Application of Two-Parameter Model Approach to Pneumatic Conveying Control Equation

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Abstract: - Pneumatic conveying and movement of powdery and bulk materials from tanks and cisterns is widely used in various industries and transportation. The most well-known methods for cleaning gas distribution grates employ pneumatic pulse regeneration. These methods typically require high specific gas flow rates at constant pressure. A more optimal method for pneumatic pulse conveying of bulk and powdery materials from pneumatic tanks has also been developed. This method utilizes the energy of forward and reverse pneumatic shock waves to move bulk and powdery materials. However, this new method has not yet received the necessary mathematical description for the development of a methodology for its optimal calculation and process control. This paper is devoted to the development of such a model, the substantiation of its two-parameter structure, and the derivation of a process control equation based on the Duffing model equation specifically for the new process design. The paper also presents the results of a preliminary analysis of the proposed model and provides identification and interpretation of the control parameters.

Key-Words: - Pneumatic conveying, pneumatic-pulse design, pneumatic shocks frequency, pressure increase time delay, two-parameter model, Duffing equation.

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1 Introduction

Pneumatic transport and handling of powdered and bulk materials from storage tanks and cisterns is widely used in various industries and transportation [1], [2], [3]. These tanks are equipped with aeration systems with gas distribution baffles and various devices that ensure the regeneration of the pores of the gas distribution grate when they become clogged [4].

The most well-known methods for cleaning gas distribution grates employ pneumatic pulse regeneration [5]. These methods typically require the use of increased specific gas flow rates at constant pressure [6]. The energy of the pneumatic pulses is used to regenerate the pores in the fabric gas distribution baffles, collapse them, and transport the bulk material to the outlet. The energy of the reverse wave is practically not utilized in this process [7].

A new method for pneumatic-pulse processing of bulk and powder materials from pneumatic tanks with regeneration of gas distribution baffles is proposed in [8]. This method uses the energy of forward and reverse pneumatic shock waves to transport bulk and powder materials along a horizontal unloading pipeline and a vertical transport line.

Tests have shown that the new method significantly improves productivity and reliability, while simultaneously reducing the specific consumption of compressed gas for processing [8]. An important feature of this method, in contrast to other known methods [9], [10], [11], is its rapid response to changes in control parameters, as well as an extended control range using pulsed pneumatic shocks under conditions of maximum exhaust pressure.

In the method under consideration, in addition to the constant-pressure compressed-gas flow [12], [13] typically applied to the material on inclined fabric gas distribution baffles, the transported material is also subjected to additional compressed-gas pulses.

These additional pulsed pneumatic impacts are delivered simultaneously with the pulsed pneumatic impacts beneath the inclined fabric gas distribution baffles of the pneumatic tank [14]. Intermittent streams of compressed gas at constant pressure alternate at a predetermined rate.

By adjusting the frequency of the pulsed pneumatic impacts, it is possible to transport an aerated material mixture along the inclined fabric baffles in the pneumatic tank, the horizontal discharge pipeline, and the initial section of the vertical transport line in a frequency-resonant mode. The basic setup is shown in Figure 1.

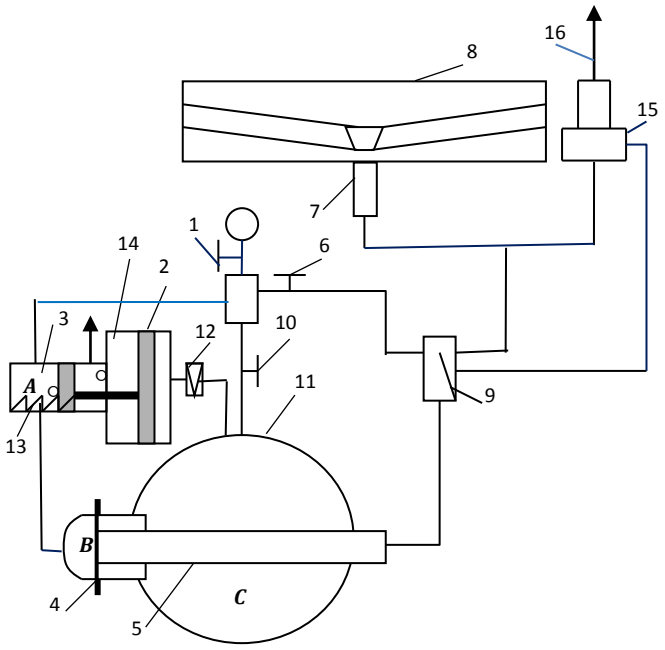


Fig. 1: Diagram of a pneumatic pulse system for handling bulk materials
Source: [8].

A detailed description of the process workflow, as well as a key to the equipment position numbers in the diagram, is provided below in subsection 2.1.

Several studies have been devoted to mathematical modeling and optimization of the process in previously known designs [15], [16]. Several completely new studies should also be noted [17], [18].

Some results in the mathematical modeling of similar processes were also obtained in [19], [20]. It should be noted that, to date, neither a reliable mathematical model nor an appropriate control methodology has been proposed for the process design claimed in the patent [8].

The main objectives of this study were to submit a mathematical model and determine the control parameters for the technological process described above. Thus, the main goal of the presented work, defended by the authors, is the heuristic analysis of which parameters can be selected and justified for use as control parameters in model equations describing the process of pulse overload in the innovative technological scheme [8].

In this case, the authors attempt to submit the two-parameter model using the experience of developing a two-parameter model in their previous works on mathematical modeling of periodic and wave processes [21].

Unlike the more theoretical work [21], this work does not claim to develop a general model. It is devoted to the problem of a mathematical description of the operating process applied to the specific technical development proposed in the patent [8]. Here, an attempt is made to select a suitable model and adapt it to a given process with reasonable control parameters.

An analysis of the sources led us to conclude that this is the first proposal of such a mathematical model for describing this process. Consequently, the model is new with respect to the object of its description.

Another result is that the proposed approach justifies using the Duffing equation as a control equation. The Duffing equation has proven its significant generality as a model for describing nonlinear oscillatory processes in continuous media [22], [23].

2 Problem Formulation

In order to present a reliable mathematical model suitable for process control, it is first necessary to clearly define the main stages of the work cycle and the corresponding control parameters.

2.1 Description of the Process Work Cycle

The operating cycle is described based on the installation diagram shown above in Figure 1.

The sequence of operating stages can be described as follows:

1. Valve 1 opens, and compressed gas, entering space ***A***, moves spool valve 2 to its far right position. In this position, orifices 3 open, and compressed gas enters space ***B***. The pressure of this gas causes membrane 4 to close the opening of exhaust pipe 5.
2. Valve 6 opens, and gas enters aeration system 7 of tank 8 and annular ejector 15 of conveyor line 16. Check valve 9 prevents pressure from building up in exhaust pipe 5.
3. Valve 10 opens, and compressed gas enters reservoir 11. The pressure increase in space ***C*** occurs with a slight delay relative to the pressure in reservoir 11, which is regulated by needle valve 12. At a certain point, the pressure on the larger-diameter piston of the spool in space ***C*** becomes sufficient to overcome the pressure force in space ***A***.

To regulate this stage, additional force is generated by spring 13. The spool then moves to its extreme left position.
4. Compressed gas from space ***B*** exits through openings 3 and 14 into the atmosphere. At this point, the gas pressure in reservoir 11 opens membrane 4, and the compressed gas enters the exhaust pipe.
5. Check valve 9 opens, and gas enters aeration system 7. This stage is carried out in a pulsed pneumatic impact mode, which causes the bulk material to rapidly collapse along the boundary of the fluidized and aerated layers. The created momentum also contributes to the additional acceleration of the aerated material in transport line 16.
5. At stage 4, the pressure in tank 11 drops sharply, and under the influence of excess pressure in tank 8, a reverse wave is generated. This reverse wave impacts the bulk material and also excites an oscillatory process on the gas distribution baffle [8].

2.2 Identification of Control Parameters

Thus, two main control parameters can be identified in the system under consideration.

The first parameter ω_0 - the frequency of the pneumatic shocks is regulated by the opening frequency of valve 11. The second parameter τ is the time delay of the pressure increase in space ***C***, which is regulated by valve 12. So, for mathematical description, a two-parameter model for controlling the oscillatory process has been selected [21], [22].

3 Problem Solution

This section describes the theoretical details of the model.

3.1 Model Concept

The oscillations occurring in this system are parametric [22]. A model based on the Duffing equation is widely used to describe nonlinear oscillatory processes in continuous media in cases of forced oscillations [23]. A distinctive feature of this model is its adaptability to two-parameter systems.

Using the more general two-parameter model developed by the authors in the paper [21] results in a more complex and cumbersome model. Therefore, this paper attempts to use the Duffing equation as a more convenient tool for preliminary analysis [24].

To do this, it is first necessary to estimate the natural frequency of the system. The simplest estimate, based on similarity theory, is as follows:

$$\omega_0 = k\frac{\sqrt{gh}}{L}, \tag{1}$$

where L is the characteristic longitudinal dimension of the system, h is the height of the material layer on the partition, and k is a coefficient depending on the bulk density and configuration of the system.

The corresponding equation of Duffing type for the forced oscillations of the material layer can be written as follows [24]:

$$\frac{d^2H}{dt^2} + \omega_0^2 H + \varepsilon H^3 = F\cos\omega^* t, \tag{2}$$

where ε is a small parameter characterizing the nonlinearity of the interaction, H is the amplitude of the disturbed motion of the material layer, F is the amplitude of the external pulsations, ω^* is the frequency of external pulsations.

Assuming that the amplitude of the disturbance waves of the surface of the moving material layer is noticeably smaller than its thickness, to estimate the small nonlinearity parameter, the following relationship can be chosen

$$\varepsilon \sim \frac{H}{h}. \tag{3}$$

The expression for the value F is taken as follows

$$F = \frac{F^* T^{*^2}}{h^2}. \tag{4}$$

Here F^* is an estimate of the characteristic pulse amplitude; T^* is the characteristic pulse period.

Justification of the adopted form of equation (2) is based on arguments and estimates from various works on modeling nonlinear oscillations using the Duffing equation [24], [25].

3.2 Preliminary Model Analysis

In accordance with the primary goal of this study, which is to define the structure of the mathematical model and identify the appropriate control parameters for describing the process within the newly developed design, the analysis of the model itself in this section is preliminary.

The stability of the solutions described below, which differ by orders of magnitude relative to the small parameters introduced, will be further studied by the authors in subsequent works using well-known mathematical tools for solving such problems [26].

The model is rearranged using asymptotic methods by expanding in terms of a small parameter. The application of the multiscale method shows that in this case, both the primary resonance and a number of secondary resonances characteristic of nonlinear interaction can arise in the system being described [27].

To simplify the analysis, the following set of variable substitutions was introduced to reduce the equations to dimensionless form:

$$\tau = \omega_0 t\,;\ \delta = \frac{h}{\omega_0^2 L}\,;\ \widetilde{F} = \frac{F}{\omega_0^2}\,;\ \omega = \frac{\omega^*}{\omega_0}\,;$$

$$H = \frac{\Delta h}{L}. \tag{5}$$

Then, instead of equation (2), the governing equation takes the form:

$$\frac{d^2 H}{d\tau^2} + H + \delta H^3 = \widetilde{F}\cos\omega\tau\,. \tag{6}$$

In this case, the role of the small parameter passes to the parameter δ.

The resonances of the described system were obtained using a multiscale method [28]. For this purpose, in addition to the fast time scale T_1, a specific slow time scale T_0 was introduced.

The following key relationship relates the introduced asymptotic time variables to the previously specified control time parameter τ:

$$\tau = T_0 + \delta T_1\,. \tag{7}$$

So, an approximate analytical solution to the equation of layer oscillations was sought in the form of an asymptotic expansion in a small parameter δ [29]:

$$H = H_0 + \sum_{n=1}^{\infty} \delta^n H_n\,. \tag{8}$$

The technique for estimating errors in such decompositions as applied to nonlinear time series is well described in the paper [30].

For the zeroth and first orders of the expansion, the following equations were obtained:

$$\frac{\partial^2}{\partial T_0^2} H_0 + H_0 = \widetilde{F}\cos\omega T_0\,, \tag{9}$$

$$\frac{\partial^2}{\partial T_0^2} H_1 + H_1 = -\frac{\partial^2}{\partial T_0 \partial T_1} H_0 - H_0^3\,. \tag{10}$$

The solution of equation (9) can be written in complex form as (here i denotes the imaginary unit) [22], [28]:

$$H_0 = A(T_1)\exp(iT_0) + \Lambda\exp(i\omega T_0) + complex conjugates \tag{11}$$

For the values A of the free oscillations and Λ of the forced ones in the system, it can be obtained, respectively:

$$A = \frac{1}{2} a \exp(i\varphi)\,; \qquad \Lambda = \frac{\widetilde{F}}{2\left(1-\omega^2\right)}\,. \tag{12}$$

Here a is an amplitude and φ is a phase variable of free oscillations.

Then equation (10) can be rewritten in the form:

$$\frac{\partial^2}{\partial T_0^2} H_1 + H_1 = -2i\frac{\partial A}{\partial T_1}\exp(iT_0) + 2i\frac{\partial \overline{A}}{\partial T_1}\exp(-iT_0) - \left[A\exp(iT_0) + \Lambda\exp(i\omega T_0) + \overline{A}\exp(-iT_0) + \Lambda\exp(-i\omega T_0)\right]^3\,. \tag{13}$$

The complex conjugate terms in (13) are denoted by a bar above the letter.

An analysis of this expression allows us to identify as secondary resonances those values of the reduced frequency at which the exponential expression yields a value equal to unity, i.e., the frequencies of the forced and natural oscillations of the system are equal [27].

After opening the brackets and applying some rearrangements and transformations, equation (13) can be expanded and brought to form (14).

$$\begin{aligned}\frac{\partial^2}{\partial T_0^2} H_1 + H_1 = &-\left[2i\frac{\partial A}{\partial T_1} + 3\left(A\overline{A} + 2\Lambda^2\right)A\right]\exp(iT_0) - \\ &\left(6A\overline{A} + 3\Lambda^2\right)\Lambda\exp(i\omega T_0) - \\ &- A^3\exp(3iT_0) - \Lambda^3\exp(3i\omega T_0) - \\ &3A^2\Lambda\exp(i(\omega+2)T_0) - 3\overline{A}^2\Lambda\exp(i(\omega-2)T_0) - \\ &- 3A\Lambda^2\exp(i(2\omega+1)T_0) - 3A\Lambda^2\exp(i(-2\omega+1)T_0) + \\ &complex conjugates\end{aligned} \tag{14}$$

In the case that corresponds to a high frequency of external pulses, it is proposed to examine the behavior of solutions near the secondary resonance in the form [22], [28]:

$$\omega_p^{(1)} = 3;\ \omega_p^{(2)} = 1/3;\ \omega_p^{(3)} = 0\ , \tag{15}$$

After some transformations, the equations for changing the amplitude and phase of the forced oscillations of the layer near the first secondary resonance can be obtained:

$$\omega = 3 + \eta\delta\,; \tag{16}$$

$$\frac{\partial a}{\partial T_1} = -\frac{3}{4}a^2\Lambda\sin(\eta T_1 - 3\varphi). \quad (17)$$

$$\frac{\partial \varphi}{\partial T_1} = -\frac{\eta}{3} + 3\Lambda^2 + \frac{3}{8}a^2 + \frac{3}{4}a\Lambda\cos(\eta T_1 - 3\varphi) \quad (18)$$

Here we introduce the new parameter η which characterizes the degree of the external frequency proximity to the value of the secondary resonance.

Analysis of the possible resulting solution of system (16), (17), (18) shows that a nontrivial steady-state solution does not exist in this case [28]. A similar method can be used to study the remaining secondary resonances.

In further research, it would be interesting to introduce some characteristic of the "renewal" of the loosened layer of material. Then, it would be possible to calculate the parameters of the renewed structure of the material layer.

For this purpose, some characteristic of energy dissipation in the layer (such as conditional viscosity) may be proposed. In this case, the main novel assumption is that the order of viscous dissipation is estimated by the same small parameter as the order of the interaction nonlinearity.

Indeed, if viscous dissipation is strong, the damping of the oscillatory motion of the layer will be rapid, and no significant renewal of the structure should be observed. Of course, this is a hypothetical concept that requires more careful study.

Regarding this concept, the fundamental equation of layer motion is then transformed to the form:

$$\frac{d^2H}{d\tau^2} + \hat{\mu}\delta\frac{dH}{dt} + H + \delta H^3 = \widetilde{F}\cos\omega\tau, \quad (19)$$

where $\hat{\mu}$ is the reduced “conditional viscosity”, i.e. the dissipation characteristic in the layer:

$$\hat{\mu} = \frac{\mu}{\rho S}. \quad (20)$$

Here μ is the physical “conditional viscosity” and ρ is the bulk density of the bulk material [31], and S is the cross-sectional area of the layer.

A detailed analysis of equation (19) is performed similarly to the analysis of equation (2) and using the same method of multiscale decompositions. However, it is now possible to study the stationary oscillatory motion modes of the system.

After some transformations, the following system has been derived:

$$\frac{\partial a}{\partial T_1} = -\frac{\hat{\mu}}{2}a - \frac{3}{4}a^2\Lambda\sin(\eta T_1 - 3\varphi); \quad (21)$$

$$\frac{\partial \varphi}{\partial T_1} = -\frac{\eta}{3} + 3\Lambda^2 + \frac{3}{8}a^2 + \frac{3}{4}a\Lambda\cos(\eta T_1 - 3\varphi). \quad (22)$$

From the stationarity conditions of system (21), (22), the amplitude-frequency characteristic of the system reads:

$$a^2 = \frac{8}{9}\left(\eta - \frac{27}{4}\Lambda^2\right) \pm \frac{8}{3}\sqrt{\frac{1}{4}\Lambda^2\left(2\eta - \frac{63}{4}\Lambda^2\right) - \frac{\hat{\mu}^2}{4}} \quad (23)$$

It follows from expression (23) that for real physical oscillations of the material layer on the tape to occur in the space of control parameters, the following inequality must be satisfied

$$2\eta - \frac{63}{4}\Lambda^2 \geq \frac{\hat{\mu}}{\Lambda}. \quad (24)$$

Relationship (24) can only be satisfied if the conditional viscosity of the material is sufficiently low and the bulk density is sufficiently high. Of course, explicit numerical limitations can only be specified for specific materials under real-world bulk material handling conditions.

The resulting relationships (23) represent the model describing the operation of a pneumatic-pulse bulk material handling system. Despite its hypothetical nature, the developed mathematical model allows for the calculation of the system's resonant characteristics using a minimum of experimental data and the evaluation of optimal control parameter values.

Of course, the approach used in this work is heuristic in nature, and a more thorough analysis of the issues of convergence of asymptotic expansions and error assessment of the obtained analytical estimates will be required. Further analysis and development of this model will enable the calculation of the system's performance depending on the specific characteristics of the materials being transported and the scale of the equipment used.

4 Conclusion

Based on the analysis of the operation of the new technological process control scheme, the control parameters necessary for a mathematical description of the process dynamics have been substantiated. This is a two-parameter mathematical model based on the Duffing equation for describing the process dynamics and controlling the pneumatic-pulse handling of powder materials according to a new energy-saving scheme.

A modification of the basic model (Equation (19)) which makes it possible to propose methods

for taking into account the loosening of the material layer during transportation has been submitted too. Based on stationarity conditions, the amplitude-frequency characteristic of the system was derived.

The purpose of this work was to present the model structure as well as identification and interpretation of the control parameters, not to conduct a rigorous mathematical study of the resulting model and formulate all necessary constraints. Therefore, further research is necessary to verify the adequacy and reliability of the model. The obtained results can then form the basis for process calculations and control.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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Umida Kayumova, Akhmetbek Mussabekov have carried out the simulation. Arnold Brener has formulated the concept and problem set up.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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