PROOF
DOI: 10.37394/232020.2022.2.3

On the Stokes Nonlinear Waves in 2D

NINO KHATIASHVILI
I. Vekua Institute of Applied Mathematics
Iv. Javakhishvili Tbilisi State University
2., University St., 0186Tbilisi
GEORGIA

Abstract: - The Stokes nonlinear waves associated with the nonlinear problem of a free boundary with peaks in incompressible heavy fluid are studied in 2D. In the early works of the author by using the conformal mapping method this problem was reduced to the nonlinear integral equation with the weakly singular kernel. In this paper one parameter of the mapping is chosen sufficiently small and the equation is linearized. The approximate solution of the linearized equation is obtained. The profile of the free boundary is plotted by means of Maple-12.

Key-Words: - Incompressible heavy fluid, Stokes waves**,** Non-linear integral equation

1 Introduction

In the incompressible heavy fluid nonlinear peaked waves are originated under certain conditions[1-5, 8, 9]. These waves are known as Stokes waves. Here we investigate the nonlinear integral equation associated with this phenomena .This equation was obtained by the author [see 4, 5] by means of the conformal mapping method from the initial problem which will be stated below. As three parameters of the mapping can be chosen arbitrarily, we choose one parameter sufficiently small. The nonlinear integral equation is simplified and the approximate solution is obtained. This solution represents peaked symmetric Stokes wave. The profile of the wave is constructed by using Maple. The approximate solution of this equation depending on some parameters is obtained. The profile of symmetric wave is constructed.

2 Problem Formulation

In the coordinate system Oxy of the Euclidian space R^2, the initial problem is stated as follows [7]

PROBLEM ST. Find the periodic curve $\Gamma : y = y(x)$ such that , if f is a conformal mapping of the area $D = \{0 < t < y(x)\}$ on the strip $\{0 < \psi < q, q = const\}$, $f(\pm\infty) = \infty$, then the following condition holds

$$\frac{1}{2}|f'(z)|^2 + gy = A, \; A = const, \tag{1}$$

where $f'(z)$, $z = x + iy$, is a complex potential, φ is a speed potential, ψ is a stream function, $f'(z)$ is a complex speed, A and q are the definite constants, g is a gravity acceleration.

Here we assume, that the bottom of the reservoir is planar and filled with incompressible heavy fluid, the wave moves with the constant speed c. We choose the mobile coordinate system moving with the wave, with the axis oy passing through the maximum point of the wave and the axis ox passing along the bottom. We consider the Problem ST for the symmetric periodic waves with the period ω in case of $f'(iq + n\omega) = 0, n = 0, \pm1, \pm2, ...$ i.e. the Stokes waves [1-9]. The case $f'(z) \neq 0$ was considered by different authors [1, 2, 3, 7, 9].

By means of the conformal mapping method in the previous works of the author [4,5] Problem ST was transformed to the following nonlinear integral equation with the weakly singular kernel

$$u(\xi)= -\frac{3g}{4\pi}\int_0^a z_1'(t)\left[\ln|z_1'(t)|-\frac{2}{3}\ln u(t)\right]K(t,\xi)dt, \quad (2)$$

where

$$K(t,\xi)=2\ln\frac{\sqrt{b^2-\xi^2}+\sqrt{b^2-t^2}}{\sqrt{b^2-a^2}+\sqrt{b^2-t^2}}+\ln\left|\frac{a^2-t^2}{t^2-\xi^2}\right|,$$
$$\xi\in[0,a].$$

$\zeta(z_1)=z_1^{-1}(\zeta)$, $\zeta=\xi+i\eta$, is a conformal mapping of one period $OABC$ of the area D on the rectangle $O_1A_1B_1C_1$ of the complex plane ζ

$$z_1(\zeta)=C\int_0^\zeta\frac{1}{(a^2-t^2)(b^2-t^2)}dt+\frac{\omega}{2}+iq,$$

$z_1=f(z), C=const$, with the following correspondence of points

$$O_1\leftrightarrow(b,0),\ A_1\leftrightarrow(-b,0),\ B_1\leftrightarrow(-a,0),$$
$$C_1\leftrightarrow(a,0),$$

where a,b are the definite constants, a is sufficiently small.

Having found the solution of the equation (2) the profile of Stokes wave will be given by [5]

$$\frac{1}{2g}\left(2A-u^{\frac{2}{3}}(\xi)\right). \quad (3)$$

Our purpose is to find the approximate solution of the equation (2) and to construct profile of Stokes wave.

3 Problem Solution

In [5] it is proved that the function $u(\xi)$ could be represented in the form

$$u(\xi)=u_0(\xi)\left((a^2-\xi^2)\ln^2(a^2-\xi^2)\right), \quad (4)$$

where $u_0(\xi)$ is bounded function of the class

$$C^1[0,a],\ m\le u_0(\xi)\le M,\ u_0(\xi)\ne 0,$$
$$M=\max u_0(\xi),$$

By using (4) the equation (2) can be rewritten in the form

$$u_0(\xi)\left((a^2-\xi^2)\ln^2(a^2-\xi^2)\right)= -\frac{3g}{4\pi}\int_0^a z'(t)\left[\ln|z_1'(t)|-\frac{2}{3}\ln\left((a^2-t^2)\ln^2(a^2-t^2)\right)-\frac{2}{3}\ln u_0(t)\ \right]K(t,\xi)dt. \quad (5)$$

We admit, that $\left(\frac{u_0}{M}-1\right)^2$ is sufficiently small. Taking into the account the formula $\ln\frac{u_0}{M}\approx\frac{u_0}{M}-1$, the equation (5) takes the form

$$u_0(\xi)\left((a^2-\xi^2)\ln^2(a^2-\xi^2)\right)= -\frac{3g}{4\pi}\int_0^a z'(t)\left[\ln|z_1'(t)|-\frac{2}{3}\ln M+\frac{2}{3}-\frac{2}{3}\ln\left((a^2-t^2)\ln^2(a^2-t^2)\right)-\frac{2}{3M}u_0(t)\ \right]K(t,\xi)dt. \quad (6)$$

We represent the right hand side of (6) in the form

$$\int_0^a z'(t)\left[\ln|z_1'(t)|-\frac{2}{3}\ln M-\frac{2}{3}\ln\left((a^2-t^2)\ln^2(a^2-t^2)\right)+\frac{2}{3}-\frac{2}{3M}u_0(t)\ \right]K(t,\xi)dt=$$
$$\sum_{i=0}^n\int_{a_i}^{a_{i+1}} z'(t)\left[\ln|z_1'(t)|-\frac{2}{3}\ln M-\frac{2}{3}\ln\left((a^2-t^2)\ln^2(a^2-t^2)\right)+\frac{2}{3}-\frac{2}{3M}u_0(t)\ \right]K(t,\xi)dt,$$
$$a_0=0,\ a_{n+1}=a,\ a_0<a_1<...<a_n<a,\ n\in N,$$

and use the approximation

$$u_0(\xi)\approx u_0\left(\frac{a_i+a_{i+1}}{2}\right)\equiv C_i,\ \ \xi\in(a_i,a_{i+1}),$$
$$i=0,1,...,n;$$

then from (6) we obtain

$$u_0(\xi)\left((a^2-\xi^2)\ln^2(a^2-\xi^2)\right)=$$
$$\frac{3g}{2\pi M}\sum_{i=0}^{n}C_i\int_{a_i}^{a_{i+1}}z_1'(t)K(t,\xi)dt-$$
$$\frac{3g}{4\pi}\sum_{i=0}^{n}\int_{a_i}^{a_{i+1}}z_1'(t)\Big[\ln|z_1'(t)|-\frac{2}{3}\ln M-$$
$$\frac{2}{3}\ln\left((a^2-t^2)\ln^2(a^2-t^2)\right)+\frac{2}{3}\Big]K(t,\xi)dt, \quad (7)$$
$$a_0=0,\ a_{n+1}=a,\ a_0<a_1<...<a_n<a,\ n\in N.$$

Hence, for the definition of $C_i, i=0,1,...,n;$ from (7) we obtain the system of algebraic equations

$$C_i\left((a^2-\xi_i^2)\ln^2(a^2-\xi_i^2)\right)=$$
$$\frac{3g}{2\pi M}\sum_{i=0}^{n}C_i\int_{a_i}^{a_{i+1}}z_1'(t)K(t,\xi_i)dt-$$
$$\frac{3g}{4\pi}\sum_{i=0}^{n}\int_{a_i}^{a_{i+1}}z_1'(t)\Big[\ln|z_1'(t)|-\frac{2}{3}\ln M-$$
$$\frac{2}{3}\ln\left((a^2-t^2)\ln^2(a^2-t^2)\right)+\frac{2}{3}\Big]K(t,\xi_i)dt, \quad (8)$$
$$\xi_i=\frac{a_i+a_{i+1}}{2},$$
$$a_0=0,\ a_{n+1}=a,\ a_0<a_1<...<a_n<a, \quad i=0,1,...,n;\ n\in N.$$

(8) is the system of algebraic equations with respect to $C_i, i=0,1,...,n;$ Having found C_i by using (7) we can construct the graph of (3) by means of Maple 12 .

Below the graphic of (3) is plotted for the parameters $b=1;\ a=10^{-3};\ M=1;\ n=1;$ (Fig.1).

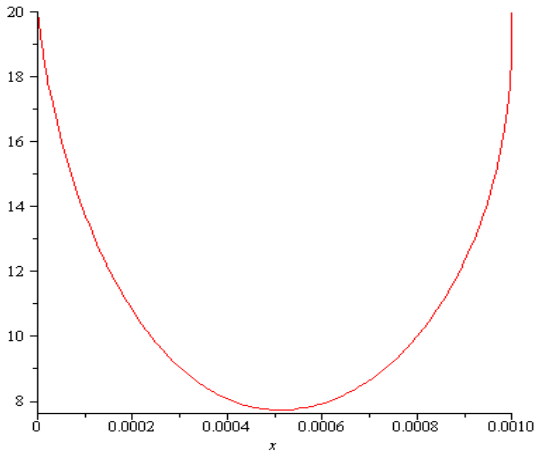

Fig. 1. The graph of $\frac{1}{2g}\left(2A-u^{\frac{2}{3}}(\xi)\right)$, one period of the Stokes wave.

NOTE . In the work [4] the Problem ST is reduced to the nonlinear integral equation with the Weierstrass kernel. In the work of the author [6] the solutions of this equation are obtained in the linearized case.

4 Conclusion

The approximate solution of the nonlinear integral equation (2) is given by (7), where the constants $C_i, i=0,1,...,n;$ are the solutions of the system of the algebraic equations (8). The function given by the formula (7) represents periodic symmetric Stokes wave with peaks.

References:

[1] TB. Benjamin, PJ.Olver, New Hamiltonian structure, symmetries and conservation lows for water waves, *Journal of Fluid Mechanics*, Vol.125, 1982, pp. 137-185.

[2] EV. Buldakov, PH.Taylor, New asymptotic description of nonlinear water waves in Lagrangian coordinates, *Journal of Fluid Mechanics*, Vol.562, 2006, pp. 431-185.

[3] SK. Chakrabarti, *Handbook of Offshore Engineering*, Elsevier, 2005.

[4] N.Khatiashvili, On the nonlinear plane boundary value problem, *Reports of VIAM*, Vol.X, No.I, 1995, pp. 46-48.

[5] N.Khatiashvili, On Stokes nonlinear integral wave equation, Integral Methods in Science and Engineering (editors B. Bertram, C. Constanda, A. Struthers), *Research Notes in Mathemathcs Series, CRC* , 2000, pp. 200-204.

[6] N.Khatiashvili, On the singular integral equation with the Weierstrass kernel, *Complex Variables and Elliptic Equations*, Vol.53, No.10, 2008, pp. 915-943.

[7] MA. Lavrent'ev and BV Shabat, *The Problems of Hydrodynamics and their Mathematical Models*, Nauka, Moscow, 1980.

[8] G. Stokes, On the Theory of Ocsillatory Waves, *Transactions of the Cambridge Philosophical Society*, Vol.VIII, No.10, 1847, pp. 197-229.

[9] GB. Whitman, *Linear and Nonlinear Waves*, Wiley-Interscience, Moscow, 1974.