Stability of Three-dimensional Cellular Auxetic Metamaterials with a Re-Entrant Structure. Part 2: Global Buckling

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Abstract: - The work is devoted to the study of global buckling of structures formed from three-dimensional re-entrant auxetic cells, and it is a companion paper to the previous part of the authors' work [to be published in WSEAS Trans Appl Theor Mech], wherein the stability of an individual auxetic cell was considered. In the present part, the problem is transferred from the level of a single cell to the level of a finite structure with due account for the number of cells in the longitudinal and transverse directions, the cross-sectional shape, boundary constraints, and the direction of compression. The structure is modeled as an equivalent orthotropic beam, the effective physical and mechanical properties of which are determined by the parameters of the periodic auxetic lattice. Analytical expressions are obtained for the global critical stresses under compression along the principal axes of orthotropy. The limit state is taken as the smallest of three quantities: the local critical stress of the cell members, the global critical stress of the structure, and the stress corresponding to the ultimate strength of the material. It is established that, under compression along the vertical axis, an increase in the re-entrant angle and in the relative height of the cell reduces the global critical stress, while the relative thickness of the members has a significant effect on both local and global stability. For the transverse directions, the dependence of the global buckling mode on the ratio of the transverse dimensions of the structure is revealed. The obtained boundaries between local and global buckling mechanisms make it possible to evaluate the load-bearing capacity of auxetic lattice structures and to select the cell parameters with regard to the loading direction, structural slenderness, and boundary constraints rationally.

Key-Words: - Auxetic metamaterials, three-dimensional cellular lattice, re-entrant structure, Timoshenko beam, buckling, elastic stability, global buckling, geometric nonlinearity

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1 Introduction

In engineering problems associated with the application of auxetic metamaterials, it is important not only to obtain the required effective elastic properties, but also to determine which limit-state mechanism is realized first in structures [1], [2], [3]. For periodic lattice metamaterials, this is especially important, since the geometry of the unit cell simultaneously determines the compliance of individual bar elements, the effective orthotropic properties of the equivalent medium, and the deformation capacity of the entire structure [4]. Consequently, in the transition from an individual auxetic cell to a structure composed of such cells, the critical state may be associated with local buckling of the lattice members, global buckling as a beam, or with attainment of the ultimate strength of the material.

The mechanics of cellular and lattice metamaterials shows that their macroscopic moduli and limit states are governed by bending, stretching, shear, and buckling of the walls or struts of the unit cell. The relationship between cell geometry and effective properties was established in classical models of two-dimensional cellular materials, where elastic properties and failure mechanisms are expressed through the deformations of the cell members [1], [2]. Subsequently, analytical models of cellular structures were extended with due account for various mechanisms of wall deformation, including

bending, stretching, and hinged rotations. This made it possible to describe not only the elastic and shear moduli, but also the directional dependence of Poisson's ratios [5]. For auxetic mechanical metamaterials, such geometry dependence is of fundamental importance, since their macroscopic properties are formed by the topology and kinematics of the periodic structure [6], [7], [8].

However, most analytical models of three-dimensional auxetic cells are focused either on determining effective elastic properties or on describing the behavior of a periodic fragment at the unit-cell level [3], [4]. For structures composed of such cells, this is insufficient. As the number of cells and the length of the structure increase, a new scale appears in the problem — the scale of the entire structure [9], [10], [11]. In this case, even with unchanged unit-cell geometry, the critical stress depends on the slenderness of the structure, the end-restraint conditions, the cross-sectional shape, and the loading direction. Therefore, the equivalent orthotropic model must be considered together with the geometric parameters of the lattice, since it is precisely these parameters that determine the effective physical and mechanical properties of the medium.

Buckling of mechanical metamaterials plays a dual role. On one hand, it could be used as a functional mechanism for structural reconfiguration, shape change, or obtaining a controlled auxetic effect [8], [12], [13], [14]. On the other hand, in structures, buckling limits the load-bearing capacity. Numerical and experimental studies of auxetic structures show that local buckling modes of the cell members may occur before the ultimate strength is reached and may significantly affect the subsequent deformation of the entire structure [15], [16], [17], [18], [19]. Therefore, when designing three-dimensional auxetic lattices, it is necessary to distinguish between local buckling modes, characteristic size of which is comparable to the size of the unit cell, and global modes corresponding to bending of the structure as a whole.

The first part of the study was devoted to criteria for local buckling of vertical struts and inclined members [20]. The results obtained provide a basis for developing a theory of stability of auxetic metamaterials at the mesoscale level. However, analysis at the cell level does not allow one to determine which of the buckling modes will govern the structure: local or global. The boundary between these mechanisms depends not only on the parameters of the auxetic cell, but also on the number of cells along the periodicity directions, the cross-sectional shape, and the effective length factor determined by the end-restraint conditions. The transition from the stability of individual cell members to the stability of the structure as a whole is considered in this part of the work.

The aim of the present part of the study is to develop an analytical scheme for assessing the global stability of a structure formed from a three-dimensional re-entrant auxetic lattice, as well as to determine the boundaries between local and global buckling mechanisms. For this purpose, the finite structure is considered as an equivalent orthotropic beam, properties of which are determined by the parameters of the periodic cell. On this basis, expressions are derived for the critical stresses under compression along the principal axes of orthotropy, and the influence of the geometric characteristics of the elementary auxetic lattice and the number of cells in the longitudinal and transverse directions on global buckling is analyzed.

The second part of the work supplements the previously developed theory of local buckling [20] and extends it to the level of the structure as a whole. This makes it possible to establish how the architecture of a three-dimensional re-entrant cell affects the overall stability of auxetic lattice structures and the selection of the governing buckling mode. The obtained relationships could be used for engineering assessment of the load-bearing capacity of such structures and for the rational selection of the geometric parameters of the cell with due account for the loading direction, structural dimensions, and end-restraint conditions.

2 Formulation of the Stability Problem for a Finite Structure

In the first part of the present study [20], a mesoscale formulation of the local stability problem for an elementary three-dimensional re-entrant auxetic cell was formulated. The geometric parameters of the periodic structure were introduced, namely: the length of the inclined member L, the height of the vertical strut H, the thicknesses of the members t, the re-entrant angle of the inclined member θ, shown in Figure 1 in [20], the Young's modulus E, the shear modulus G, as well as the dimensionless quantities $\lambda = H/L$, $\eta = t/L$ and $g = G/E$, through which both the dimensions of a single cell and the overall dimensions of the structure composed of a finite number of such cells are described. For the beam model, in the authors' previous work [21], effective orthotropic moduli E_z and $E_y = E_x$, as well as tangent moduli $E_z^{\tan}$, $E_y^{\tan} = E_x^{\tan}$ were obtained, considering the quadratic geometric nonlinearity of the preliminary stress strain state. Besides, a

variational formulation of the local stability problem for the elementary cell was established in [20] through the total potential energy and the tangent stiffness matrix $\mathbf{K}_t^{(c)} = \mathbf{K}_M^{(c)} - \mathbf{K}_G^{(c)}$, where $\mathbf{K}_t^{(c)}$ is the tangent stiffness matrix of the elementary cell, $\mathbf{K}_M^{(c)}$ is the elastic part of the stiffness matrix of the elementary cell, and is the geometric stiffness matrix of the elementary cell. The critical state of the cell was determined by the condition of loss of positive definiteness of this matrix, that is $\det \mathbf{K}_t^{(c)} = 0$. In the engineering approximation, this result was reduced to local buckling criteria for inclined members and vertical struts [20]. Under compression along the Z-axis, two possible local mechanisms have been discussed: buckling of the inclined member and buckling of the vertical strut, whereas under compression along the Y- or X-axes, the local mechanism is associated with buckling of the inclined members.

Thereby, the results in [20] define the lower, mesoscale level of the limit state of the structure for each loading direction $i \in \{x, y, z\}$.

For structures, the formulation is completely analogous. Let $\mathbf{Q}$ be the vector of generalized displacements representing the degrees-of-freedom of the structure. Then the total potential energy of the structure Σ, formed from a finite number of periodically repeated three-dimensional re-entrant auxetic cells, under compression along the i-direction, and its second variation are defined as [22]

$$\Pi_\Sigma^{(i)}(\mathbf{Q}, \sigma_i) = U_\Sigma^{(i)}(\mathbf{Q}) - W_\Sigma^{(i)}(\mathbf{Q}, \sigma_i), \quad \delta^2 \Pi_\Sigma = \delta \mathbf{Q}^{\mathrm{T}} K_t^{(i)} \delta \mathbf{Q}, \ \mathbf{K}_t^{(i)} = \mathbf{K}_M^{(i)} - \mathbf{K}_G^{(i)}, \tag{1}$$

and global buckling occurs when the tangent stiffness matrix loses positive definiteness [22]

$$\det \mathbf{K}_t^{(i)} = 0. \tag{2}$$

Here $U_\Sigma^{(i)}(\mathbf{Q})$ is the internal elastic energy of the structure as a function of the degrees-of-freedom of $\mathbf{Q}$. It is determined by the deformations of the elements of the computational model and by the effective elastic properties of the auxetic medium. When the structure is modeled as an equivalent orthotropic three-dimensional beam, the relations for $U_\Sigma^{(i)}$ include the strain energies of axial, bending, shear, and, if necessary, torsional components, calculated through the corresponding effective moduli. In the linear formulation, the initial moduli E_i are used, whereas the tangent moduli $E_i^{\tan}$ are used when pre-deformation is considered. $W_\Sigma^{(i)}(\mathbf{Q}, \sigma_i)$ is the work of the external compressive load reduced to the loading i-direction, where σ_i is the macroscopic compressive stress. $\mathbf{K}_t^{(i)}$ is the tangent stiffness matrix of the structure under compression along direction i. It represents the second derivative of the total potential energy with respect to the global degrees-of-freedom and relates small increments of the generalized displacements $\mathbf{Q}$ to the corresponding increments of generalized forces in the prestressed state. $\mathbf{K}_M^{(i)}$ is the elastic part of the tangent stiffness matrix. It is determined by the elastic properties of the equivalent orthotropic medium, the geometry of the structure, and the selected discretization scheme. For the beam model, this matrix contains the axial, bending, shear, and torsional stiffnesses of the elements. In the linear formulation, $\mathbf{K}_M^{(i)}$ does not depend on the level of the compressive load. $\mathbf{K}_G^{(i)}$ is the geometric stiffness matrix, which arises due to the action of the initial compressive load and depends on the axial forces in the structure under loading along the i- direction.

Therefore, the limit (critical) state of the structure should be determined as

$$\sigma_{cr}^{(i)} = \min\left\{\sigma_{cr,cell}^{(i)}, \sigma_{cr,glob}^{(i)}, \sigma_{str}^{(i)}\right\}, \quad i \in \{x, y, z\}, \tag{3}$$

where $\sigma_{cr,cell}^{(i)}$ is the critical stress of local buckling at the cell level, $\sigma_{cr,glob}^{(i)}$ is the critical stress of global buckling of the structure, and $\sigma_{str}^{(i)}$ is the ultimate strength for the members of the auxetic lattice. This relationship is important because, in a three-dimensional auxetic structure, loss of load-bearing capacity and buckling do not necessarily coincide, and the actual limit (critical) load level is determined by the smallest of these mechanisms.

3 Global Stability of Structures Made of a Periodic Auxetic Lattice

For the linear formulation, let us now consider a finite structure made of an auxetic metamaterial in the form of a prismatic body formed from a large number of cells and compressed along one of the principal axes of orthotropy. If the length of the structure significantly exceeds its transverse dimensions, while the cross section remains constant, then the global stability problem is reduced to the buckling problem of an equivalent orthotropic beam. For the loading direction $i \in \{x, y, z\}$, the design slenderness is introduced as

$$\Lambda_g^{(i)} = \frac{\mu_i L_\Sigma^{(i)}}{i_{\min}^{(i)}}, \tag{4}$$

where $L_{\Sigma}^{(i)}$ is the length of the beam along the compression axis, $i_{\min}^{(i)}$ is the minimum radius of gyration of the corresponding cross section, and μ_i is the effective length factor depending on the end-restraint conditions.

To investigate global buckling, slender structures are considered, for which the linear global critical stress is given by Euler's formula

$$\sigma_{cr,glob,0}^{(i)} = \frac{\pi^2 E_i}{\left(\Lambda_g^{(i)}\right)^2}. \tag{5}$$

Formula (5) shows that the local geometry of the cell affects global stability not only through the dimensions of the structure, but also through the effective moduli E_i, that is, twice, first by determining the compliance of the periodic lattice itself, and then through the external slenderness of the finite structure.

For a square cross section of the beam composed of $n_x = n_y = n$ cells along the width and depth, as well as n_z cells along the height in the Z-axis direction, we have

$$L_x = L_y = L_i = 2nL\sin\theta,$$
$$L_z = 2n_z\left(H - L\cos\theta\right), \quad i_{\min}^{(i)} = \frac{L_i}{\sqrt{12}}. \tag{6}$$

Hence,

$$\Lambda_g^{(z)} = \mu_z\sqrt{12}\frac{n_z}{n}\frac{\lambda-\cos\theta}{\sin\theta}. \tag{7}$$

Substituting the expression for the design slenderness (7) and the expression for the effective modulus E_z (see formula (5) in [20]) into (5) yields

$$\sigma_{cr,glob,0}^{(z)} = \frac{\dfrac{\pi^2 E\eta^2}{12\mu_z^2}\left(\dfrac{n}{n_z}\right)^2}{\left(\lambda-\cos\theta\right)\left[2\lambda+\left(\dfrac{1}{2\eta^2}+\dfrac{3}{5g}\right)\sin^2\theta\right]}. \tag{8}$$

It follows from (8) that the global critical stress depends on the re-entrant angle, the relative height of the cell, the thickness of the members, as well as on the ratio of the number of cells along the height to that in the transverse direction.

To analyze the effect of θ on the critical stresses, it is convenient to denote the numerator of the fraction in Eq. (8) by the function $f_z\left(\theta,\lambda,\eta,g\right)$:

$$f_z = \left(\lambda-\cos\theta\right)\left[2\lambda+\left(\frac{1}{2\eta^2}+\frac{3}{5g}\right)\sin^2\theta\right], \tag{9}$$

Differentiating the function f_z with respect to the re-entrant angle θ yields

$$\frac{\partial f_z}{\partial\theta} = \sin\theta\left[2\lambda+\left(\frac{1}{2\eta^2}+\frac{3}{5g}\right)\sin^2\theta + 2\left(\frac{1}{2\eta^2}+\frac{3}{5g}\right)\left(\lambda-\cos\theta\right)\cos\theta\right]. \tag{10}$$

Inasmuch as

$$0<\theta<\frac{\pi}{2}, \quad \lambda>\cos\theta, \quad \eta>0, \quad g>0, \tag{11}$$

each term in the square brackets is positive, and therefore

$$\frac{\partial f_z}{\partial\theta}>0 \;\Rightarrow\; \frac{\partial\sigma_{cr,glob,0}^{(z)}}{\partial\theta}<0. \tag{12}$$

Consequently, the global stability of the structure under compression along the Z-axis deteriorates monotonically with an increase in the re-entrant angle. At the same time, the angle θ simultaneously changes the overall length of one cell along the Z-axis, the width of the structure, and the value of the modulus E_z itself.

To analyze the effect of λ on the critical stresses, let us analyze the first derivative of the function f_z with respect to λ

$$\frac{\partial f_z}{\partial\lambda} = 4\lambda - 2\cos\theta + \left(\frac{1}{2\eta^2}+\frac{3}{5g}\right)\sin^2\theta > 0. \tag{13}$$

Since $\dfrac{\partial f_z}{\partial\lambda}>0$, we obtain the following estimate:

$$\frac{\partial\sigma_{cr,glob,0}^{(z)}}{\partial\lambda}<0. \tag{14}$$

Thus, for a fixed ratio n/n_z an increase in λ leads to a decrease in $\sigma_{cr,glob,0}^{(z)}$, since the beam simultaneously becomes longer and is formed from a cell architecture that is more compliant in bending.

For thin-walled lattices $\eta = t/L \ll 1$, the linear global critical stress along the Z-axis is transformed into the form:

$$\frac{\sigma_{cr,glob,0}^{(z)}}{E} \sim \frac{\pi^2}{6\mu_z^2}\left(\frac{n}{n_z}\right)^2\frac{\eta^4}{\left(\lambda-\cos\theta\right)\sin^2\theta}. \tag{15}$$

The same power of η^4 is also present in the local criteria for $\sigma_{cr,loc}^{(z,s)}$ and $\sigma_{cr,loc}^{(z,v)}$. Consequently, the relative thickness of the members of the auxetic lattice is the main parameter for stability problems, since a decrease in t/L sharply reduces both the local and global critical stresses.

Global stability of the structure under compression along the Y-axis. Under compression of the structure along the Y-axis, the shape of the cross section in the XZ plane becomes an additional factor. Therefore, global buckling is determined by the ratio of two

transverse dimensions, L_x or L_z. The global buckling mode that is realized is the one for which the corresponding bending stiffness is minimal [18], [19]. Since the structure contains n_x, n_y, and n_z cells in the corresponding directions, then if $L_x \le L_z$, the smallest transverse dimension is L_x. Therefore, the minimum bending stiffness corresponds to the global mode with lateral deflection along the X-axis, that is, bending about the Z-axis. In this case,

$$\Lambda_g^{(y,x)} = \mu_y \sqrt{12}\frac{n_y}{n_x}. \qquad (16)$$

Substitution of the effective modulus $E_y \equiv E_x$ (see formula (5) in [20]) into (5) yields

$$\frac{\sigma_{cr,glob,0}^{(y,x)}}{E} = \frac{\dfrac{5\pi^2 g\eta^4}{6\mu_y^2(5g+6\eta^2)}\left(\dfrac{n_x}{n_y}\right)^2}{(\lambda-\cos\theta)\cos^2\theta}. \qquad (17)$$

If, however, $L_z \le L_x$, then the smaller transverse dimension is equal to L_z. Consequently, the global buckling mode with lateral deflection along the Z-axis is realized, that is, bending about the X-axis. Then

$$\Lambda_g^{(y,z)} = \mu_y \sqrt{12}\frac{n_y}{n_z}\frac{\sin\theta}{\lambda-\cos\theta}, \qquad (18)$$

whence

$$\frac{\sigma_{cr,glob,0}^{(y,z)}}{E} = \frac{5\pi^2 g\eta^4}{6\mu_y^2\left(5g+6\eta^2\right)}\left(\frac{n_z}{n_y}\right)^2\frac{\lambda-\cos\theta}{\sin^2\theta\cos^2\theta}. \qquad (19)$$

Under compression along the Y-axis, formulas (17) and (19) are determined not only by the cell parameters. The cross-sectional shape is also included in the calculation, since it determines the minimum radius of gyration of the section and, consequently, the weak bending axis about which the first global buckling mode is realized. Therefore, one and the same periodic body with the same cell architecture, but with different ratios n_x/n_z, could transit to different global buckling modes. As already noted, at the level of a single cell, the properties along the X- and Y-axes are equivalent; however, at the level of a structure formed from a finite number of such cells, this equivalence is lost due to the difference in transverse dimensions and, consequently, due to the selection of the most flexible direction, or axis, of global bending.

Influence of θ, λ, and η on global stability under compression along the Y-axis. In the case of greater flexibility of the structure along the X-axis in (17)an increase in λ reduces the global critical stress level, since the denominator contains the factor $(\lambda-\cos\theta)$. Thus, along the Y-axis, the same qualitative effect is observed as along the Z-axis: an increase in the cell height reduces the stability of the structure, despite the change in the effective moduli at the level of a single cell.

The dependence of the global critical stress on the angle θ in relationship (17) is no longer monotonic. The most dangerous area is determined by the maximum of the function

$$h(c) = (\lambda - c)c^2,\ c = \cos\theta. \qquad (20)$$

If $0 < \lambda < 1.5$, this function reaches its maximum at $c = 2\lambda/3$, that is, the minimum global critical stress is realized at an intermediate angle. If $\lambda \ge 1.5$, the area of small angles becomes the most dangerous.

In the case of more flexibility of the structure along the Z-axis in formula (19), the dangerous area shifts, since the denominator contains $\sin^2\theta\cos^2\theta$, whereas the numerator contains $(\lambda-\cos\theta)$. Hence, for the Y-direction, it is impossible to unambiguously determine either the best or the worst re-entrant angle θ without considering the shape of the structure itself.

With respect to the thickness of the members, the general rule is preserved:

$$\sigma_{cr,glob,0}^{(y)} = O\left(E\eta^4\right),\ \eta \ll 1, \qquad (21)$$

that is, a decrease in t/L reduces global stability in this direction as well.

Nonlinear correction using the tangent modulus. If the pre-deformation is no longer small, the formula (5) must be corrected by replacing the initial modulus E_i with the tangent modulus $E_i^{\tan}$ (see formula (11) in [20]). Then the global critical stress level should be determined from the implicit equation

$$\sigma_{cr,glob}^{(i)} = \frac{\pi^2 E_i^{\tan}\left(\sigma_{cr,glob}^{(i)}\right)}{\left(\Lambda_g^{(i)}\right)^2}. \qquad (22)$$

For the Z-direction, with due account for formula (11) form [20]), this gives

$$\sigma_{cr,glob}^{(z)} = \frac{\pi^2}{\left(\Lambda_g^{(z)}\right)^2}\frac{1}{A_z - 2B_z\sigma_{cr,glob}^{(z)}}, \qquad (23)$$

that is,

$$2B_z\left(\sigma_{cr,glob}^{(z)}\right)^2 - A_z\sigma_{cr,glob}^{(z)} + \frac{\pi^2}{\left(\Lambda_g^{(z)}\right)^2} = 0, \qquad (24)$$

where A_z is the linear compliance in the direction of the Z axis, and B_z is the coefficient of quadratic geometric nonlinearity.

Selecting the root that tends to the linear solution as $B_z \to 0$ yields

$$\sigma_{cr,glob}^{(z)} = \frac{A_z - \sqrt{A_z^2 - \frac{8\pi^2 B_z}{\left(\Lambda_g^{(z)}\right)^2}}}{4B_z}. \quad (25)$$

For the Y-direction, we have

$$\sigma_{cr,glob}^{(y)} = \frac{\pi^2}{\left(\Lambda_g^{(y)}\right)^2} \frac{1}{A_y + 2B_y \sigma_{cr,glob}^{(y)}}, \quad (26)$$

that is,

$$2B_y \left(\sigma_{cr,glob}^{(y)}\right)^2 + A_y \sigma_{cr,glob}^{(y)} - \frac{\pi^2}{\left(\Lambda_g^{(y)}\right)^2} = 0, \quad (27)$$

and

$$\sigma_{cr,glob}^{(y)} = \frac{-A_y + \sqrt{A_y^2 + \frac{8\pi^2 B_y}{\left(\Lambda_g^{(y)}\right)^2}}}{4B_y}. \quad (28)$$

Formulas (25) and (28) represent important theoretical results for the stability problem of a structure made of a periodic auxetic lattice. They show that the geometric nonlinearity of the initial stress-strain state affects global stability differently depending on the direction of compression.

Under compression along the Z-axis, the coefficient $B_z > 0$ enters the deformation law

$$\varepsilon_z = A_z \sigma_z - B_z \sigma^2{}_z. \quad (29)$$

This means geometric hardening of the structure and an increase in the preliminary stress–strain state, as well as an increase in the global critical stress compared with the linear Euler formula.

Under compression along the Y- or X-axes, we have the deformation law

$$\varepsilon_y = A_y \sigma_y + B_y \sigma^2{}_y, \quad (30)$$

that is, geometric softening. Therefore, the nonlinear correction, on the contrary, reduces the global stability margin. This is one of the most significant differences between buckling along the Z- and Y-axes.

4 Boundaries Between Local and Global Buckling

To distinguish between local and global buckling mechanisms, it is appropriate to equate the linear global Euler critical stresses to the local critical stresses of the corresponding member of the auxetic lattice. As a result, values of the design slenderness of the beam structure are determined, the exceedance of which indicates that the global mechanism becomes more dangerous than the local one.

Compression along the Z-axis. If global buckling is compared with local buckling of the inclined member, then

$$\Lambda_*^{(z,s)} = \pi \sqrt{\frac{E_z}{\sigma_{cr,loc}^{(z,s)}}}. \quad (31)$$

Substituting the expression for the effective modulus E_z and the dimensionless local criterion $\frac{\sigma_{cr,loc}^{(z,s)}}{E}$ (see formulas (5) and (30) in [20]), yields

$$\Lambda_*^{(z,s)} = \frac{\sqrt{15}\beta\sqrt{g\cos\theta\left(\lambda - \cos\theta\right)}}{\sqrt{\chi_s}\sqrt{20\eta^2 g\lambda + \sin^2\theta\left(6\eta^2 + 5g\right)}}, \quad (32)$$

where $\beta = 1 - \frac{\eta}{2\sin\theta}$ is a correction to the effective design length of the inclined strut, and $\chi_s \in \left(0,\ 1\right]$ is a coefficient accounting for the imperfection of 3D printing of the inclined member.

If global buckling is compared with local buckling of the vertical strut (see formula (31) in [20]), then

$$\Lambda_*^{(z,v)} = \pi \sqrt{\frac{E_z}{\sigma_{cr,loc}^{(z,v)}}}, \quad (33)$$

whence

$$\Lambda_*^{(z,v)} = \frac{2\sqrt{15g}\lambda\sqrt{\lambda - \cos\theta}}{\sqrt{\chi_v}\sqrt{20\eta^2 g\lambda + \sin^2\theta\left(6\eta^2 + 5g\right)}}, \quad (34)$$

where $\chi_v \in \left(0,\ 1\right]$ is a coefficient accounting for the imperfection of 3D printing of the vertical strut.

Since, for practically significant values of λ, the local buckling mechanism in the Z-direction is usually governed precisely by the strut, the main parameter separating the mechanisms for the Z-direction is $\Lambda_*^{(z,v)}$.

Compression along the Y- or X-axes**.** For the Y- or X-direction, we have

$$\Lambda_*^{(y,s)} = \pi \sqrt{\frac{E_y}{\sigma_{cr,loc}^{(y,s)}}} = \frac{\sqrt{15g}\beta}{\sqrt{\chi_s \cos\theta\left(6\eta^2 + 5g\right)}}. \quad (35)$$

Formula (35) deserves special attention. In the linear approximation, the boundary between local and global buckling under compression along the Y- or X-direction does not depend on $\lambda = H/L$. It means that an increase in the strut height equally reduces both the local critical stress of the member and the global critical stress along the Y- or X-axis. This is due to the fact that the same geometric factor containing λ enters both the expression for the effective modulus E_y (see formula (5) in [20]) and the expression for the local critical stress $\sigma_{cr,loc}^{(y,s)}$ (see formula (32) in study [20]). Therefore, in this direction, the transition from the

local buckling mechanism to the global mechanism is determined mainly by the thickness of the members, the re-entrant angle, the ratio G/E, and the slenderness of the structure itself.

Nonlinearly corrected transition boundaries between buckling mechanisms. The linear boundaries obtained in the previous sections through $\Lambda_*^{(z,v)}$, $\Lambda_*^{(z,s)}$ and $\Lambda_*^{(y,s)}$ could be refined by considering the quadratic nonlinearity of the preliminary stress-strain state. For this purpose, it is sufficient to equate the local stability criterion to the nonlinear global criterion. For the Z-direction, where two local mechanisms $m \in \{s, v\}$ are possible, we obtain

$$\Lambda_{*,nl}^{(z,m)} = \frac{\pi}{\sqrt{A_z \sigma_{cr,loc}^{(z,m)} - 2B_z \left(\sigma_{cr,loc}^{(z,m)}\right)^2}}, \qquad (36)$$

and for the Y-direction:

$$\Lambda_{*,nl}^{(y,s)} = \frac{\pi}{\sqrt{A_y \sigma_{cr,loc}^{(y,s)} + 2B_y \left(\sigma_{cr,loc}^{(y,s)}\right)^2}}. \qquad (37)$$

As B_z and $B_y \to 0$, formulas (36) and (37) are reduced to the linear boundaries. However, their qualitative influence is different: $\Lambda_{*,nl}^{(z,m)} > \Lambda_*^{(z,m)}$, since the denominator decreases, whereas $\Lambda_{*,nl}^{(y,s)} < \Lambda_*^{(y,s)}$, since the denominator increases.

Consequently, under compression along the direction of the Z-axis, nonlinearity shifts the buckling boundary toward the local mechanism, whereas under compression along the Y- or X-direction, it shifts the boundary toward the global mechanism.

5 Boundaries in Terms of Cell Number Ratios for Characteristic Cases

For a square cross section of a beam consisting of $n \times n \times n_z$ elementary auxetic cells and compressed along the Z-axis, the linear boundary between the local strut mechanism and global Euler buckling has the form

$$\left(\frac{n_z}{n}\right)_*^{(z)} = \frac{\Lambda_*^{(z,v)} \sin\theta}{\mu_z \sqrt{12}\left(\lambda - \cos\theta\right)}. \qquad (38)$$

Global buckling is realized if

$$\frac{n_z}{n} > \left(\frac{n_z}{n}\right)_*^{(z)}, \qquad (39)$$

while local buckling is realized if the opposite inequality is satisfied. If a more accurate nonlinear boundary is required, then $\Lambda_*^{(z,v)}$ in (38) should be replaced by $\Lambda_{*,nl}^{(z,v)}$.

For a bar compressed along the Y-axis, when the structure is more flexible along the X-axis, we obtain

$$\left(\frac{n_y}{n_x}\right)_*^{(y,x)} = \frac{\Lambda_*^{(y,s)}}{\mu_y \sqrt{12}}. \qquad (40)$$

When the structure is more flexible along the Z-axis:

$$\left(\frac{n_y}{n_z}\right)_*^{(y,z)} = \frac{\Lambda_*^{(y,z)}\left(\lambda - \cos\theta\right)}{\mu_y \sqrt{12}\sin\theta}. \qquad (41)$$

Formulas (38) – (41) make it possible to pass from the design slenderness to requirements for the overall dimensions of the finite structure through the number of cells along the length and width.

Thin-walled asymptotics of the boundaries between mechanisms. Within the limit $\eta \ll 1$, relationship (38) is additionally simplified:

$$\left(\frac{n_z}{n}\right)_{*,thin}^{(z)} \sim \frac{\lambda}{\mu_z \sqrt{\chi_v\left(\lambda - \cos\theta\right)}}. \qquad (42)$$

It is noteworthy that, within this approximation, the contributions of t/L and G/E practically disappear. That is, for a sufficiently thin-walled lattice, the relationship between local and global buckling is determined mainly by the parameters λ, θ, the end-restraint conditions, and the ratio of the beam dimensions, rather than by the thickness as such.

Under compression along the Y-axis, let us consider the case in which the first global buckling mode corresponds to lateral deflection in the X-direction, that is, bending about the Z-axis. If the thin-walled asymptotic for $\Lambda_*^{(y,s)}$ is considered in relation (40), then it is reduced to the form

$$\left(\frac{n_y}{n_x}\right)_{*,thin}^{(y,x)} \sim \frac{\beta}{2\mu_y \sqrt{\chi_s \cos\theta}}. \qquad (43)$$

It follows from (43) that as θ increases, the criterion for transition to global buckling increases, since $\cos\theta$ decreases. However, at the same time, the absolute value of the global critical stress itself also changes through E_y. Therefore, for the Y-direction, one length ratio is not enough, since a combined analysis of local buckling, global buckling, and strength criteria is always required.

Interpretation of the boundaries between mechanisms. If

$$\Lambda_g^{(z)} < \Lambda_*^{(z,v)}, \qquad (44)$$

then, under compression along the Z-axis, the structure is not yet sufficiently slender for global buckling to occur, and local buckling of the vertical struts is realized first, or the ultimate strength is reached in the members of the auxetic lattice.

If

$$\Lambda_g^{(z)} > \Lambda_*^{(z,v)}, \quad (45)$$

then global buckling of the entire structure as a beam occurs earlier.

Similarly, for the Y-direction:

$$\Lambda_g^{(y)} < \Lambda_*^{(y,s)}, \quad (46)$$

then the local mechanism prevails, if

$$\Lambda_g^{(y)} > \Lambda_*^{(y,s)}, \quad (47)$$

then the global buckling mechanism prevails.

Using the same parameters as in [20]

$$L = 5\,\text{mm},\ H = 10\,\text{mm},\ t = 1\,\text{mm},\ \theta = 60^\circ,\ E = 200\,\text{GPa},\ G = 77.4\,\text{GPa}, \quad (48)$$

we obtain the following buckling characteristics:

$$\Lambda_*^{(z,s)} \approx 1.23,\ \Lambda_*^{(z,v)} \approx 7.87,\ \Lambda_*^{(y,s)} \approx 2.04. \quad (49)$$

These estimates allow a meaningful conclusion to be drawn. For the selected geometry and material, local buckling along the Z-axis will prevail for almost all short and medium-length beams. The transition to global buckling begins only for truly slender structures with a design slenderness on the order of eight and higher. For the Y- or X-direction, the boundary between the local and global buckling mechanisms is significantly lower, so even moderately slender structures may undergo global buckling.

For a beam of square cross section with hinged supports at both ends $(\mu_z = 1)$, we obtain $\left(\frac{n_z}{n}\right)_*^{(z)} \approx 1.31$ and about 1.38 when the quadratic nonlinear correction is taken into account. Consequently, a beam with $n_z/n = 1$ will lose stability locally by buckling of the struts, whereas a beam with $n_z/n = 2$ will already undergo global buckling.

For fixed-fixed end supports $\mu_z = 0.5$, the same boundary shifts approximately to $\frac{n_z}{n} \approx 2.67$ for the linear case, and to 2.76 when the nonlinear correction is taken into account. For $\mu_z = 2$, it shifts to $\frac{n_z}{n} \approx 0.66$ and 0.69 for the linear and nonlinear cases, respectively.

This means that the same set of cells could undergo either local or global buckling depending on the end-restraint scheme.

For the Y-direction, when the structure has greater flexibility along the X-axis, from (40) we obtain, for the same geometry, $\left(\frac{n_y}{n_x}\right)_*^{(y,x)} \approx 0.59$ for hinged end supports. Therefore, even a beam, the length along the Y-axis of which is only comparable to the cross-sectional dimension, already falls within the region where global buckling must be checked before local buckling of the inclined members.

The geometry of the re-entrant auxetic cell determines not only the effective physical and mechanical properties, but also the limit-state mechanism that is realized first: local buckling of the lattice members or global buckling of the entire structure. Moreover, this boundary depends on the loading direction in different ways, which cannot be established within a continuum (equivalent orthotropic) model without explicitly returning to the cell architecture.

6 Conclusion

The model for assessing the global stability of slender structures composed of a finite number of three-dimensional re-entrant auxetic cells has been constructed. In contrast to the analysis of an individual cell, which was performed in the first part of the study [20] the scale of the entire structure is considered here, including the number of cells in the longitudinal and transverse directions, the shape of the cross section, the end-support conditions, and the direction of compression.

It is shown that the limit state of such a structure should be determined as the minimum of three possible levels: local buckling of the auxetic cell members, global buckling of the entire structure, and attainment of the ultimate strength of the material. Therefore, calculation of only local criteria or only equivalent elastic properties is insufficient for assessing the load-bearing capacity of structures made of auxetic metamaterials.

To describe global buckling, an equivalent orthotropic beam was considered, the effective moduli of which are determined by the geometry of the periodic re-entrant cell. On this basis, expressions were obtained for the global critical stresses under compression along the principal axes of orthotropy. It was established that the cell parameters affect global stability through the effective physical and mechanical properties of the material and through the overall dimensions of the structure composed of a finite number of cells.

Under compression along the Z-axis, it was shown that an increase in the re-entrant angle θ leads to a monotonic decrease in the global critical stress. An increase in the parameter λ also reduces global stability, since the structure becomes more flexible and is simultaneously formed from a more compliant cell architecture. The global critical stress, like the local stability criteria, is proportional to the fourth

power of the relative member thickness η. This means that a decrease in the ratio t/L is one of the most significant factors reducing the stability of the auxetic lattice.

Under compression along the Y-direction (or X), a more complex dependence of global stability on the geometry of the structure was revealed. In this case, not only does the cell architecture become governing, but also which of the transverse dimensions of the structure is smaller. Hence, one and the same periodic structure could realize different global buckling modes depending on the ratio of the number of cells in the transverse directions. At the level of an individual cell, the properties along the X- and Y-directions are equivalent; however, at the level of a finite structure, this equivalence is violated due to the difference in transverse dimensions and the selection of the most flexible direction of global bending.

The obtained nonlinear corrections based on the tangent modulus showed a fundamental difference between the compression directions. Under compression along the Z-axis, the initial stress-strain state increases the critical stress: as the level of compression along the Z-axis increases, the tangent modulus increases; that is, the longitudinal stiffness of the equivalent orthotropic medium increases, which leads to an increase in the global critical stress compared with the critical stress given by the Euler formula. Under compression along the Y- or X-axes, on the contrary, the geometric nonlinearity of the initial stress-strain state reduces the critical stress: as the level of compression increases, the tangent modulus decreases, that is, the incremental longitudinal stiffness of the equivalent orthotropic medium decreases. As a result, the nonlinear correction reduces the global stability domain.

The boundaries for the transition between local and global buckling have been defined. These boundaries are expressed in terms of the designed slenderness of the structure, and make it possible to determine which mechanism will be governing for a given cell geometry, number of cells, and end-restraint conditions. For the Z-direction, the main local mechanism for practically significant parameter values is buckling of the vertical struts. For the Y-direction (or X), the governing local mechanism is buckling of the inclined members.

For the characteristic set of parameters considered in the present paper, it has been established that, under compression along the Z-axis, local buckling of the vertical struts prevails in short and medium-length beams, whereas global buckling occurs only in sufficiently slender structures. Under compression along the Y-direction (or X), the boundary between the local and global mechanisms is significantly lower; therefore, even moderately slender structures require verification for global buckling.

Thus, the geometry of the re-entrant auxetic cell determines not only the effective elastic properties, but also the mechanism by which the limit state of the structure is reached. The loading direction, re-entrant angle, relative cell height, relative member thickness, number of cells, and end-restraint conditions jointly define the region in which either local buckling of the lattice members or global buckling of the structure occurs first. The obtained relationships could be used for engineering assessment of the load-bearing capacity of three-dimensional cellular auxetic metamaterials with a re-entrant structure and for the rational selection of their geometric parameters in structural design. Future work will focus on extending the present theory to structures for which shear deformations must be taken into account.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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