A Note on Taylor Series Representation of the Nield-Kuznetsov Function of the Second Kind

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Abstract: - Taylor and Maclaurin series, and polynomial approximations of the Standard Nield-Kuznetsov function of the second kind are obtained in this work. Convergence and error criteria are developed. The obtained series represent alternatives to existing asymptotic and ascending series approximations of this integral function, and are expected to provide an efficient method of computation.

Key-Words: - Taylor and Maclaurin series, Standard Nield-Kuznetsov function of the second kind

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1 Introduction

In a recent article, Alderson and Hamdan, [1], investigated Taylor and Maclaurin series and polynomial representations of the standard Nield-Kuznetsov function of the first kind, $Ni(x)$. This function arises in the solution to Airy's inhomogeneous ordinary differential equation, [2], of the form

$$\frac{d^2y}{dx^2} - xy = R \tag{1}$$

where $R \in \Re$. Hamdan and Kamel, [3], showed that the general solution to (1) is expressible in the form:

$$y = c_1 Ai(x) + c_2 Bi(x) - \pi R Ni(x) \tag{2}$$

where c_1 and c_2 are arbitrary constants, $Ai(x)$ and $Bi(x)$ are the linearly independent Airy's functions of the first and second kind, whose Wronskian is given by, [4,5]:

$$W\big(Ai(x), Bi(x)\big) = Ai(x)\frac{dBi(x)}{dx} - Bi(x)\frac{dAi(x)}{dx} = \frac{1}{\pi} \tag{3}$$

and the standard Nield-Kuznetsov function of the first kind, $Ni(x)$, is defined by:

$$Ni(x) = Ai(x)\int_0^x Bi(t)dt - Bi(x)\int_0^x Ai(t)dt \tag{4}$$

When the forcing function is a differentiable function, $f(x)$, that replaces R in (1), general solution to (1) is expressible in the form, [3]:

$$y = c_1 Ai(x) + c_2 Bi(x) + \pi\{Ki(x) - f(x)Ni(x)\} \tag{5}$$

where the function $Ki(x)$ is the Standard Nield-Kuznetsov Function of the second kind. This function takes the following equivalent forms, [3]:

$$Ki(x) = Ai(x)\int_0^x \left\{\int_0^t Bi(\tau)d\tau\right\} f'(t)dt - B_i(x)\int_0^x \left\{\int_0^t Ai(\tau)d\tau\right\} f'(t) \tag{6}$$

$$Ki(x) = f(x)Ni(x) - \left\{Ai(x)\int_0^x f(t)Bi(t)\,dt - Bi(x)\int_0^x f(t)Ai(t)\,dt\right\} \tag{7}$$

Computations of the general solutions (2) and (5) necessitate evaluations of $Ni(x)$ and $Ki(x)$. Methods of evaluation of these functions typically involve asymptotic or ascending series representations. These methods have been developed in the literature, and received considerable validation (*cf.* [6-8] and the references therein).

Taylor and Maclaurin series representations for $Ni(x)$, [1], provided a convenient, time-saving method of computations pf $Ni(x)$, for values of x that are small enough for ascending series evaluations. Accurate results can be obtained using a Maclaurin polynomial of degree less than 15. Motivated by the findings of Alderson and Hamdan, [1], the current work aims at obtaining Taylor and Maclaurin series expansions and polynomial representations of $Ki(x)$. To this end, higher derivatives of $Ki(x)$, together with Airy's polynomials, are needed in the analysis that follows.

2 Series Expansion of *Ki(x)*

The function $Ki(x)$ is a smooth function with an n^{th} derivative expressible in terms of integral terms whose coefficients are polynomials. Two of these polynomials are in fact Airy's polynomials, as will be discussed below. The function $Ki(x)$ can thus be expanded in a Taylor series, about $x = x_0$, of the form:

$$Ki(x) = \sum_{j=0}^{\infty} C_j\,(x - x_0)^j = \sum_{j=0}^{\infty} \frac{Ki^{(j)}(x_0)}{j!}(x - x_0)^j \qquad (8)$$

and $Ki^{(j)}(x_0)$ denotes the j^{th} derivative of $Ki(x)$ evaluated at $x = x_0$.

If $x_0 = 0$ then Taylor series becomes Maclaurin series, namely:

$$Ki(x) = \sum_{j=0}^{\infty} C_j\,(x)^j = \sum_{j=0}^{\infty} \frac{Ki^{(j)}(0)}{j!}(x)^j \qquad (9)$$

Equations (8) and (9) point to the need for higher derivatives of $Ki(x)$. These are discussed in what follows.

3 Derivatives of *Ki(x)*

One way in which higher derivatives of $Ki(x)$ can be obtained is to differentiate the following expression:

$$Ki(x) = f(x)Ni(x) - \left\{Ai(x)\int_0^x f(t)Bi(t)\,dt - Bi(x)\int_0^x f(t)Ai(t)\,dt\right\} \qquad (10)$$

Using (10), the following first few derivatives of $Ki(x)$ are obtained:

$$Ki'(x) = f'(x)Ni(x) + f(x)Ni'(x) - \left\{A'i(x)\int_0^x f(t)Bi(t)\,dt - B'i(x)\int_0^x f(t)Ai(t)dt\right\} \qquad (11)$$

$$Ki''(x) = 2f'(x)Ni'(x) + f''(x)Ni(x) + xKi(x) \qquad (12)$$

$$Ki'''(x) = 3f''(x)Ni'(x) + [f'''(x) + 2xf'(x)]Ni(x) + Ki(x) + xKi'(x) - 2f'(x)W\big(Ai(x), Bi(x)\big) \qquad (13)$$

Continuing in this manner, the following n^{th} derivative is obtained:

$$Ki^{(n)}(x) = [f(x)Ni(x)]^{(n)} + p_n(x)\left\{Ai(x)\int_0^x f(t)Bi(t)\,dt - Bi(x)\int_0^x f(t)Ai(t)\,dt\right\} + q_n(x)\left\{Ai'(x)\int_0^x f(t)Bi(t)\,dt - Bi'(x)\int_0^x f(t)Ai(t)dt\right\} + r_n(x)W(Ai(x), Bi(x)) \qquad (14)$$

where $p_n(x), q_n(x)$ and $r_n(x)$ are the polynomial coefficients of the integral terms and of the Wronskian that appear in the *nth* derivative, namely

$p_n(x)$ is coefficient of $\left\{Ai(x)\int_0^x f(t)Bi(t)\,dt - Bi(x)\int_0^x f(t)Ai(t)\,dt\right\}$

$q_n(x)$ is coefficient of $\left\{Ai'(x)\int_0^x f(t)Bi(t)\,dt - Bi'(x)\int_0^x f(t)Ai(t)dt\right\}$

$r_n(x)$ is coefficient of $W\big(Ai(x), Bi(x)\big) = \frac{1}{\pi}$.

The $n+1^{st}$ derivative of $Ki(x)$, obtained by differentiating (14), takes the form:

$$Ki^{(n+1)}(x) = \sum_{k=0}^{n+1}\binom{n+1}{k} Ni^{(n+1-k)}f^{(k)}(x)$$
$$+[p_n'(x) + xq_n(x)]\{Ai(x)\int_0^x f(t)Bi(t)\,dt - Bi(x)\int_0^x f(t)Ai(t)\,dt\}$$
$$+[p_n(x) + q_n'(x)\{Ai'(x)\int_0^x f(t)Bi(t)\,dt - Bi'(x)\int_0^x f(t)Ai(t)dt\}$$
$$+[r_n'(x) - f(x)q_n(x)]W(Ai(x), Bi(x)) \quad (15)$$

Following Alderson and Hamdan, [1], and Jayyousi-Dajani and Hamdan, [9], relationships between polynomials $p_n(x), q_n(x)$ and $r_n(x)$ are given by:

$$p_{n+1}(x) = p'_n(x) + xq_n(x) \quad (16)$$
$$q_{n+1}(x) = q'_n(x) + p_n(x) \quad (17)$$
$$r_{n+1}(x) = r'_n(x) + q_n(x) \quad (18)$$

Now, using (10), expression (15) can be written in the following form:

$$Ki^{(n+1)}(x) = \sum_{k=0}^{n+1}\binom{n+1}{k} Ni^{(n+1-k)}(x)f(x)^{(k)}$$
$$+[p_n'(x) + xq_n(x)]\{f(x)Ni(x) - Ki(x)\}$$
$$+[p_n(x) + q_n'(x)]\{f(x)Ni(x) - Ki(x)\} + \frac{1}{\pi}[r_n'(x) - f(x)q_n(x)] \quad (19)$$

Replacing $n+1$ by n in (16)-(19), the n^{th} derivative of $Ki(x)$ takes the following form:

$$Ki^{(n)}(x) = \sum_{k=0}^{n}\binom{n}{k} Ni^{(n-k)}(x)f^{(k)}(x)$$
$$+[p_{n-1}(x) + p'_{n-1}(x) + xq_{n-1}(x) + q'_{n-1}(x)]\{f(x)Ni(x) - Ki(x)\}$$
$$+\frac{1}{\pi}[r'_{n-1}(x) - f(x)q_{n-1}(x)]; \quad n = 1,2,3,\ldots \quad (20)$$

Polynomials $p_n(x)$, $q_n(x)$ and $r_n(x)$ are associated with the n^{th} derivative of $K_i(x)$, where n refers to the order of the derivative and not the degree of the polynomial. These polynomials are the negatives of the polynomials associated with the n^{th} derivatives of Airy's functions, $Ai(x)$ and $Bi(x)$, and the n^{th} derivative of the standard Nield-Kuznetsov function of the first kind, $Ni(x)$, [1]. **Table 1**, below, lists the polynomials $p_n(x)$, $q_n(x)$ and $r_n(x)$, for $n = 0,1,2,\ldots,10$.

Table 1. Coefficient Polynomials

$n=0$	$p_n(x)$	$q_n(x)$	$r_n(x)$
0	-1	0	0
1	0	-1	0
2	$-x$	0	0
3	-1	$-x$	0
4	$-x^2$	-2	$-x$
5	$-4x$	$-x^2$	-3
6	$-4-x^3$	$-6x$	$-x^2$
7	$-9x^2$	$-10-x^3$	$-8x$
8	$-28x-x^4$	$-12x^2$	$-x^3-18$
9	$-28-16x^3$	$-52x-x^4$	$-15x^2$
10	$-100x^2-x^5$	$-80-20x^3$	$-x^4$

Degrees of the coefficient polynomials may be determined for arbitrary order of derivative, n, and are provided in the following **Table 2** in terms of the floor function.

Table 2. Degrees of Coefficient Polynomials

Polynomial	Degree
$p_n(x)$	$3\left\lfloor\frac{n-2}{2}\right\rfloor - n + 3,\ n \geq 2$
$q_n(x)$	$3\left\lfloor\frac{n-3}{2}\right\rfloor - n + 4,\ n \geq 3$
$r_n(x)$	$3\left\lfloor\frac{n-4}{2}\right\rfloor - n + 5,\ n \geq 4$

4 Taylor and Maclaurin Series and Ploynomials

Using (20) in (8), Taylor series expansion of $Ki(x)$ can be written in the following form:

$$Ki(x) = Ki(x_0) + \sum_{n=1}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k} Ni^{(n-k)}(x_0)f^{(k)}(x_0) + \frac{1}{\pi}[r'_{n-1}(x_0) - f(x_0)q_{n-1}(x_0)] + [p_{n-1}(x_0) + p'_{n-1}(x_0) + x_0 q_{n-1}(x_0) + q'_{n-1}(x_0)]\{f(x_0)Ni(x_0) - Ki(x_0)\}\right\}\frac{(x-x_0)^n}{n!} \quad (21)$$

Writing (18) in the form

$$r'_{n-1}(x) = r_n(x) - q_{n-1}(x) \quad (22)$$

followed by using (22) in (21), gives

$$Ki(x) = Ki(x_0) + \sum_{n=1}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k} Ni^{(n-k)}(x_0)f^{(k)}(x_0) + \frac{1}{\pi}[r_n(x_0) - \{1 + f(x_0)\}q_{n-1}(x_0)] + [p_n(x_0) + q_n(x_0)]\{f(x_0)Ni(x_0) - Ki(x_0)\}\right\}\frac{(x-x_0)^n}{n!} \quad (23)$$

Taking $x_0 = 0$ in (23), and using $Ki(0) = K'i(0) = Ni(0) = Ni'(0) = 0$, the following Maclaurin series is obtained:

$$Ki(x) = \sum_{n=2}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k} Ni^{(n-k)}(0)f^{(k)}(0) + \frac{1}{\pi}[r_n(0) - \{1 + f(0)\}q_{n-1}(0)]\right\}\frac{(x)^n}{n!} \quad (24)$$

Equations (23) and (24) represent the final forms of the Taylor and Maclaurin series expansions of $Ki(x)$, respectively. If the Taylor series of $Ki(x)$ is terminated after $N+1$ terms, then a Taylor polynomial, $T_N(x)$, of degree N results. This polynomial approximates the function $Ki(x)$ *near* $x = x_0$, namely

$$Ki(x) \approx T_N(x) = Ki(x_0) + \sum_{n=1}^{N}\left\{\sum_{k=0}^{n}\binom{n}{k} Ni^{(n-k)}(x_0)f^{(k)}(x_0) + \frac{1}{\pi}[r_n(x_0) - \{1 + f(x_0)\}q_{n-1}(x_0)] + [p_n(x_0) + q_n(x_0)]\{f(x_0)Ni(x_0) - Ki(x_0)\}\right\}\frac{(x-x_0)^n}{n!} \quad (25)$$

If $x_0 = 0$, the following N^{th} degree Maclaurin polynomial is obtained:

$$Ki(x) \approx M_N(x) = \sum_{n=2}^{N}\left\{\sum_{k=0}^{n}\binom{n}{k} Ni^{(n-k)}(0)f^{(k)}(0) + \frac{1}{\pi}[r_n(0) - q_{n-1}(0)\{1 + f(0)\}]\right\}\frac{(x)^n}{n!} \quad (26)$$

4.1 Determination of the N^{th} Degree Maclaurin Polynomial

Equation (26) suggests that, for a given $f(x)$, obtaining a Maclaurin polynomial of degree N for $Ki(x)$ requires the following as input:

$f^{(m)}(0)$ for $m = 0,1,2,3,\dots,N$

$Ni^{(m)}(0)$ for $m = 0,1,2,3,\dots,N$

$r_m(0)$ and $q_{m-1}(0)$ for $m = 1,2,3,\dots,N$.

Since $f(x)$ is a smooth, continuously differentiable function, evaluating $f^{(m)}(0)$ is accomplished by repeated differentiation of $f^{(m)}(x)$ and computing it at $x = 0$.

In order to evaluate $Ni^{(m)}(0)$, the following recursive relation can be used (*cf.* Hamdan and Kamel, [3]):

$$Ni^{(m)}(0) = (m-2)Ni^{(m-3)}(0);\ m = 3,4,\dots,N \quad (27)$$

Where $Ni(0) = Ni'(0) = 0$; $Ni''(0) = -\frac{1}{\pi}$.

In order to evaluate $r_m(0)$ and $q_{m-1}(0)$ for $m = 1,2,3,\dots,N$, **Table 1** is reproduced when $x_0 = 0$. The resulting values are shown in **Table 3**, below. If $N > 10$, then more entries in **Tables 1** and **2** can be generated using relationships (16)-(18).

Table 3. Coefficient Polynomials at $x_0 = 0$

$n = 0$	$p_n(0)$	$q_n(0)$	$r_n(0)$
0	−1	0	0
1	0	−1	0

2	0	0	0
3	-1	0	0
4	0	-2	0
5	0	0	-3
6	-4	0	0
7	0	-10	0
8	0	0	-18
9	-28	0	0
10	0	-80	0

4.2 Determination of the N^{th}Degree Taylor Polynomial

Equation (25) suggests that, for a given $f(x)$, obtaining a Taylor polynomial of degree N for $Ki(x)$ requires the following as input:

$Ni(x_0)$ and $Ki(x_0)$

$f^{(m)}(x_0)$ for $m = 0,1,2,3,\ldots,N$

$Ni^{(m)}(x_0)$ for $m = 0,1,2,3,\ldots,N$

$p_m(x_0),\ r_m(x_0),\ q_m(x_0)$ for $m = 1,2,3,\ldots,N$.

Evaluation of $f^{(m)}(x_0)$ follows a similar procedure to that discussed for the case of Maclaurin polynomial, above.

Evaluation of $p_m(x_0)$, $r_m(x_0)$ and $q_m(x_0)$ for $m = 1,2,3,\ldots,N$, is accomplished by evaluating the polynomials of **Table 1** at $x = x_0$.

Values of $Ni^{(m)}(x_0)$, for $m = 2,3,\ldots,N$, can be evaluated using the following derivative formula, (*cf.* [6-8]):

$$Ni^{(m)}(x_0) = P_m(x_0)Ni(x_0) + Q_m(x_0)Ni'(x_0) - R_m(x_0)/\pi \qquad (28)$$

where$P_m(x) = -p_m(x);\ Q_m(x) = -q_m(x);\ R_m(x) = -r_m(x)$, for all x, can be obtained from **Table 1.**

Clearly, to use (27) the values of $Ni(x_0)$ and $Ni'(x_0)$ are needed. In addition, $Ki(x_0)$ is still needed as an input to finding the N^{th} degree Taylor polynomial for $Ki(x)$. Although there are a number of ways to evaluate these functions, a most viable approach is to use Maclaurin polynomials for $Ki(x)$, (equation (26), above), and for $Ni(x)$, (as derived in Alderson and Hamdan, [1]). The value of $Ni'(x_0)$ can be obtained by differentiating Maclaurin polynomial for $Ni(x)$ and evaluating it at x_0.

In order to illustrate this process, consider the determination of a tenth degree Maclaurin polynomial for $Ki(x)$ when $f(x) = e^x$. The form of this polynomial is given by:

$$Ki(x) \approx M_N(x) = \sum_{n=2}^{10}\left\{\sum_{k=0}^{n}\binom{n}{k} Ni^{(n-k)}(0) + \frac{1}{\pi}[r_n(0) - 2q_{n-1}(0)]\right\}\frac{(x)^n}{n!} \qquad (29)$$

The following values are used as input in (28):

$f^{(m)}(0) = e^0 = 1$, for $m = 0,1,2,\ldots,10$.

Using (27), the following values of $N^{(m)}(0)$ are obtained, for $m = 0,1,2,\ldots,10$:

$Ni(0) = Ni'(0) = Ni'''(0) = Ni^{(4)}(0) = Ni^{(6)}(0) = Ni^{(7)}(0) = Ni^{(9)}(0) = Ni^{(10)}(0) = 0$,
$Ni''(0) = -\frac{1}{\pi};\ Ni^{(5)} = -\frac{3}{\pi};\ Ni^{(8)} = -\frac{18}{\pi}$.

Using the values of $r_m(0)$ and $q_m(0)$ of **Table 3** and the above input in (28) results in:

$$Ki(x) \approx M_{10}(x) = \frac{1}{\pi}\left[\frac{2(x)^2}{2!} - \frac{3(x)^3}{3!} - \frac{6(x)^4}{4!} - \frac{12(x)^5}{5!} - \frac{90(x)^6}{6!} - \frac{63(x)^7}{7!} - \frac{212(x)^8}{8!} - \frac{576(x)^9}{9!} - \frac{1611(x)^{10}}{10!}\right] \qquad (30)$$

Equation (29) gives the following values for $Ki(2)$ and $Ki(5)$:

$Ki(2) \approx -6.1766770$
$Ki(5) \approx -796.65898$

By comparison, the corresponding values obtained by Alzahrani *et.al.* [10,11], using ascending series expansion of $Ki(x)$ with 10 terms are:

$Ki(2) \approx -3.02839377$
$Ki(5) \approx -891.543059$

While the discrepancy is quite large when $x = 2$, it is less as x is increased. Furthermore, the work of

Alzahrani *et.al.* [10] is the only source to compare results with. Therefore, it is hard to know which results are more accurate. Further analysis and investigations in this regard are needed.

5 Convergence of Taylor Series of $Ki(x)$

Series (23) converges for values of x satisfying $|x - x_0| < R$, where R is the radius of convergence defined by

$$\frac{1}{R} = \lim_{j\to\infty}\left|\frac{C_{j+1}}{C_j}\right| = \lim_{j\to\infty}\left|\frac{Ki^{(j+1)}(x_0)}{(j+1)Ki^{(j)}(x_0)}\right| = \lim_{j\to\infty}\left|\frac{L}{(j+1)}\right| = 0 \qquad (31)$$

where $L = \frac{Ki^{(j+1)}(x_0)}{Ki^{(j)}(x_0)}$ is finite since the maximum degrees of the polynomials involved in $Ki^{(j)}(x_0)$ and $Ki^{(j+1)}(x_0)$ are comparable. Hence, the radius of convergence R is infinite and series (23) converges for all x. The same is true for Maclaurin series (24). This furnishes the following Theorem on convergence.

Theorem 1.
The Taylor series expansion, (23), of $Ki(x)$ about x_0 converges for all values of x.

6 Remainder and Error Terms

In this analysis, the work of Alderson and Hamdan, [1], is followed closely. When approximating $Ki(x)$ by an N^{th} degree Taylor polynomial, $T_N(x)$, an error term, $E_N(x) = Ki(x) - T_N(x)$, is introduced. Explicitly, we have

$$E_N(x) = \sum_{k=0}^{\infty}\frac{Ki^{(k)}(x_0)}{k!}(x-x_0)^k - \sum_{k=0}^{N}\frac{Ki^{(k)}(x_0)}{k!}(x-x_0)^k = \sum_{k=N+1}^{\infty}\frac{Ki^{(k)}(x_0)}{k!}(x-x_0)^k \qquad (32)$$

On an arbitrary interval $[a, b]$ around x_0, continuity of $Ki(x)$ and each of it's derivatives deems that $Ki^{(N+1)}(x)$ is bounded, say $|Ki^{(N+1)}(x)| \le M$. As such, *Taylor's inequality* provides:

$$|E_N(x)| \le M\frac{|x-\tau|^{N+1}}{(M+1)!} \qquad (33)$$

For all $\tau \in [a, b]$. Consequently, we have

$$0 \le |E_N(x)| \le M\frac{|x-\tau|^{N+1}}{(N+1)!} \le M \cdot \frac{(b-a)^{N+1}}{(N+1)!} \qquad (34)$$

Taking limits in (34) shows

$$\lim_{N\to\infty} E_N = 0 \qquad (35)$$

This implies that $Ki(x)$ is equal to it's Taylor Series (everywhere).

7 Tangent Line Approximation

Tangent line approximation is needed here in light of the fact that Maclaurin series and polynomial approximations for $Ki(x)$ involve x^2 as its first term. If *N=1* then Taylor polynomial approximation to $Ki(x)$ becomes:

$$Ki(x) \approx T_1(x) = Ki(x_0) + Ki'(x_0)(x - x_0) \qquad (36)$$

Using (10) and (11) gives

$$Ki(x) \approx T_1(x) = [f(x_0)Ni(x_0) + \{f'(x_0)Ni(x_0) + f(x_0)N'i(x_0)\}(x-x_0)] + \int_0^{x_0} f(t)Ai(t)\,dt[Bi(x_0) + B'i(x_0)(x-x_0)] - \int_0^{x_0} f(t)Bi(t)\,dt[Ai(x_0) + A'i(x_0)(x-x_0)] \qquad (37)$$

Equation (37) is the tangent line approximation to $Ki(x)$ near $x = x_0$. It is written here in terms of Airy's functions and integrals. Equation (36) also gives an approximation to the slope of the tangent line, $Ki'(x_0)$, in terms of the slope of the secant line, namely

$$Ki'(x_0) \approx \frac{T_1(x) - Ki(x_0)}{(x - x_0)} \qquad (38)$$

8 Conclusion

In this work, Taylor and Maclaurin series expansions of the Standard Nield-Kuznetsov function of the second kind, $Ki(x)$, were obtained in order to provide further insight into the behavior of this integral function. Convergence criteria were also investigated in order to show that Taylor series representation of $Ki(x)$ converges for all x. Errors incurred in representing this function by Taylor and Maclaurin polynomials were quantified and tangent line approximation was obtained. Further investigations are needed due to explain the difference between computed results using this work's tenth degree Maclaurin polynomial and what is reported in the literature using ten terms of ascending series expansion of $Ki(x)$.

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Contribution of individual authors

Both authors reviewed the literature, formulated the problem, provided independent analysis, and jointly wrote and revised the manuscript.

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