# Li-Yorke Chaotic Eigen Set of Direct Sum of Linear Operators

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Abstract: The Li-Yorke chaotic eigen set of an operator consisting of all λ's such that T- λI is Li-Yorke chaotic. In this paper, the Li-Yorke chaotic eigen set of the direct sum of linear operators is found to be the union of Li-Yorke chaotic sets of the corresponding operators. Also we discuss about the Li-Yorke chaotic eigen set of compact operators, normal operators and self adjoint operators.

Keywords: Direct sum, Li-Yorke chaos, Li-Yorke chaotic eigen set.

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## 1. Introduction

The notion of Li-Yorke chaos in a topological dynamical system was first used by T. Y. Li and J. A. Yorke in their paper [3]. There are several other types of chaos available in the literature. Devaney chaos, Distributional chaos, Bruckner-Ceder chaos, Wiggins chaos etc. are some of them (See [19, 5, 16, 10]). Relationship between these chaos are analysed by several researchers. See for example; Huang and Ye [11] has shown that Devaney's chaos implies Li-Yorke chaos. Research work has been carried out in the areas of Li-Yorke chaos and Devaney chaos in linear dynamics also. The introduction of Irregular vectors make it possible to bring Li-Yorke chaos in linear dynamics setting whereas hypercyclic vectors and denseness of periodic points brings chaos in the sense of Devaney.

The Li-Yorke chaotic eigen set of a bounded linear operator has recently been intoduced in [4] and found that the Li-Yorke chaotic eigen set of a positive integer multiple of the backward shift operator on $\ell^2(\mathbb{N})$ is a disk in the complex plane $\mathbb{C}$. In this paper, we will be concerned with the Li-Yorke chaotic eigen set of the direct sum of linear operators, compact, normal and self adjoint operators.

Let $X$ be an infinite-dimensional separable complex Banach space, $H$ be a separable complex Hilbert space. $B(X)$ and $B(H)$ denote the space of all bounded linear operators on $X$ and $H$ respectively.

## 2. Preliminaries

*Lemma II. 1 ([1]):* Let $K$ be a compact subset of $\mathbb{C}$ and let $C$ be a connected component of $K$. Assume that $C$ is contained in some open set $\Omega \subset \mathbb{C}$. Then one can find a clopen (i.e, closed and open) subset $\sigma \subset K$ such that $C \subset \sigma \subset K$.

*Theorem II. 2 ([1]) (RIESZ DECOMPOSITION THEOREM):* Let $T \in B(X)$, and assume that the spectrum of $T$ can be decomposed as $\sigma(T) = \sigma_1 \cup \sigma_2 \cdots \cup \sigma_N$, where the sets $\sigma_i$ are closed and pairwise disjoint. Then one can write $X =$

$X_1 \oplus X_2 \cdots \oplus X_N$, where each $X_i$ is a closed $T$-invariant subspace and $\sigma(T_{|X_i}) = \sigma_i$ for each $i \in \{1, 2, \cdots, N\}$

*Definition II.3 ([19]) (Devaney Chaos)* Let $f : X \longrightarrow X$ be a continuous map acting on some metric space $(X, d)$. The map $f$ is said to be Devaney chaotic if

(1) $f$ is topologically transitive : for any pair $U, V$ of non-empty open subsets of $X$, there exists some $n \geq 0$ such that $f^n(U) \cap V \neq \varnothing$

(2) $f$ has a dense set of periodic points ($x \in X$ is a periodic point of $f$ if $f^k(x) = x$ for some $k \geq 1$);

(3) $f$ has a sensitive dependence on initial conditions : there exists $\delta > 0$ such that, for any $x \in X$ and every neighbourhood $U$ of $x$, one can find $y \in U$ and an integer $n \geq 0$ such that $d(f^n(x), f^n(y)) \geq \delta$.

*Definition II. 4 ([7]) (Li - Yorke chaotic pair)* Let $f : X \longrightarrow X$ be a continuous map acting on some metric space $(X, d)$. The set $\{x, y\} \subset X$ is said to be a Li-Yorke chaotic pair, if $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$ and $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

*Definition II. 5 ([7]) (scrambled set)* A subset $\Gamma \subseteq X$ is called a scrambled set if each pair of two distinct points in $\Gamma$ is a Li - Yorke chaotic pair.

*Definition II. 6 ([7]) (Li - Yorke chaos )* Let $f : X \longrightarrow X$ be a continuous map acting on some metric space $(X, d)$. The map $f$ is said to be Li-Yorke chaotic, if there exists an uncountable scrambled set.

*Theorem II.7 ([12])* $f$ Devaney chaotic $\Rightarrow f$ Li-Yorke chaotic

*Definition II. 8 ([9]) (Irregular vector)* Let $T \in B(X)$. An irregular vector for an operator $T$ is an $x \in X$ such that $\limsup_n \parallel T^n x \parallel = \infty$ and $\liminf_n \parallel T^n x \parallel = 0$.

*Theorem II. 9 ([5])* Let $T : X \longrightarrow X$ be an operator. The following assertions are equivalent:

(i) $T$ is Li-Yorke chaotic.

(ii) $T$ admits a Li-Yorke pair.

(iii) $T$ admits an irregular vector.

*Corollary II. 10([5])* Let $T : X \longrightarrow X$ be a Li-Yorke chaotic operator. The following assertions hold

(i) $\sigma(T) \cap \partial\mathbb{D} \neq \emptyset$

(ii) $T^n$ is Li-Yorke chaotic for every $n \in \mathbb{N}$

(iii) $T$ is not compact

(iv) $T$ is not normal

*Definition II. 11 ([4]) (Li-Yorke Chaotic Eigen Set)* Let $T \in B(X)$. The Li-Yorke chaotic eigen set of $T$, denoted by $LY(T)$, is defined by $LY(T) = \Big\{\lambda \in \mathbb{C} \;\Big|\; T - \lambda I \text{ is Li-Yorke chaotic}\Big\}$.

*Theorem 11. 12 ([4])*: The Li - Yorke chaotic eigen set of the positive integer multiple of backward shift operator on $\ell^2(\mathbb{N})$:

$$T(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$$

is $LY(nT) = (n + 1)\mathbb{D}$ for any $n > 1$, $n \in \mathbb{N}$

# 3. Direct Sum of Linear Operators

*Definition III. 1 ([2])* Let $X_n, n \geq 1$ be a countable collection of separable Banach spaces. The direct sum of the spaces $X_n$ is defined as $\oplus_{n=1}^{\infty} X_n = \{(x_n)_{n\geq 1} : x_n \in X_n\}$

*Definition III. 2 ([2])* Let $T_n, n \geq 1$ be operators on separable Banach spaces $X_n, n \geq 1$. Then the direct sum of the operators $T_n$, defined by $\left(\oplus_{n=1}^{\infty} T_n\right)(x_n)_n = (T_n x_n)_n$, is an operator on $\oplus_{n=1}^{\infty} X_n$.

*Remark III. 3*

$(T_1 \oplus T_2)(x_1, x_2) = (T_1(x_1), T_2(x_2))$

$(T_1 \oplus T_2)^2 (x_1, x_2) = \left(T_1^2(x_1), T_2^2(x_2)\right)$

In general, $(T_1 \oplus T_2)^n (x_1, x_2) = (T_1^n(x_1), T_2^n(x_2))$

# 4. Main Theorem

In this section, we prove that the Li-Yorke chaotic eigen set of $T_1 \oplus T_2$ is exactly same as the Li-Yorke chaotic eigen sets of $T_1$ and $T_2$ taken together.

*Theorem IV. 1 : If* $T_1, T_2 \in B(X)$, *then* $LY(T_1 \oplus T_2) = LY(T_1) \bigcup LY(T_2)$

*Proof.* By definition, we have

$LY(T_1 \oplus T_2) = \Big\{\lambda \in \mathbb{C} \;\Big|\; T_1 \oplus T_2 - \lambda I \text{ is Li-Yorke chaotic}\Big\}$

Assume that $T_1 \oplus T_2 - \lambda I$ is Li-Yorke chaotic and $(x, y)$ is an irregular vector for $T_1 \oplus T_2 - \lambda I$. Then there exist a sequence

$a_n$ such that $(T_1 \oplus T_2 - \lambda I)^{a_n}(x, y) \longrightarrow 0$ (see properties 2 in [9]). This implies that $(T_1 - \lambda I)^{a_n}(x) \longrightarrow 0$ and $(T_2 - \lambda I)^{a_n}(y) \longrightarrow 0$. On the otherhand, there is a sequence $b_n$ such that $\left\|(T_1 \oplus T_2 - \lambda I)^{b_n}(x, y)\right\| \longrightarrow \infty$ which means that

$$\begin{aligned} &\left\|(T_1 - \lambda I)^{b_n}(x) \oplus (T_2 - \lambda I)^{b_n}(y)\right\|^2 \longrightarrow \infty \\ \Leftrightarrow &\left\|(T_1 - \lambda I)^{b_n}(x)\right\|^2 + \left\|(T_2 - \lambda I)^{b_n}(y)\right\|^2 \longrightarrow \infty \end{aligned}$$

(see Lemma 2. 1 in [9]) which implies that atleast one of them has a subsequence converging to $\infty$. Thus we see that there exist sequences $a_n$ and $b_n$ such that either

$$\lim \|(T_1 - \lambda I)^{a_n}(x)\| = 0 \text{ and } \lim \left\|(T_1 - \lambda I)^{b_n}(x)\right\| = \infty \quad (1)$$

or

$$\lim \|(T_2 - \lambda I)^{a_n}(y)\| = 0 \text{ and } \lim \left\|(T_2 - \lambda I)^{b_n}(y)\right\| = \infty \quad (2)$$

Equations (1) and (2) says that $x$ is an irregular vector for $T_1 - \lambda I$ and $y$ is an irregular vector for $T_2 - \lambda I$. Thus we have shown that $(x, y)$ is an irregular vector for $T_1 \oplus T_2 - \lambda I$ if and only if $x$ is an irregular vector for $T_1 - \lambda I$ or $y$ is an irregular vector for $T_2 - \lambda I$. Therefore, those $\lambda$'s which makes $T_1 \oplus T_2 - \lambda I$ as Li-Yorke chaotic will makes either $T_1 - \lambda I$ or $T_2 - \lambda I$ as Li-Yorke chaotic and hence $LY(T_1 \oplus T_2) = LY(T_1) \bigcup LY(T_2)$.

# 5. Li-yorke Chaotic Eigen Set of Compact, Normal and Self Adjoint Operators

Our goal in this section is to investigate the Li-Yorke chaotic eigen set of self adjoint operators, which plays a vital role, especially in the field of quantum mechanics. Also we discuss the Li-Yorke Chaotic Eigen Set of compact and normal operators.

*Lemma V. 1 ([5]):* Let $T \in B(H)$ be a compact operator. Then $T$ is not Li-Yorke chaotic.

*Lemma V. 2 ([5]):* Let $T \in B(H)$. If $T$ is Li-Yorke chaotic, then $\sigma(T) \cap \mathbb{T} \neq \emptyset$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$.

*Theorem V. 3 :* Let $T$ be a compact operator. Then $LY(T) \subseteq \mathbb{T}$.

*Proof.* Let $\sigma(T)$ be the spectrum of $T$. Since $T$ is compact, then $\sigma(T)$ is countable and $\sigma(T) \supset \{0\}$. We have any countable subset of $\mathbb{C}$ is totally disconnected, so that $\{0\}$ is a connected component of $\sigma(T)$. If we let $B_1 = \{0\}$, then $B_1 \cap \mathbb{T} = \emptyset$ so that $B_1 \subset \mathbb{D}$ or $B_1 \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. By lemma II. 1, we can find a clopen set $\sigma_1 \subset \sigma(T)$ such that $B_1 = \{0\} \subset \sigma_1 \subset \mathbb{D}$ or $B_1 = \{0\} \subset \sigma_1 \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Thus we have $\sigma(T) = \sigma_1 \cup \sigma_2$ where $\sigma_1 = \sigma(T_1)$ and $\sigma_2 = \sigma(T) \setminus \sigma_1$. Now, for any $\lambda \notin \mathbb{T}$, by the application of Riesz's decomposition theorem, we have $T - \lambda I = T_1 \oplus T_2 : X_1 \oplus X_2 \longrightarrow X_1 \oplus X_2$, where $X = X_1 \oplus X_2$, $\sigma(T_1) \cap \mathbb{T} = \emptyset$ and $X_2$ is finite dimensional. By applying lemma V. 2, $T_1$ is not Li-Yorke chaotic. Since $T_2$ is a finite - dimensional operator, then it is compact. By applying Lemma V. 1, $T_2$ is not Li-Yorke chaotic. Thus we have $T_1$ as well as $T_2$ is not Li-Yorke chaotic and therefore $T - \lambda I$ is not Li-Yorke chaotic. Thus, we have shown that for $\lambda \notin \mathbb{T}$, $T - \lambda I$ is not Li-Yorke chaotic which implies that $\lambda \notin LY(T)$. Hence, $LY(T) \subseteq \mathbb{T}$.

*Example V. 4:-* Let $T$ be the unilateral forward weighted shift operator on $\ell^2(\mathbb{N})$

$$T(x_1, x_2, \cdots,) = (0, w_1 x_1, w_2 x_2, \cdots, w_n x_n, \cdots,)$$

with weight sequence $(w_n)_{n \geqslant 1} \longrightarrow 0$. Then $LY(T) \subseteq \mathbb{T}$.

*Proof.* Since the weight sequence $(w_n)$ is converging to 0 as $n \longrightarrow \infty$, we can easily approximate $T$ by compact operator $T_n(x) = (0, w_1 x_1, w_2 x_2, \cdots, w_n x_n, 0, 0, \cdots)$ and therefore $T$ becomes compact. Then by Theorem V. 3, $LY(T) \subseteq \mathbb{T}$.

*Remark V. 5:-*

If $T$ is a normal operator, then $T - \lambda I$ is also a normal operator.

*Proof.* If $T$ is normal, then we have $TT^\star = T^\star T$.

Now, Consider

$$\begin{aligned} (T - \lambda I)^\star (T - \lambda I) &= T^\star T - \overline{\lambda} T - \lambda T^\star + |\lambda|^2 I \\ &= TT^\star - \lambda T^\star - \overline{\lambda} T + |\lambda|^2 I \\ &= (T - \lambda I)(T - \lambda I)^\star \end{aligned}$$

which shows that $T - \lambda I$ is normal.

*Theorem V. 6:* Let $T \in B(H)$ be a normal operator. Then $LY(T) = \emptyset$

*Proof.* By Corollary II. 10 and using Remark V. 5, $T - \lambda I$ is not Li-Yorke chaotic. Hence by definition II. 11, $LY(T) = \emptyset$.

*Theorem V. 7:* Let $T \in B(H)$ be a self adjoint operator. Then $LY(T) = \emptyset$

*Proof.* Since every self adjoint operators are normal and hence by *Theorem V. 6* , $LY(T) = \emptyset$.

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