\frac{\partial}{\partial x} + (x + u + \alpha t)\frac{\partial}{\partial u}$ then $f = e^{2u_x}$, and $g = \alpha u_x$
3. If $X_5 = (x + t)\frac{\partial}{\partial u}$ then $f = c$, and $g = u_x$
4. If $X_5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial u}$ then $f = u_x^{-2}$, and $g = -\ln u_x$
5. If $X_5 = x\frac{\partial}{\partial x} + \alpha u\frac{\partial}{\partial u}$ then $f = u_x^{\frac{2}{\alpha-1}}$, and $g = u_x^{\frac{2}{\alpha-1}}$ for $\alpha \neq 1$
6. If $X_5 = x\frac{\partial}{\partial x} + (\alpha u + \beta t)\frac{\partial}{\partial u}$ then $f = u_x^{\frac{2}{\alpha-1}}$, and $g = \alpha^{-1}(u_x^{\frac{2}{\alpha-1}} - \beta)$ for $\alpha \neq \beta$

Each extension will give us a five-dimensional Lie algebra L_5. From the above we will concentrate on the first four whose equations are given by the following

$$u_{tt} + u_t = e^{2u_x} u_{xx} + c. \tag{21}$$

$$u_{tt} + u_t = e^{2u_x} u_{xx} + \alpha u_x \tag{22}$$

$$u_{tt} + u_t = c u_{xx} + u_x. \tag{23}$$

$$u_{tt} + u_t = u_x^{-2} u_{xx} + \ln u_x. \tag{24}$$

From the latter we have five-dimensional Lie algebras for each of the equations (21) to (24). We will only construct optimal systems of one-dimensional Lie subalgebras for the first three equations. We will then calculate the invariant solutions using some of these one-dimensional subalgebras.

1.4 Invariant Solutions

Consider the equation

$$u_{tt} + u_t = e^{2u_x} u_{xx} + c, \tag{25}$$

whose set of generators is given by $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial u}$, $X_4 = e^{-t}\frac{\partial}{\partial u}$, $X_5 = x\frac{\partial}{\partial x} + (u + x)\frac{\partial}{\partial u}$.

We will use the one dimensional subalgebra $X = X_1 + (1 + \rho) X_3$ i.e.

$$X = \frac{\partial}{\partial x} + (1 + \rho)\frac{\partial}{\partial u}. \tag{26}$$

The characteristic equation of the above generator (26) is given by

$$\frac{dt}{0} = \frac{du}{k} = \frac{dx}{1} \qquad \text{where } k = 1 + \rho. \tag{27}$$

From equation (27) the invariants are given by

$$I_1 = u - kx \quad ; \quad I_2 = t. \tag{28}$$

If we define $I_1 = \phi(I_2)$ for some function ϕ, then

$$u(t, x) = kx + \phi(t). \tag{29}$$

The substitution of (29) into equation (25) asserts that

$$\begin{aligned} u_t &= \phi'(t) \\ u_{tt} &= \phi''(t) \\ u_x &= k \\ u_{xx} &= 0 \end{aligned}$$

hence

$$u_{tt} + u_t - e^{2u_x} u_{xx} - c = \phi''(t) + \phi'(t) - c = 0. \tag{30}$$

The equation (30) simplifies to

$$\phi''(t) + \phi'(t) = c, \tag{31}$$

which is a second order ODE whose solution is given by

$$\phi(t) = c_1 + c_2 e^{-t} + ct - c. \tag{32}$$

Thus the invariant solution of (25) is given by

$$u(t, x) = kx + c_1 + c_2 e^{-t} + ct - c, \tag{33}$$

where $k = 1 + \rho$.

Consider the equation

$$u_{tt} + u_t = e^{2u_x} u_{xx} + \alpha u_x \tag{34}$$

which has the following set of generators $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial u}$, $X_4 = e^{-t}\frac{\partial}{\partial u}$, $X_5 = x\frac{\partial}{\partial x} + (u + x + \alpha t)\frac{\partial}{\partial u}$.

We will use the one dimensional subalgebra $X = X_1 + X_4$ i.e.

$$X = \frac{\partial}{\partial x} + e^{-t}\frac{\partial}{\partial u}. \tag{35}$$

The characteristic equation of the above generator (35) is given by

$$\frac{dt}{0} = \frac{du}{e^{-t}} = \frac{dx}{1} \tag{36}$$

From equation (36) the invariants are given by

$$I_1 = u - xe^{-t} \qquad ; \qquad I_2 = t. \tag{37}$$

If we define $I_1 = \phi(I_2)$ for some function ϕ, then

$$u(t, x) = xe^{-t} + \phi(t). \tag{38}$$

The substitution of (38) into equation (34) asserts that

$$\begin{array}{rcl} u_t &=& -xe^{-t} + \phi^{'}(t) \\ u_{tt} &=& xe^{-t} + \phi^{''}(t) \\ u_x &=& e^{-t} \\ u_{xx} &=& 0, \end{array}$$

hence

$$u_{tt} + u_t - e^{2u_x} u_{xx} - \alpha u_x = \phi^{''}(t) + \phi^{'}(t) - \alpha e^{-t} = 0. \tag{39}$$

The equation (39) simplifies to

$$\phi^{''}(t) + \phi^{'}(t) = \alpha e^{-t}, \tag{40}$$

which is a non-linear second order ODE whose solution is given by

$$\phi(t) = c_1 + c_2 e^{-t} + \alpha e^{-t} - \alpha t e^{-t}.$$

The invariant solution of $u_{tt} + u_t = e^{2u_x} u_{xx} + \alpha u_x$ is given by

$$u(t, x) = xe^{-t} + c_1 + c_2 e^{-t} + \alpha e^{-t} - \alpha t e^{-t}. \tag{41}$$

Consider the equation

$$u_{tt} + u_t = c u_{xx} + u_x \tag{42}$$

whose set of generators is given by $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial u}$, $X_4 = e^{-t}\frac{\partial}{\partial u}$, $X_5 = (x + t)\frac{\partial}{\partial u}$.

We will use the one dimensional subalgebras $X = \alpha X_1 + X_5$ and $X = \beta X_2 + X_5$ i.e. $X = \alpha\frac{\partial}{\partial x} + (x + t)\frac{\partial}{\partial u}$, and $X = \beta\frac{\partial}{\partial t} + (x + t)\frac{\partial}{\partial u}$ respectively to calculate the invariant solutions of (42).

Consider the one dimensional subalgebra

$$X = \alpha\frac{\partial}{\partial x} + (x + t)\frac{\partial}{\partial u}. \tag{43}$$

The characteristic equation of () is given by

$$\frac{dx}{\alpha} = \frac{du}{x + t} = \frac{dt}{0}. \tag{44}$$

From equation () the invariants are given by $I_1 = \alpha u - \frac{1}{2}(x + t)^2$, $I_2 = t.$

If we let I_1 be a function of I_2,

$$u(t, x) = \frac{1}{\alpha}\left\{\frac{(x + t)^2}{2} + \phi(t)\right\} \text{ where } \phi(t) = I_1 \text{ i.e } I_1 = \phi(I_2). \tag{45}$$

The substitution of (45) into (42) asserts that

$$\begin{array}{rcl} u_t &=& \frac{1}{\alpha}\left\{(x + t) - \phi^{'}(t)\right\} \\ u_{tt} &=& \frac{1}{\alpha}(1 - \phi^{''}(t)) \\ u_x &=& \frac{1}{\alpha}(x + t) \\ u_{xx} &=& \frac{1}{\alpha}. \end{array} \tag{46}$$

Hence $u_{tt} + u_t - c u_{xx} - u_x = \frac{1}{\alpha}\left\{1 - \phi^{''}(t) + (x + t) - (x + t) - c - \phi^{'}(t)\right\} = 0$, simplifies to

$$\phi^{''}(t) + \phi^{'}(t) = 1 - c. \tag{47}$$

Solving the equation (47) we obtain that

$$\phi(t) = c_1 - c_2 e^{-t} + (1 - t)(1 - c). \tag{48}$$

Therefore the invariant solution of (42) is given by

$$u(t, x) = \frac{1}{\alpha}\left\{\frac{(x + t)^2}{2} + c_1 - c_2 e^{-t} + (1 - t)(1 - c)\right\}. \tag{49}$$

Acknowledgment

The author would like to acknowledge the assistance of fellow colleagues in going through the manuscript and for their invaluable inputs.

References

[1] Blumen,G.W. Kumei,S. 1989. Symmetries and Differential Equations. New York, Springer-Verlag.

[2] Blumen,G.W. Anco,S.C. 2002. Symmetries and Integration methods for Differential Equations. New York. Springer-Verlag.

[3] Hydon, P.E 2000. Symmetry methods for Differential Equations. New York. Cambridge University Press .

[4] Ibragimov,N.H. & Torrisi,M. and A. Valenti,.(1991). Preliminary Group Classification of the equations $u_{tt} = f(x, u_x) u_{xx} + g(x, u_x)$. J. Math. Phys.32, 2988-2995. June

[5] Ibragimov, N.H.1999. Elementary Lie Group Analysis and Ordinary Differential Equations. London. J. Wiley & Sons Ltd.