Geometric Probabilities in Euclidean Space E_3

G. CARISTI, A. PUGLISI, E. SAITTA
University of Messina
Department of Economics
Via dei Verdi, 75, 98122- Messina
ITALY

Abstract: In the last year G. Caristi and M. Stoka [2] have considered Laplace type problem for different lattice with or without obstacles and compute the associated probabilities by considering bodies test not-uniformly distributed. We consider a lattice with fundamental cell a parallelepiped in the Ecuclidean Space E_3. We compute the probability that a random segment of constant length, with exponential distribution, intersects a side of the lattice.

Key–Words: Geometric Probability, stochastic geometry, random sets, random convex sets and integral geometry.

1 Introduction

Classical geometry defined the Euclidean plane and Euclidean three-dimensional space using certain postulates, while the other properties of these spaces were deduced as theorems.

When algebra and mathematical analysis became developed enough, this relation reserved and now it is more common to define Euclidean space using Cartesian coordinates and the ideas of analytic geometry.

It means that points of the space are specified with collection of real numbers, and geometric shapes are defined as equation and inequalities. This approach brings the tools algebra and calculus to bear on question of geometry and has the advantage that it generalizes easily to Euclidean spaces of more than three dimensions.

In Euclidean space E_3, a lattice with fundamental cell formed by parallelepiped have three sides and it's possible to show that a random segment with exponential distribution of constant length intersects a side of a lattice and a side of fundamental cell, too.

Buffon's problem for an arbitrary convex body K and a lattice of parallelograms in the Euclidean space is considered for two different types of lattices in the space, for those lattices whose fundamental cell in a triangle or a regular hexagon.

In Buffon's problem is solved for a lattice of right-angled parallelepipeds in the 3-dimensional space (which will be denoted here by R_1) and an arbitrary convex body of revolution.

Let K be an arbitrary convex body of revolution with centroid S and oriented axis of rotation d. Clearly, the axis d is determined by the angle θ between d and the z-axis and by the angle ϕ between the projection of d on the xy-plane and the x-axis and we express this by writing $d = d(\theta, \phi)$. If for a given $d = d(\theta, \phi)$, the body K is tangent to the xy-plane such that the centroid S lies in the upper half-space, we denote by $p(\theta, \phi)$ the distance from S to the xy-plane. Then the length of the projection of K on the z-axis is given by $L(\theta, \phi) = p(\theta, \phi) + p(\pi - \theta, \phi)$.

Note that $p(\theta, \phi)$ does actually depend only on the angle θ and moreover, since K is a body of revolution about the axis d the value $p(\theta, \phi)$ is invariant to any rotation about this axis, say by an ψ. Now let F be a fundamental cell of the lattice R and assume that the two 3-dimensional random variables defined by the coordinates of S and by the triple (θ, ϕ, ψ) are uniformly distributed in the cell F and in $[0, \pi] \times [0, 2\pi] \times [0, 2\pi]$ respectively. We are interested in the probability $p_{K,R}$ that the body K intersects the lattice R. Furthermore, we will assume, as it is done in all papers cited here, that the body K is small with respect to the lattice R. In order to recall briefly this concept, consider for fixed $(\theta, \phi)[0, \pi] \times [0, 2\pi]$ the set of all points $P \in F$ for which the body K with centroid P and rotation axis $d = d(\theta, \phi)$ does not intersect the boundary ∂F and let $F(\theta, \phi)$ be the closure of this open subset of F. We say that the body K is small with respect to R, if the polyhedrons sides of $F(\theta, \phi)$ and F are then clearly pairwise parallel.

Denote by M_F the set of all test bodies K whose centroid S lies in F and by N_F the set of bodies K that are completely contained in F. Of course, we can identify these sets with subsets of R^6 and if μ denotes the Lebesgue measure then the probability is given by

$$p_{K,R} = 1 - \frac{\mu(N_F)}{\mu(M_F)}. \quad (1)$$

Using the cinematic measure

$$dK = dx \wedge dy \wedge dz \wedge d\Omega \wedge d\psi \quad (2)$$

where x,y,z are the coordinates of S, $d\Omega = \sin\theta d\theta \wedge d\varphi$ and ψ is an angle of rotation about d we can compute

$$\mu(M_F) = 8\pi^2 Vol(F), \quad (3)$$

$$\mu(N_F) = 2\pi \int_0^{2\pi} \left(\int_0^{\pi} VolF(\theta,\varphi) \sin\theta d\theta \right) d\varphi,$$

which leads to

$$p_{K,R} = 1 - \frac{1}{4\pi Vol(F)}.$$

$$\cdot \int_0^{2\pi} \left(\int_0^{\pi} VolF(\theta,\varphi) \sin\theta d\theta \right) d\varphi. \quad (4)$$

2 Main results

Now, we consider, in Euclidean space E_3, a lattice with fundamental cell a parallelepiped and we determine the probability that a segment of constant length and random direction of exponential distribution intersects a side of the lattice.

Theorem 1. Let, in Euclidean place E_3, a lattice $\Re(a_1, a_2, a_3)$ with fundamental cell P_0 a parallelepiped of sides a_1, a_2, a_3 and $volP_o = a_1a_2a_3$ The probability that a segment s of constant length $l < \frac{1}{2}\inf(a_1, a_2, a_3)$ and random direction of exponential distribution intersects a side of lattice $\Re$, i.e. the probability P_{int} that the segment s intersects a side of fundamental cell P_0 is:

$$P_{int} = \frac{2l}{a_1a_2a_3\left(1-e^{-\frac{\pi}{2}}\right)^2} \left\{ \frac{1}{10}a_1a_2\left(1-e^{-\frac{\pi}{2}}\right) \right.$$

$$\left(1+3e^{-\frac{\pi}{2}}\right) + \frac{1}{10}a_1a_3\left(1+e^{-\frac{\pi}{2}}\right)\left(1+3e^{-\frac{\pi}{2}}\right) +$$

$$\frac{1}{5}a_2a_3\left(1-e^{-\frac{\pi}{2}}\right) - \frac{l}{10}\left[\frac{a_1}{5}\left(1+e^{-\frac{\pi}{2}}\right)\left(3-4e^{-\frac{\pi}{2}}\right) + \right.$$

$$\frac{a_2}{2}\left(1-e^{-\frac{\pi}{2}}\right)\left(1+2e^{-\frac{\pi}{2}}\right) + \frac{a_3}{2}\left(1+e^{-\frac{\pi}{2}}\right)$$

$$\left.\left.\left(1+2e^{-\frac{\pi}{2}}\right) + \frac{l^2}{75}\left(1+e^{-\frac{\pi}{2}}\right)\left(1+2e^{-\frac{\pi}{2}}\right)\right]\right\}.$$

Proof. The position of the segment s and determined by its middle point and by the directors cosines of the line support of s:

$$\alpha_1 = \cos\theta, \qquad \alpha_2 = \sin\theta\cos\varphi, \qquad \alpha_3 = \sin\theta\sin\varphi.$$

To compute the probability P_{int} we consider the parallelepiped P_0^* with the vertex $A\left(\frac{l}{2}\cos\theta,\ \frac{l}{2}\sin\theta\cos\varphi,\ \frac{l}{2}\sin\theta\sin\varphi\right)$, with parallel sides of P_0 and with lengths

$$a_1 - l\cos\theta,\ \ a_2 - l\sin\theta\cos\varphi,\ \ a_3 - l\sin\theta\sin\varphi.$$

Then

$$volP_o^* = (a_1 - l\cos\theta)(a_2 - l\sin\theta\cos\varphi)$$

$$(a_3 - l\sin\theta\sin\varphi). \quad (5)$$

We denote with M, the set of segments s that they have the middle point in the cell P_0, and with N the set of segments s entirely conteined in the cell P_o,we have [3]:

$$P_{int} = 1 - \frac{\mu(N)}{\mu(M)}, \quad (6)$$

where μ is the Lebesgue measure in the space E_3.

To compute the measure $\mu(M)$ and $\mu(N)$ we use the kinematic measure of Blaschke in the space [1]:

$$dk = d\psi \wedge d\varphi \wedge \sin\theta d\theta \wedge dx \wedge dy \wedge dz,$$

where $\varkappa$, y, z are the coordinate of middle point of s, φ and θ the fixed angle and ψ an angle of rotation. We have

$$0 \le \varphi \le \frac{\pi}{2}, \qquad 0 \le \theta \le \frac{\pi}{2}, \qquad 0 \le \psi \le 2\pi.$$

We suppose that φ and θ are indipendent random variables with the same distribution of probability $f(\varphi) = e^{-\varphi}$, $f(\theta) = e^{-\theta}$, we have

$$\mu(M) = 2\pi volP_0 \int_0^{\frac{\pi}{2}} e^{-\varphi} d\varphi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin\theta d\theta =$$

$$2\pi a_1 a_2 a_3 \int_0^{\frac{\pi}{2}} e^{-\varphi} d\varphi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin\theta d\theta. \quad (7)$$

Then, considering formula (5), we can write

$$\mu(N) = 2\pi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin\theta d\theta \int_0^{\frac{\pi}{2}} e^{-\varphi} \left(volP_0^*\right) d\varphi =$$

$$2\pi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin\theta d\theta \int_0^{\frac{\pi}{2}} [a_1 a_2 a_3 -$$

$$l \left(a_1 a_2 \sin\theta \sin\varphi + a_1 a_3 \sin\theta \cos\varphi + a_2 a_3 \cos\theta\right) +$$

$$l^2 \left(a_1 \sin^2\theta \sin\varphi \cos\varphi + a_2 \sin\theta \cos\theta \sin\varphi + \right.$$

$$\left. a_3 \sin\theta \cos\theta \cos\varphi\right) - l^3 \sin^2\theta \cos\theta \sin\varphi \cos\varphi \big] e^{-\varphi} d\varphi. \quad (8)$$

Replacing in the (6), formulas (7) and (8) we have that:

$$P_{int} = \frac{l}{a_1 a_2 a_3 \int_0^{\frac{\pi}{2}} e^{-\varphi} d\varphi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin\theta d\theta}$$

$$\int_0^{\frac{\pi}{2}} e^{-\theta} \sin\theta d\theta \int_0^{\frac{\pi}{2}} [a_1 a_2 \sin\theta \cos\varphi + a_1 a_3 \sin\theta \cos\varphi +$$

$$a_2 a_3 \sin\theta - l \left(a_1 \sin^2\theta \sin\varphi \cos\varphi + \right.$$

$$\left. a_2 \sin\theta \cos\theta \sin\varphi + a_3 \sin\theta \cos\theta \cos\varphi\right)$$

$$+ l^2 \sin^2\theta \cos\theta \sin\varphi \cos\varphi] e^{-\varphi} d\varphi.$$

First, we compute the integral

$$I_1(\theta) = \int_0^{\frac{\pi}{2}} [a_1 a_2 \sin\theta \sin\varphi + a_1 a_3 \sin\theta \cos\varphi +$$

$$a_2 a_3 \cos\theta - l \left(\frac{a_1}{2} \sin^2\theta \sin 2\varphi + a_2 \sin\theta \cos\theta \sin\varphi + \right.$$

$$\left. a_3 \sin\theta \cos\theta \cos\varphi\right)$$

$$+ \frac{l^2}{2} \sin^2\theta \cos\theta \sin 2\varphi \Big] e^{-\varphi} d\varphi. \quad (9)$$

We have that:

$$\int_0^{\frac{\pi}{2}} e^{-\varphi} d\varphi = 1 - e^{-\frac{\pi}{2}};$$

$$\int_0^{\frac{\pi}{2}} e^{-\varphi} \sin\varphi d\varphi = \frac{1}{2}\left(1 - e^{-\frac{\pi}{2}}\right),$$

$$\int_0^{\frac{\pi}{2}} e^{-\varphi} \cos\varphi d\varphi = \frac{1}{2}\left(1 + e^{-\frac{\pi}{2}}\right);$$

$$\int_0^{\frac{\pi}{2}} e^{-\varphi} \sin 2\varphi d\varphi = \frac{2}{5}\left(1 + e^{-\frac{\pi}{2}}\right),$$

$$\int_0^{\frac{\pi}{2}} e^{-\varphi} \cos 2\varphi d\varphi = \frac{1}{5}\left(1 + e^{-\frac{\pi}{2}}\right). \quad (10)$$

With these values we have

$$I_1(\theta) = \frac{1}{2} a_1 \left[a_2 \left(1 - e^{-\frac{\pi}{2}}\right) + a_3 \left(1 + e^{-\frac{\pi}{2}}\right)\right]$$

$$\sin\theta + a_2 a_3 \left(1 - e^{-\frac{\pi}{2}}\right) \cos\theta -$$

$$\frac{l}{2} \left\{ \frac{2}{5} a_1 \left(1 + e^{-\frac{\pi}{2}}\right) \sin^2\theta + \left[a_2 \left(1 - e^{-\frac{\pi}{2}}\right) + \right.\right.$$

$$\left.\left. a_3 \left(1 + e^{-\frac{\pi}{2}}\right)\right] \sin\theta \cos\theta \right\} +$$

$$\frac{l^2}{5} \left(1 + e^{-\frac{\pi}{2}}\right) \sin^2\theta \cos\theta.$$

Then

$$I_2 = \int_0^{\frac{\pi}{2}} e^{-\theta} I_1(\theta) \sin\theta d\theta =$$

$$\int_0^{\frac{\pi}{2}} e^{-\theta} \left(\frac{1}{2} a_1 \left[a_2 \left(1 - e^{-\frac{\pi}{2}}\right) + a_3 \left(1 + e^{-\frac{\pi}{2}}\right)\right]\right.$$

$$\sin^2\theta + \frac{1}{2} a_2 a_3 \left(1 - e^{-\frac{\pi}{2}}\right) \sin 2\theta -$$

$$\frac{l}{2}\left\{\frac{2}{5}a_1\left(1+e^{-\frac{\pi}{2}}\right)\sin^3\theta+\left[a_2\left(1-e^{-\frac{\pi}{2}}\right)+\right.\right.$$

$$\left.\left.a_3\left(1+e^{-\frac{\pi}{2}}\right)\right]\sin^2\theta\cos\theta\right\}+$$

$$\frac{l^2}{5}\left(1+e^{-\frac{\pi}{2}}\right)\sin^3\theta\cos\theta)d\theta.$$

Replacing θ with φ, in (11) we have that

$$\int_0^{\frac{\pi}{2}} e^{-\theta}\sin^2\theta d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}} e^{-\theta}\left(1-\cos 2\theta\right)d\theta=$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} e^{-\theta}d\theta-\frac{1}{2}\int_0^{\frac{\pi}{2}} e^{-\theta}\cos 2\theta d\theta=$$

$$\frac{1}{2}\left(1-e^{-\frac{\pi}{2}}\right)-\frac{1}{10}\left(1+e^{-\frac{\pi}{2}}\right)=\frac{2}{5}-\frac{3}{5}e^{-\frac{\pi}{2}}. \quad (11)$$

In the same way

$$\int_0^{\frac{\pi}{2}} e^{-\theta}2\sin\theta d\theta=\frac{2}{5}\left(1+e^{-\frac{\pi}{2}}\right). \quad (12)$$

To compute the integral

$$J=\int_0^{\frac{\pi}{2}} e^{-\theta}\sin^3\theta d\theta,$$

we denote with

$$J_1=\int_0^{\frac{\pi}{2}} e^{-\theta}\cos^2\theta\sin\theta d\theta,$$

$$J_2=\int_0^{\frac{\pi}{2}} e^{-\theta}\cos 2\theta\cos\theta d\theta,$$

$$J_3=\int_0^{\frac{\pi}{2}} e^{-\theta}\sin 2\theta\sin\theta d\theta,$$

$$J_4=\int_0^{\frac{\pi}{2}} e^{-\theta}\sin 2\theta\cos\theta d\theta.$$

By integration by parts we obtain

$$J_1=-\frac{1}{10}+\frac{3}{10}e^{-\frac{\pi}{2}},$$

$$J_2=\frac{3}{10}+\frac{1}{10}e^{-\frac{\pi}{2}},$$

$$J_3=\frac{1}{5}+\frac{2}{5}e^{-\frac{\pi}{2}},$$

$$J_4=\frac{2}{5}-\frac{1}{5}e^{-\frac{\pi}{2}}. \quad (13)$$

With these values follows

$$J=\frac{1}{2}\int_0^{\frac{\pi}{2}} e^{-\theta}\left(1-\cos 2\theta\right)\sin\theta d\theta=$$

$$\frac{1}{2}\left[\int_0^{\frac{\pi}{2}} e^{-\theta}\sin\theta d\theta-\int_0^{\frac{\pi}{2}} e^{-\theta}\cos 2\theta\sin\theta d\theta\right]=$$

$$\frac{1}{2}\left[\frac{1}{2}\left(1-e^{-\frac{\pi}{2}}\right)-J_1\right]=\frac{3}{10}-\frac{2}{5}e^{-\frac{\pi}{2}}. \quad (14)$$

Now we compute the integral

$$L=\int_0^{\frac{\pi}{2}} e^{-\theta}\sin^2\theta\cos\theta d\theta.$$

Considering formula (11) and (14) we have that:

$$L=\frac{1}{2}\int_0^{\frac{\pi}{2}} e^{-\theta}\left(1-\cos 2\theta\right)\cos\theta d\theta=$$

$$\frac{1}{2}\left[\int_0^{\frac{\pi}{2}} e^{-\theta}\cos\theta d\theta-\int_0^{\frac{\pi}{2}} e^{-\theta}\cos 2\theta\cos\theta d\theta\right]=$$

$$\frac{1}{2}\left[\frac{1}{2}\left(1+e^{-\frac{\pi}{2}}\right)-\left(\frac{3}{10}+\frac{1}{10}e^{-\frac{\pi}{2}}\right)\right]=$$

$$\frac{1}{10}+\frac{1}{5}e^{-\frac{\pi}{2}}. \quad (15)$$

In the end, we compute the integral

$$M=\int_0^{\frac{\pi}{2}} e^{-\theta}\sin^3\theta\sin\theta\cos\theta d\theta.$$

We have that

$$M=\int_0^{\frac{\pi}{2}} e^{-\theta}\sin^2\theta\sin\theta\cos\theta d\theta=$$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} e^{-\theta}\left(1-\cos 2\theta\right)\sin 2\theta d\theta=$$

$$\frac{1}{4}\left(\int_0^{\frac{\pi}{2}} e^{-\theta}\sin 2\theta d\theta-\frac{1}{2}\int_0^{\frac{\pi}{2}} e^{-\theta}\sin 4\theta d\theta\right).$$

By two integrations by parts we obtain

$$\int_0^{\frac{\pi}{2}} e^{-\theta} \sin 4\theta d\theta = \frac{4}{15}\left(1 - e^{-\frac{\pi}{2}}\right)$$

This formula and fourth formula (11) give us

$$M = \frac{1}{4}\left[\frac{2}{5}\left(1 + e^{-\frac{\pi}{2}}\right) - \frac{2}{15}\left(1 - e^{-\frac{\pi}{2}}\right)\right] = \frac{1}{15} + \frac{2}{15}e^{-\frac{\pi}{2}}. \quad (16)$$

Considering relations (11), (12), (15), (16) and (17) we have

$$I_2 = \frac{1}{10}a_1a_2\left(1 - e^{-\frac{\pi}{2}}\right)\left(1 + 3e^{-\frac{\pi}{2}}\right) + \frac{1}{10}a_1a_3\left(1 + e^{-\frac{\pi}{2}}\right)\left(1 + 3e^{-\frac{\pi}{2}}\right) + \frac{1}{5}a_2a_3\left(1 - e^{-\frac{\pi}{2}}\right) - \frac{l}{10}\left[\frac{a_1}{5}\left(1 + e^{-\frac{\pi}{2}}\right)\left(3 - 4e^{-\frac{\pi}{2}}\right) + \frac{a_2}{2}\left(1 - e^{-\frac{\pi}{2}}\right)\left(1 + 2e^{-\frac{\pi}{2}}\right) + \frac{a_3}{2}\left(1 + e^{-\frac{\pi}{2}}\right)\left(1 + 2e^{-\frac{\pi}{2}}\right)\right] + \frac{l^2}{75}\left(1 + e^{-\frac{\pi}{2}}\right)\left(1 + 2e^{-\frac{\pi}{2}}\right).$$

By formulas (9), (11) and (18) follows

$$P_{int} = \frac{2l}{a_1a_2a_3\left(1 - e^{-\frac{\pi}{2}}\right)^2}\left\{\frac{1}{10}a_1a_2\left(1 - e^{-\frac{\pi}{2}}\right)\left(1 + 3e^{-\frac{\pi}{2}}\right) + \frac{1}{10}a_1a_3\left(1 + e^{-\frac{\pi}{2}}\right)\left(1 + 3e^{-\frac{\pi}{2}}\right) + \frac{1}{5}a_2a_3\left(1 - e^{-\frac{\pi}{2}}\right) - \frac{l}{10}\left[\frac{a_1}{5}\left(1 + e^{-\frac{\pi}{2}}\right)\left(3 - 4e^{-\frac{\pi}{2}}\right) + \frac{a_2}{2}\left(1 - e^{-\frac{\pi}{2}}\right)\left(1 + 2e^{-\frac{\pi}{2}}\right) + \frac{a_3}{2}\left(1 + e^{-\frac{\pi}{2}}\right)\left(1 + 2e^{-\frac{\pi}{2}}\right) + \frac{l^2}{75}\left(1 + e^{-\frac{\pi}{2}}\right)\left(1 + 2e^{-\frac{\pi}{2}}\right)\right]\right\}.$$

If $a_2 \to +\infty$, $a_3 \to +\infty$, the lattice R becomes a lattice of parallel planes with distance a_1 and the probability P_{int} can be write

$$P_{int} = \frac{2l}{5a_1}\frac{1 + e^{-\frac{\pi}{2}}}{1 - e^{-\frac{\pi}{2}}}.$$

3 Conclusion

The aim of the paper was to study a particular Laplace type problem in the Euclidean Space E3 with body test not-uniformly distributed. The result remark the interest for the geometric probability and its applications in the 20th century. Infact, geometric probability can help teachers to answer convincingly the usual student s question What is it for? . We can have important applications in biomedicine (dermatology, nephrology, oncology, cardiology etc), material engineering (quantitative analysis of metals, composites, concrete, ceramic), geology etc.

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