Non-stationary response of a beam for a new rheological model

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Abstract: The paper is intended to provide the quasi-static and dynamic analysis of beam with fractional order viscoelastic material model, which was derived from integer order description using the Boltzmann superposition principle. The results were obtained for a fractional Zener model by the techniques of Laplace transform and binomial series. An example proves the accuracy of the solution for a simply-supported beam subjected to a uniform distributed load. Theoretical and numerical solutions can be easily extending to the complex structures configurations.

Key-words: viscoelastic beam, Euler-Bernoulli beam theory, fractional derivative, Laplace transform, the constitutive law in hereditary integral form, correspondence principle

1. Introduction

Engineering design of the past thirty years are frequently present the structures with viscoelastic components due to their ability to dampen out the vibrations. The metals at elevated temperatures, rubbers, polymers that have the characteristic of both elastic solids as well as viscous solids are examples of viscoelastic materials. There are remarkable theoretical studies on these materials of Cristescu [7], Mainardi and Spada [7], Flugge [10], Freundlich [10]-[11], Reddy [13], Kennedy [14]. These are complemented by the works dealing with the analysis of viscoelastic structures from both mathematical and engineering points of views: [1], [8], [16], [17], [20] - [22].

Using the Euler-Bernoulli beam theory, we present in our paper the governing equation for a simply supported viscoelastic beam under a uniform distributed load, [19]. This equation is accompanied by a constitutive law presented in a hereditary integral form. Then, extending the procedures of the classical Zener model with the denomination of Standard Linear Solid (*SLS*) to a fractional Zener model. In order to obtain the quasi-static exact solution (i.e. the solution ignoring inertia effects), we will use the correspondence principle for classical Zener model [15], [22]. This principle relates mathematically the solution of a linear, viscoelastic boundary value problem to an analogous problem of an elastic body of the same geometry and under the same initial boundary conditions. Mention that not all problems can be solved by this principle, but only those for which the boundary conditions do not vary with the time. Applying the principle of d'Alembert, this structure will be analyzed in the dynamic case with a mixed algorithm that is based on the Galerkin's method for the spatial domain and then, the Laplace transform and binomial series expansion for the time domain.

Numerical results for both quasi-static and dynamic analysis are presented for classical model and fractional model, [2], [4], [10], [15]. The first rheological model will be accompanied of the comparative studies made with the help of graphical representations.

2. Beam governing equation

The differential equation of the transverse oscillations of a beam, which is subjected to uniformly distributed forces $\overline{p}$, is obtained from the dynamic equilibrium of an element having the length dx. If all the forces acting on the beam element are projected on the axis Oz, we find:

$$\overline{p}dx + T - q_i\, dx - T - \frac{\partial T}{\partial x}dx = 0 \qquad (1)$$

so

$$\overline{p} - q_i = \frac{\partial T}{\partial x} \qquad (2)$$

where T is the shearing force and q_i are the inertial force. Accordance with the d'Alembert's principle, (2) becomes

$$\frac{\partial T}{\partial x} = \overline{p} - \rho S\,\frac{\partial^2 w}{\partial t^2} \qquad (3)$$

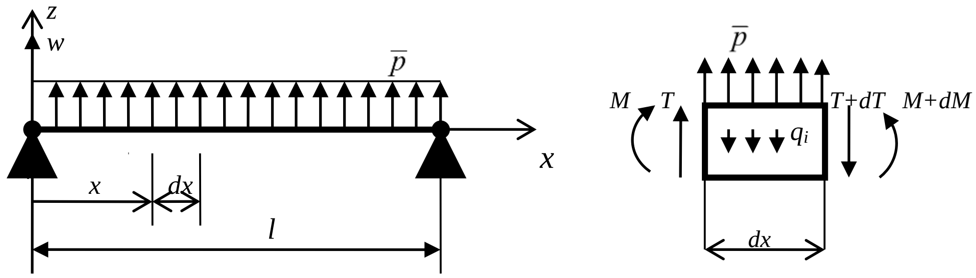

Fig. 1

with the following notations: ρ - density of material; S – cross sectional area; w - transverse displacement of the beam and t – time.
Let us now consider the moments in relation to section x + dx:

$$M - M - \frac{\partial M}{\partial x} dx + T dx - \overline{p}\frac{(dx)^2}{2} = 0 \qquad (4)$$

and approximating $(dx)^2 \approx 0$, is obtained

$$\frac{\partial M}{\partial x} = T \qquad (5)$$

and (2) becomes the differential equation:

$$\frac{\partial^2 M}{\partial x^2} = \overline{p} - \rho S \frac{\partial^2 w}{\partial t^2} \qquad (6)$$

Next, we will study the deformation of the beam element. The strain on the fiber ***cd*** of the deformed element is tensile (positive) and the strain on the fiber ***ab*** is compressive (negative). Because the strain is continuous throughout the cross section, will exist an axis where the strain is zero. This is named neutral axis and is rs in the Fig. 2.
If line mn there is at a distance z from the neutral axis, then the strain corresponding to mn is defined as:

$$\varepsilon = \frac{(\rho + z)\mathrm{d}\theta - \rho \mathrm{d}\theta}{\mathrm{d}x} = \frac{z\mathrm{d}\theta}{\rho\,\mathrm{d}\theta} = \frac{z}{\rho} \qquad (7)$$

where ρ is the curvature radius of the neutral axis and θ is the angle subtended by the deformed element.

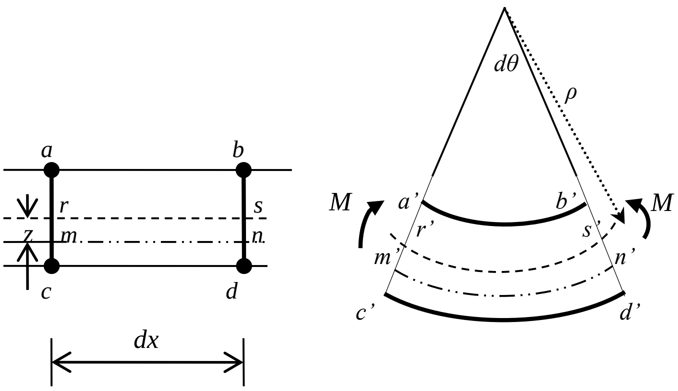

Fig. 2

The sections ac and bd remain plane and normal on the deformed axis rs of the beam after deformation, according to Bernoulli's hypothesis. In differential geometry, the expression of the curvature radius is the following:

$$\frac{1}{\rho} = \frac{\dfrac{\partial^2 w}{\partial x^2}}{\left[1 + \left(\dfrac{\partial w}{\partial x}\right)^2\right]} \qquad (8)$$

Practical applications show that $\frac{\partial w}{\partial x}$ is very small with respect to unity and so we can consider

$$\frac{1}{\rho} = \frac{\partial^2 w}{\partial x^2} \qquad (9)$$

Introducing (9) in (7), we get

$$\varepsilon(x,t) = z\,\frac{\partial^2 w(x,t)}{\partial x^2} \tag{10}$$

Let us consider a constitutive law in the hereditary integral form, [21]:

$$\sigma(x,t) = G(0)\varepsilon(x,t) - \int_0^t \frac{dG(t-\tau)}{\mathrm{d}\tau}\,\varepsilon(x,\tau)\,d\tau \tag{11}$$

where G is the relaxation modulus for the beam material, $G(0) = E$ (elasticity modulus) and σ the stress corresponding to the strain ε.

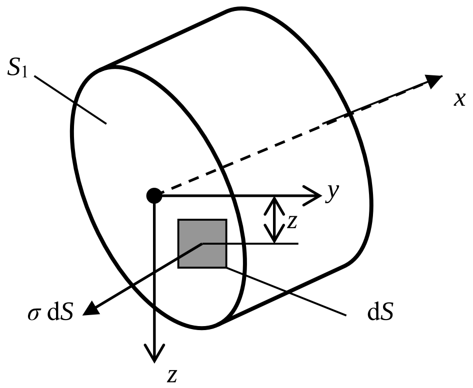

Fig. 3

The bending moment M at the beam cross section x may be expressed in terms of stress in the form of an integral:

$$M = \iint_S \sigma(x,t)\, z\, dydz \tag{12}$$

and using (11) this becomes:

$$M = G(0)I\,\frac{\partial^2 w(x,t)}{\partial x^2} - I\int_0^t \frac{dG(t-\tau)}{d\tau}\,\frac{\partial^2 w(x,\tau)}{\partial x^2}\,d\tau \tag{13}$$

where I is the moment of inertia of the section S with respect to the axis Oy.
Thus, after substitution of M in (6), we obtain the following integro - differential equation:

$$\rho S\,\frac{\partial^2 w(x,t)}{\partial t^2} + G(0)I\,\frac{\partial^4 w(x,t)}{\partial x^4} - \\ - I\int_0^t \frac{dG(t-\tau)}{d\tau}\,\frac{\partial^4 w(x,\tau)}{\partial x^4}\,d\tau = \overline{p} \tag{14}$$

Solving of equation (13) will lead to the finding of the transverse displacements for any boundary and initial conditions given for the viscoelastic beam subjected to a uniformly distributed loading p.

2. Rheological model

The stretching - relaxation process is analyzed using a Poynting model that contains two spring elements and one damping and consists in a serial connection of a spring and a Kelvin - Voigt model (Fig. 4). Now, formulating equilibrium and taking the kinematics of the rheological model into account, we get the following equalities:

$$\begin{aligned} \sigma &= k_1\varepsilon_1 & \varepsilon &= \varepsilon_1 + \varepsilon_2 \\ \sigma &= k_2\varepsilon_2 + \eta\dot{\varepsilon}_2 & \dot{\varepsilon} &= \dot{\varepsilon}_1 + \dot{\varepsilon}_2 \end{aligned} \tag{15}$$

where k_1, k_2 are the elastic modulus of the springs and η is the coefficient of viscosity of the dashpot

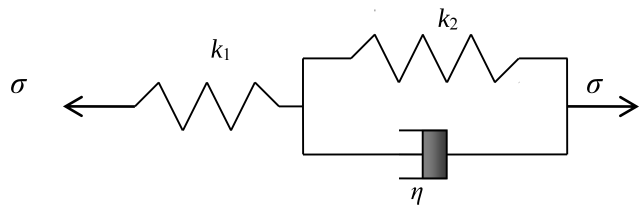

Fig. 4

These equations (15) lead to the differential equation

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{k_1} + \frac{\sigma - k_2\left(\varepsilon - \dfrac{\sigma}{k_1}\right)}{\eta} \tag{16}$$

or

$$(k_1 + k_2)\sigma + \eta\dot{\sigma} = k_1 k_2\,\varepsilon + k_1\eta\,\dot{\varepsilon} \tag{17}$$

where σ and ε depend on the time t.
Let us consider that the material of beam is in its relaxation phase, so, under constant strain $\varepsilon = \varepsilon_0$

the stress will decrease. In this case, the equation (17) becomes

$$\left(k_1 + k_2\right)\sigma + \eta\dot{\sigma} = k_1 k_2\, \varepsilon_0 \tag{18}$$

For the condition: $\sigma(0) = k_1\, \varepsilon_0$, the solution σ of equation (18) will be of the form

$$\sigma(t) = \frac{k_1 k_2 \varepsilon_0}{k_1 + k_2}\left(1 - e^{-\frac{t}{\tau_a}}\right) + k_1 \varepsilon_0 e^{-\frac{t}{\tau_a}} \tag{19}$$

where the relaxation time is

$$\tau_a = \frac{\eta}{k_1 + k_2} \tag{20}$$

Using the definition of the relaxation modulus $G(t)$, we have in this case:

$$\sigma(t) = G(t)\,\varepsilon_0 \tag{21}$$

From (19), we get

$$G(t) = \frac{k_1 k_2}{k_1 + k_2}\left(1 - e^{-\frac{t}{\tau_a}}\right) + k_1 e^{-\frac{t}{\tau_a}} \tag{22}$$

that is a function that depends on the beam material.

It will expand this result to the fractional calculus using the Mittag - Leffler functions, [15]:

$$E_\nu(-(t/\tau)) = \sum_{n=0}^{\infty}(-1)^n \frac{(t/\tau)^{\nu n}}{\Gamma(\nu n + 1)}\ , \qquad 0 < \nu < 1, \quad \tau > 0 \tag{23}$$

that for $\nu = 1$ reduce to $\exp(-t/\tau)$.
In the case $0 < \nu < 1$, the differential equation (16) becomes:

$$\sigma(t) + \frac{\eta}{k_1 + k_2}\frac{d^\nu \sigma}{dt^\nu} = = \frac{k_1 k_2}{k_1 + k_2}\varepsilon(t) + \frac{k_1 \eta}{k_1 + k_2}\frac{d^\nu \varepsilon}{dt^\nu} \tag{24}$$

The relaxation modulus will be now of the form:

$$G(t) = \frac{k_1 k_2}{k_1 + k_2}\left(1 + \frac{k_1}{k_2} E_\nu(-(t/\tau_a))\right) \tag{25}$$

where $E_\nu(0) = 1$.

4. Mixed method for solving integro – differential equation (14)

To solve the equation (14) associated with the boundary and initial conditions, we express the transverse deflection w by an expansion of the form

$$w_n(x,t) = \sum_{j=1}^{n} a_j(t)\varphi_j(x) \tag{26}$$

where $\varphi_j(x)$ is the j^{th} shape function and $a_j(t)$ is the corresponding time-dependent amplitude. For the spatial domain will be used the Galerkin's method and then, the techniques of the Laplace transform for the time domain. The shape functions are chosen to be linearly independent, orthonormal

$$\int_0^L \varphi_i(x)\varphi_j(x)\,dx = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{27}$$

and must satisfy all boundary condition for the convergence of Galerkin's method.
Although w_n satisfies the boundary conditions, it generally, does not satisfy equation (14). If the expansion (26) is substituted into (14) will result the residual function

$$\overline{R}_n(x,t) = = \rho S\left(\sum_{i=1}^{n}\ddot{a}_i(t)\varphi_i(x)\right) + G(0)I\left(\sum_{i=1}^{n}a_i(t)\varphi_i^{(4)}(x)\right) - -I\int_0^t \frac{dG(t-\tau)}{d\tau}\left(\sum_{i=1}^{n}a_i(\tau)\varphi_i^{(4)}(x)\right)d\tau - \overline{p} \tag{28}$$

Let us now consider the shape functions of the form

$$\varphi_j(x) = \sqrt{\frac{2}{l}}\sin\frac{j\pi x}{l},\ j = 1,2,....,n$$

and

$$\varphi_j^{(4)}(x) = \left(\frac{j\pi}{l}\right)^4 \varphi_j(x) = \lambda_j \varphi_j(x)$$

The Galerkin's method requires that the residual to be orthogonal to each of the chosen shape functions, so

$$\iint_{\Omega} \overline{R}_n(x,t)\varphi_j(x)\,dx\,dt = 0,\ i = 1,2,...,n \qquad (29)$$

where $\Omega = [0, L] \times [0, t]$. This leads to n equations verified by the functions $a_j(t)$:

$$\rho S \ddot{a}_j(t) - \lambda_j I \int_0^t \frac{dG(t-\tau)}{d\tau} a_j(\tau)\,d\tau + \\ + \lambda_j I G(0) a_j(t) = \int_0^L \overline{p}\varphi_j(x)dx \qquad (30)$$

Using the Laplace transform techniques and the initial conditions

$$\frac{d^k a_j}{dt^k}\Big|_{t=0} = \int_0^L \frac{\partial^k w(x,0)}{\partial t^k} \varphi_j(x)dx,\ k = 0,1,2,.... \qquad (31)$$

the functions $a_j(t)$ are determined independent of one another. Finally, an approximate value of the transverse deflection $w(x,t)$ will be found by (26).

5. Numerical example

Let us consider a simply supported beam under the uniform distributed load $\overline{p}$ = 4 N/m, which is applied as a creep load at $t = 0$ (the load is applied suddenly at $t = 0$ and then help constant). The length of the beam is l = 4 m, width b = 0.08 m and height h = 0.23 m. These input data lead to the moment of inertia of the rectangular section: $I = bh^3/12 = 8\cdot 10^{-5}\,\text{m}^4$. The material is taken to have the density of 1200 kg/m^3. For this example, we employ the three – parameter solid model (*SLS* model) with the relaxation modulus expressed by (19) - (20) [15], [17], where

$k_1 = 9.8\cdot 10^7$ N/m^2 , $k_2 = 2.45\cdot 10^7$ N/m^2 ,

$\eta = 2.74\cdot 10^8$ N-sec/ m^2

Classical quasi - static analysis

For ν = 1, the relaxation modulus will be

$$G(t) = 1.96\cdot 10^7 + 7.84\cdot 10^7 e^{-t/2.24}\ \text{N/m}^2 \qquad (32)$$

with t in seconds. To get the creep compliance $D(t)$, will be calculated first, [17]:

$$D_1(t) = 1 - \beta e^{-\frac{t}{\lambda}}$$

that corresponds to

$$G_1(t) = \frac{G(t)}{1.96\cdot 10^7} = 1 + \alpha e^{-\frac{t}{\tau}}, \quad \alpha = 4, \tau = 2.24$$

Then

$$D_1(t) = D(t)\cdot 1.96\cdot 10^7 = 1 - 0.8 e^{-\frac{t}{11.2}}$$

where

$$\beta = \frac{\alpha}{1+\alpha} = 0.8 \text{ and } \lambda = \tau(1+\alpha) = 11.2$$

Finally, the creep compliance that corresponds to the relaxation modulus (32), will be of the form:

$$D(t) = 0.51\cdot 10^{-7} - 0.408\cdot 10^{-7} e^{-\frac{t}{11.2}} \qquad (33)$$

The solving of the problem (30) - (31) will be accompanied by the appropriate boundary conditions for the simply supported beams:

$$w(0,t) = w(l,t) = 0 \\ M(0,t) = M(l,t) = 0 \qquad (34)$$

In the quasi-static case the inertial forces are ignored. An exact solution for the creep loading applied at t = 0 can be computed using the correspondence principle. We get the transverse displacements of the following form:

$$w(x,t) = \overline{w}(x)D(t) \qquad (35)$$

where $D(t)$ is given in (33) and $\overline{w}$ is the solution for a similar elastic structure that has the modulus of elasticity $E = 1$, [19]:

$$\overline{w}(x) = \frac{\overline{p}x(x^3 - 2Lx^2 + L^3)}{24I} \tag{36}$$

Classical dynamic analysis

For above input data, the equations (30) become:

$$28.8\ddot{a}_j(t) - 1065.4\, j^4 \int_0^t e^{-(t-\tau)/2.24} a_j(\tau)\,d\tau +$$
$$+ 2983 j^4 a_j(t) = \frac{2.7}{j}\left[(-1)^{j+1} + 1\right]$$

or

$$\begin{aligned} &\ddot{a}_j(t) - 37 j^4 \int_0^t e^{-(t-\tau)/2.24} a_j(\tau)\,d\tau + \\ &+ 104 j^4 a_j(t) = \frac{0.1}{j}\left[(-1)^{j+1} + 1\right] \end{aligned} \tag{37}$$

We remark from the form of (37) that $a_j(t)$ are zero for even values of j. Hence, only odd values of j need to be considered in the finding of the Galerkin's solution. These equations will have exact solutions determined with the techniques of Laplace transform. Since the initial conditions on w and its derivatives are zero, then, and the conditions on a_j and its derivatives are also zero. If $A_j(p)$ is Laplace transform ($\xrightarrow{L}$) of $a_j(t)$ and we use the Multiplication Theorem, we get

$$\int_0^t e^{-\frac{t-\tau}{2.24}} a_j(\tau)\,d\tau \xrightarrow{L} \frac{A_j(s)}{s + (1/2.24)} \tag{38}$$

and (37) becomes:

$$A_j(p) = \frac{0.2}{j} \cdot \frac{s + 0.45}{s(s^3 + 0.45s^2 + 104j^4 + 9.43j^4)} \tag{39}$$

Using the Newton iterative formula, we find a root of a polynomial that there is into parenthesis: $p_1 = 0.091$. Then, the partial fraction decomposition will lead to the solution $a_j(t)$ of (37). Finally, the transverse displacements w are approximated by (26) and for $n = 9$.

The Fig. 5 shows how the classical dynamic results oscillate around of the quasi-static results for the midpoint transverse deflection ($x = 2\ m$). It may be noted that the amplitude of vibration decreases with increasing time, due to the presence of the viscoelastic damping. In general, the amplitude of the oscillations depends upon the material density, the rate at which the loading is applied and the amount of the viscoelastic damping.

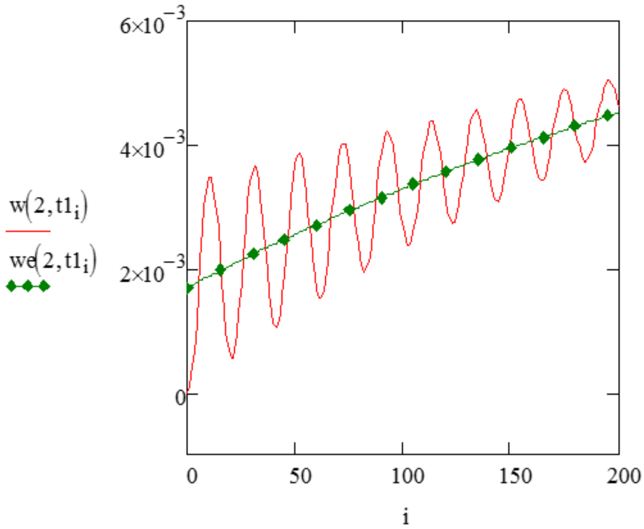

Fig. 5

Fractional dynamic model

Using the relaxation modulus (25) in the equation (30), we get

$$\begin{aligned} &\rho S \ddot{a}_j(t) - \lambda_j I \int_0^t \frac{d^\nu G(t-\tau)}{d\tau^\nu} a_j(\tau)\,d\tau + \\ &+ \lambda_j I G(0) a_j(t) = \int_0^L \overline{p}\varphi_j(x)dx, \quad \nu \in (0,1] \end{aligned} \tag{40}$$

Appealing to the theory of Laplace transform, equation (40) becomes:

$$\rho S s^2 A_j(s) - \lambda_j I\, L\left(\frac{d^\nu G(t-\tau)}{d\tau^\nu}\right) A_j(s) +$$
$$+ \lambda_j I G(0) A_j(s) = \frac{1}{s}\int_0^L \overline{p}\varphi_j(x)dx, \quad \nu \in (0,1]$$

where

$$\frac{d^{\nu}}{dt^{\nu}}f(t) \quad \xrightarrow{L} \quad s^{\nu}F(s)-\sum_{k=1}^{n} f^{(k-1)}(0)s^{\nu-k},$$

$F(s)$ – Laplace transform of $f(t)$ and

$\lambda_j = (j\pi / l)^4$.

According to the paper [15], if the

$$L\left(\frac{d^{\nu}G(t-\tau)}{d\tau^{\nu}}\right) = -s^{\nu}[\widetilde{G}(s)-s^{\nu-1}G(0)] = \tag{41}$$

$$= \mu\,\frac{-s^{\nu}}{s}\frac{1+\tau_2^{\nu}s^{\nu}}{1+\tau_a^{\nu}s^{\nu}} + s^{\nu-1}\mu\left(1+\frac{k_1}{k_2}\right)E_{\nu}(0)$$

where

$\widetilde{G}(s)$ - Laplace transformation of $G(u)$,
$u = t-\tau$,

$$\mu = \frac{k_1k_2}{k_1+k_2} = 1.96\cdot 10^7,\ \tau_2^{\nu} = \frac{\eta}{k_2} = 11.2\,\text{sec},$$
$$\tau_a^{\nu} = \frac{\eta}{k_1+k_2} = 2.24\,\text{sec},$$

$\tau_2^{\nu}, \tau_a^{\nu}$ are the retardation time and relaxation time.

Thus, the Laplace transformation (41) becomes:

$$L\left(\frac{d^{\nu}G(t-\tau)}{d\tau^{\nu}}\right) = 1.96\cdot s^{\nu-1}\,\frac{-1-11.2s^{\nu}+5+11.2s^{\nu}}{1+2.24s^{\nu}}$$

and from (32), we get

$$G(0) = 1.96\cdot 10^7 + 7.84\cdot 10^7 = 9.8\cdot 10^7 .$$

Finally, the equation (40) will be of the form

$$28.8\cdot s^2A(s) - 2391j^4\,\frac{s^{\nu-1}}{1+2.24s^{\nu}}\,A(s) + + 2989j^4A(s) = \frac{2.27}{j} \tag{42}$$

So

$$A_j(s) = \frac{0.2}{j}\cdot C_j(s),$$
$$C_j(s) = \frac{0.45s^{1-\nu}+s}{s(s^3+104sj^4-37j^4+0.45s^{3-\nu}+46.4j^4s^{1-\nu})} \tag{43}$$

We notice that for ν = 1 we have (39).

To obtain the original function that corresponds to (43), we write $C_j(s)$ for $\beta = 0.45$ in the following form:

$$C_j(s) = \frac{1+\beta s^{-\nu}}{(s^3+104sj^4-37j^4)\left[1+\dfrac{\beta s^{-\nu}(s^3+104j^4s)}{s^3+104sj^4-37j^4}\right]} =$$

$$= (1+\beta s^{-\nu})\sum_{k=0}^{\infty}\frac{(-1)^k s^{-k\nu}\beta^k(s^3+104j^4s)^k}{(s^3+104j^4s-83\beta j^4)^{k+1}} =$$

$$= (1+\beta s^{-\nu})\sum_{k=0}^{\infty}\frac{(-1)^k s^{-k\nu}\beta^k}{(s^3+104j^4s)\left(1-\dfrac{83j^4\beta}{s^3+104j^4s}\right)^{k+1}} =$$

$$= (1+\beta s^{-\nu})\sum_{k=0}^{\infty}\frac{(-1)^k\beta^k s^{-k\nu}}{s^3+104j^4s}\sum_{i=0}^{\infty}\binom{k+i}{i}\frac{83^i\beta^i j^{4i}}{(s^3+104j^4s)^i}$$

which was used equality

$$(1-z)^{-m} = \sum_{r=0}^{\infty}\binom{m+r-1}{r}z^r \ \text{ and } \ \binom{m+r-1}{r}$$

is binomial coefficient. Still obtain

$$C_j(s) = (1+\beta s^{-\nu})\sum_{k=0}^{\infty}(-1)^k\sum_{i=0}^{\infty}\binom{k+i}{i}\frac{83^i\beta^{i+k}j^{4i}s^{-k\nu}}{s^{3i+3}(1+104j^4s^{-2})^{i+1}} =$$

$$= (1+\beta s^{-\nu})\sum_{k=0}^{\infty}(-1)^k\sum_{i=0}^{\infty}\binom{k+i}{i}83^i\beta^{k+i}j^{4i}s^{-k\nu-3i-3}\cdot$$

$$\cdot\sum_{m=0}^{\infty}(-1)^m\binom{i+m}{m}104^m j^{4m}s^{-2m} =$$

$$= (1+\beta s^{-\nu})\cdot$$

$$\cdot\sum_{k=0}^{\infty}\left[\begin{array}{l}\sum_{i=0}^{\infty}\binom{i+k}{i}\beta^{k+i}83^i\cdot \\ \cdot\left(\sum_{m=0}^{\infty}\binom{m+i}{m}104^m j^{4(m+i)}(-1)^{k+m}s^{-k\nu-3i-2m-3}\right)\end{array}\right] =$$

$$= A1_j(s) + A2_j(s)$$

Applying the inverse Laplace transformation formula:

$$s^{-n-1}\xrightarrow{L^{-1}}\frac{t^n}{n!}$$

we get

$$A1_j(s) = \sum_{k=0}^{\infty}(-1)^k\sum_{i=0}^{\infty}\binom{k+i}{i}\frac{83^i\beta^{i+k}j^{4i}s^{-k\nu}}{s^{3i+3}(1+104j^4s^{-2})^{i+1}}$$

$$a1_j(t) = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(k+i)!}{i!k!}\beta^{k+i}83^i\sum_{m=0}^{\infty}\frac{(m+i)!}{m!i!}104^m j^{4(m+i)}\cdot$$
$$\cdot(-1)^{k+m}\frac{t^{k\nu+3i+2m+2}}{\Gamma(k\nu+3i+2m+3)}$$

Similarly, we find for

$$A2_j(s) = \beta s^{-\nu}\sum_{k=0}^{\infty}(-1)^k\sum_{i=0}^{\infty}\binom{k+i}{i}\frac{83^i\beta^{i+k+1}j^{4i}s^{-(k+1)\nu}}{s^{3i+3}(1+104j^4s^{-2})^{i+1}}$$

$$a2_j(t) = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(k+i)!}{i!k!}\beta^{k+i+1}83^i\sum_{m=0}^{\infty}\frac{(m+i)!}{m!i!}104^m j^{4(m+i)}$$
$$\cdot$$
$$\cdot(-1)^{k+m}\frac{t^{(k+1)\nu+3i+2m+2}}{\Gamma((k+1)\nu+3i+2m+3)}$$

Finally, the transverse deflection is equal to

$$w_n(x,t) = \sum_{j=1}^{n}\frac{0.2}{j}\cdot(a1_j(t)+a2_j(t))\,\varphi_j(x)$$

6. Conclusions

In this paper is presented an efficient method for numerical simulation of a quasi-static and dynamic response of viscoelastic beam both classical Zener model and for fractional Zener model. The proposed method can be easily extended to the complex viscoelastic structures calculus.

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