

Singular Perturbation Modeling of the Flexible Joint Manipulator and Electric Vehicles Thermal Management System

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Abstract: - The paper is a research on flexible-joint robot manipulator robust tracking control and thermal management system for Electric Vehicles(EV), where the dynamical uncertainty is modeled and evaluated as an Initial Value Problem (IVP), done through the Singular Perturbation Method (SPM). SPM solution leads to two branches of solutions, (i) boundary-layer correction (BLC) solution reviving the initial conditions lost to suboptimal stability, and (ii) an outer series solution that improves the steady-state performance. In the case of the Initial Value Problem (IVP) under investigation, where a flexible-joint robot manipulator and EV are approximated to second order, numerical simulations are provided and contrasted on a case where all the parameters are non-zero and the opposite case where all parameters are zero. The findings are shown to prove the effectiveness of the proposed SPM framework in the case of a brushless DC motors which drives a three joint articulated flexible-joint robot manipulator and the applicability of the Peltier-effect heat pumps for the temperature regulation of an EV.

Key-Words: - Singular Perturbation Method, Robust, Initial Value Problem, Boundary layer correction, Flexible joint robot manipulators, Thermal Management System, Electric Vehicles.

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1 Introduction

The singularly perturbed methods(SPM) are ailing-ordered systems by working out the hardness. Thus, to acquire an exact solution, particular mathematical methods are required to conquer this hardness. The SPM eliminates the hardness of the system by suppressing the small parameters and reducing the order of the system, and satisfying all particular boundary conditions of the system. Thus, the modelled SPM thereby gives the order of solution very close to the exact solution.

In earlier days, robot manipulators were controlled by torque-based mechanisms. This mode of approach has a few limitations corresponding to the exclusion of actuator dynamics in the controller and complexity in manipulator dynamics. These limitations are characterized by coupling, linearity, uncertainty, the largeness of the model, and joint flexibility. This paper proposes a novel approach which free up manipulator dynamics by a voltage

controller mechanism. This method of approach overcomes all the limitations of the torque-based approach and fetches the advantage of input-based control in actuator and manipulator dynamics. To overcome the shortcomings of that kind, a great attention to the proper uncertainty bound parameter is to be paid for improving robust control of Flexible Joint Robot Manipulators (FJRM). The concept flow in the present work starts with an introduction, followed by modelling of BLDC motor-driven FJRM, and The conventional dynamic systems under a two-time scale with small parameter ε are inherently slow and fast dynamical systems. Thermal capacitance from the heat sinks, Thermal capacitance from the battery compartment and large-scale power systems fall under two timescale systems. Further the ε , which is a scalar dimensionless parameter and a small positive, is determined from the physical properties of a two-time scale system.

The very next section demonstrates a robust control strategy followed by stability analysis, performance evaluation, and uncertainty estimation. Then the efficacy of simulation results is substantiated by comparing the order of solutions versus the actual exact solutions, with concluding remarks.

2 Literature Review

The perturbation method is a multi-temporal scale measurement, where the single-parameter singular perturbation system is generated by the two-time scale system, and the two-parameter singular perturbation system is generated by the three-time scale system. Similarly, multi-time scale (MTS) systems can be single perturbation systems with n parameters [1]. MTS systems are very rigid and exhibit chaotic behaviour similar to the butterfly phenomenon. Clearly, these systems need alternatives SPM [2]. Singular perturbation (SP) theory is well developed in continuous-time control [3], [4], [5], [6], [7], compared to discrete-time control [8], [9], [10]. Chaos [11] addresses the butterfly phenomenon, which poses the problem of finding a solution, at various time scales.

Usually, electric motors run at high speed and low torque in contrary to the requirements of the robot mechanism. To overcome this, a power transmission conversion system is provided which runs an electric motor driving a robot arm at high torque and low speed. These power transmission conversion systems are very sensitive and, for any slight deviation, might wear out the flexibility of the joints. This sensitive nature is the major cause of pulsation in industrial robot manipulators [12], [13], [14]. Thereafter, the aided advantage of such manipulators when compared to rigid robots is that the degree of freedom will be twice as flexible in the joints in terms of control actions. The flexible joint robot applied for high precision seems to have difficulty in link positioning and therefore cannot trace the actuator position [15]. One vehicle (EV) is the way to ensure and maintain a good thermal comfort in the passenger compartment and to keep the temperature of the battery compartment within certain limits to ensure optimum performance of the batteries [16]. There are several battery performance problems associated with a lack of thermal management. Hence, to avoid the unwanted oscillations, a compensator is to be provided to improve the performance.

3 Singular Perturbation Method

3.1 Statement of Problem

The conventional dynamic systems under a three-time scale with small parameter ε are inherently slow, medium, and fast dynamical systems. The real-time flexible joint robots, flexible link robots and large-scale power systems fall under three-time-scale systems. And Thermal capacitance from the heat sinks, Thermal capacitance from the battery compartment, and large-scale power systems fall under two timescale systems. Further the ε , which is a scalar dimensionless parameter and a small positive, is determined from the physical properties of the time-scale system.

i. For the Flexible joint manipulator:

For the Flexible joint manipulator, a perturbation parameter is selected as shown below. The flexible manipulator with time varying dynamical state is obtained [15] as:

$$\ddot{\theta} = \frac{K_{stiff}}{J_{hub}} \alpha - \frac{K_m^2 K_g^2}{R J_{hub}} \dot{\theta} + \frac{K_m K_g}{R J_{hub}} V_m \quad (1a)$$

$$\ddot{\alpha} = -\frac{K_{stiff}(J_{link} + J_{hub})}{J_{hub} J_{link}} \alpha + \frac{K_m^2 K_g^2}{R J_{hub}} \dot{\theta} - \frac{K_m K_g}{R J_{hub}} V_m \quad (1b)$$

The stiffness constant associated by the scaling factor is defined as

$$\tilde{K}_{stiff} = \mu^2 K_{stiff}$$

$$\text{Let } \alpha = \mu^2 \beta$$

Hence we obtain the model as:

$$\ddot{\theta} = \frac{K_{stiff}}{J_{hub}} \mu^2 \beta - \frac{K_m^2 K_g^2}{R J_{hub}} \dot{\theta} + \frac{K_m K_g}{R J_{hub}} V_m \quad (2a)$$

$$\ddot{\alpha} = -\frac{K_{stiff}(J_{link} + J_{hub})}{J_{hub} J_{link}} \mu^2 \beta + \frac{K_m^2 K_g^2}{R J_{hub}} \dot{\theta} - \frac{K_m K_g}{R J_{hub}} V_m \quad (2b)$$

Let us define the state variables as

$$\begin{aligned} x_1 &= \theta & z_1 &= \beta \\ x_2 &= \dot{\theta} & z_2 &= \varepsilon \dot{\beta} \end{aligned} \quad (3)$$

where ε is the perturbation factor or small parameter.

Hence we obtain the SP Model as:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{\tilde{K}_{stiff}}{J_{hub}} z_1 - \frac{K_m^2 K_g^2}{RJ_{hub}} x_2 + \frac{K_m K_g}{RJ_{hub}} u \\ \varepsilon \dot{z}_1 &= z_2 \\ \varepsilon \dot{z}_2 &= -\frac{\tilde{K}_{stiff}(J_{link} + J_{hub})}{J_{hub} J_{link}} z_1 + \frac{K_m^2 K_g^2}{RJ_{hub}} x_2 - \frac{K_m K_g}{RJ_{hub}} u\end{aligned}\quad (4)$$

We can rewrite (4) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \varepsilon \dot{z}_1 \\ \varepsilon \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{K_m^2 K_g^2}{RJ_{hub}} & \frac{\tilde{K}_{stiff}}{J_{hub}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{K_m^2 K_g^2}{RJ_{hub}} & -\frac{\tilde{K}_{stiff}(J_{link} + J_{hub})}{J_{hub} J_{link}} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_m K_g}{RJ_{hub}} \\ 0 \\ -\frac{K_m K_g}{RJ_{hub}} \end{bmatrix} u\quad (5)$$

We can rewrite (5) as

$$\begin{aligned}\dot{x}(t) &= A_{11} x(t) + A_{12} z(t) + B_1 u(t) \\ \varepsilon \dot{z}(t) &= A_{21} x(t) + A_{22} z(t) + B_2 u(t)\end{aligned}\quad (6)$$

Where

$$\begin{aligned}A_{11} &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{K_m^2 K_g^2}{RJ_{hub}} \end{bmatrix}, & A_{12} &= \begin{bmatrix} 0 & 0 \\ \frac{\tilde{K}_{stiff}}{J_{hub}} & 0 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0 \\ \frac{K_m K_g}{RJ_{hub}} \end{bmatrix} \\ A_{21} &= \begin{bmatrix} 0 & 0 \\ 0 & \frac{K_m^2 K_g^2}{RJ_{hub}} \end{bmatrix}, & A_{22} &= \begin{bmatrix} 0 & 1 \\ -\frac{\tilde{K}_{stiff}(J_{link} + J_{hub})}{J_{hub} J_{link}} & 0 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0 \\ -\frac{K_m K_g}{RJ_{hub}} \end{bmatrix}\end{aligned}$$

From this we obtain the time-scale model of the flexible manipulator in Table 1.

Table 1. Specifications of the flexible joint manipulator

Specification	Symbol	Value
Inertia of the link	J_{link}	0.0026 [kgm ²]
Gear Ratio	K_g	70
Inertia oh hub	J_{hub}	0.0035[kgm ²]
Link mass	m	0.403 [Kg]
Grav. Const.	g	9.81 [N/m]
Joint stiffness(Spring)	K_{stiff}	1.2485 N/m
Motor resistance	R_m	2.6 Ω
Motor constant	K_m	0.00767 [N/rad/Sec]

Gearbox Efficiency	η_g	0.9[pu]
Motor efficiency	η_m	0.69[pu]

Source: [14]

ii. For the thermal management system:

For on a thermal management system for EV, a perturbation parameter is selected as shown below. A thermal management system for EV with time-varying dynamical state is obtained [16]. As shown in Figure 1 (Appendix) illustrates the equivalent electrical circuit that reflects the behavior of the thermal management unit in the linear region.

$$\begin{bmatrix} V_A \\ V_B \end{bmatrix} = \begin{bmatrix} \frac{1}{R_A C_A} + \frac{1}{R_B C_A} & \frac{1}{R_B C_A} \\ \frac{1}{R_B C_B} & \frac{1}{R_B C_B} + \frac{1}{R_C C_B} \end{bmatrix} \begin{bmatrix} V_A \\ V_B \end{bmatrix} + \begin{bmatrix} \frac{1}{R_A C_A} & 0 \\ 0 & \frac{1}{R_C C_B} \end{bmatrix} \begin{bmatrix} e_1 \\ e_0 \end{bmatrix}\quad (7)$$

From this we obtain the two-timescale model of the Thermoelectric Units in Table 2.

Table 2. Specifications of the Thermoelectric Units

Specification	Symbol	Value
Thermal resistance of the heat sink	R_A	30K Ω
Thermal resistance through the forced convection	R_B	80K Ω
	R_C	120K Ω
Thermal capacitance from the heat sinks	C_A	10mF
Thermal capacitance from the battery compartment	C_B	3mF

Source: [16]

3.2 Perturbation Method

Discrete Multi Parameter(DMP) Problem: The extensive study of discrete control systems pertaining to multi-parameter with SPM is realized as DMPSPM in state space form [11]. The linear DMPSPM model is mathematically represented as:

$$x_j(k+1) = A [\mu_0 \dots \mu_j x_j(k)] + B u(k) \quad (8.a)$$

$$x_j(k+1) = A [\varepsilon_0 \dots \varepsilon_j x_j(k)] + B u(k) \quad (8.b)$$

$$x_j(k=0) = x_j(0), j=0, 1, \dots, n.$$

$$\text{where } A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix} \text{ and}$$

state vector $x_{j-1}(k) \in R^{n_j}$, $j=1, 2, \dots, m$; $m=n+1$.

A_{ij} and B_i are chosen dimensionality matrices.

The free parameterized control vector is $u(k)$, $an \in R^r$. Now $\varepsilon_1 = \mu_1$ and $\varepsilon_j = \frac{\mu_j}{\mu_{j-1}}$ where $j = 2, \dots, n$ is redefined as fast momentum time scale parameters. The parameter $\varepsilon_0 = \mu_0 \triangleq 1$ is initiated to make possible the appearance of the multi parameter systems. The satisfying initial conditions, the respective system (8a) are

$$x_j(k=0) = x_j(0), j = 0, 1, \dots, n. \quad (8c)$$

The $(n_1+n_2+\dots+n_m)$ order degenerate TPBVP based singularly perturbed form of (8) is as follows:

$$\begin{bmatrix} x_0^{0\dots 0}(k+1) \\ x_j^{0\dots 0}(k+1) \end{bmatrix} = A \begin{bmatrix} x_0^{0\dots 0}(k) \\ 0 \end{bmatrix} + B u(k), \quad j = 1, 2, \dots, n. \quad (9a)$$

The boundary conditions of slow rate which is acquired by restraining parameters in (9a) is of order n_1 and are:

$$x_0^{0\dots 0}(k=0) = x_0(0) \text{ and } x_j^{0\dots 0}(k=0) \neq x_j(0), j = 1, 2, \dots, n. \quad (9b)$$

Modified SPM shown below restores $(n_2+\dots+n_m)$ initial condition that are unavailable in the process of degeneration.

Singular-perturbation-method has combined the outer solution and boundary layer solution with initial and boundary conditions, and recovered the fast variable in discrete systems. Outer solution is taken by stifling the small parameter ε . Then boundary layers are produced due to the loss of fast variable and to improve the solution [5]. First start the flow chart with the degenerate solution and then the zero-order solution to improve in Flow Chart for the singular perturbation method it as shown in Figure 2 (Appendix). Here, Outer and BLC solutions are to be found using outer and BLC equations and conditions. Then they are added according to total series solution (TSS) to get the zero-order solution. Similar procedure to be followed for first and higher order approximate solutions for further improvement.

4 Results

i. Two parameters SPM for Initial Value Problem (IVP):

The eighth order plant with a sampling time of 0.01s [15] is considered for evaluation. The control system is there, by resulted as follows equation (10)

$$\begin{bmatrix} x_{01}(k+1) \\ x_{02}(k+1) \\ x_{03}(k+1) \\ x_{04}(k+1) \\ x_{05}(k+1) \\ x_{11}(k+1) \\ x_{21}(k+1) \\ x_{22}(k+1) \end{bmatrix} = \begin{bmatrix} 1.0214 & 0.0393 & -0.0214 & 0.0355 & 0.0045 & 0.0607 & 0.0369 & 0.0044 \\ 1.4712 & 0.9472 & -1.4712 & 0.0296 & 0.0036 & 0.0528 & 0.0303 & 0.0036 \\ 0.0742 & 0.0981 & 0.9258 & 0.001 & 0.0001 & 0.0019 & 0.001 & 0.0001 \\ 0.0000 & 0.0000 & 0.0000 & 0.9087 & 0.0006 & 0.0000 & 0.0048 & 0.0006 \\ 0.0000 & 0.0001 & 0.0000 & 0.0039 & 0.9049 & 0.0001 & -0.0012 & 0.0001 \\ 0.5890 & 0.6557 & -0.589 & 0.2024 & 0.029 & 0.3441 & 0.2268 & 0.027 \\ -0.0003 & -0.0003 & 0.0003 & -0.9187 & -0.0002 & -0.0002 & -0.005 & -0.0006 \\ 0.001 & 0.0011 & -0.001 & 0.0485 & -0.9051 & 0.0005 & 0.0019 & 0.0000 \end{bmatrix} \begin{bmatrix} x_{01}(k) \\ x_{02}(k) \\ x_{03}(k) \\ x_{04}(k) \\ x_{05}(k) \\ x_{11}(k) \\ x_{21}(k) \\ x_{22}(k) \end{bmatrix} + \begin{bmatrix} 0.0071 \\ 0.4904 \\ 0.0247 \\ -0.0000 \\ 0.0000 \\ 0.1963 \\ -0.0001 \\ 0.0003 \end{bmatrix} u(k) \quad (10a)$$

$$\begin{bmatrix} x_{01}(k+1) \\ x_{02}(k+1) \\ x_{03}(k+1) \\ x_{04}(k+1) \\ x_{05}(k+1) \\ x_{11}(k+1) \\ x_{21}(k+1) \\ x_{22}(k+1) \end{bmatrix} = \begin{bmatrix} 1.0214 & 0.0393 & -0.0214 & 0.0355 & 0.0045 & 0.607 & 0.369 & 0.44 \\ 1.4712 & 0.9472 & -1.4712 & 0.0296 & 0.0036 & 0.528 & 0.303 & 0.36 \\ 0.0742 & 0.0981 & 0.9258 & 0.001 & 0.0001 & 0.019 & 0.01 & 0.01 \\ 0.0000 & 0.0000 & 0.0000 & 0.9087 & 0.0006 & 0.0000 & 0.048 & 0.06 \\ 0.0000 & 0.0001 & 0.0000 & 0.0039 & 0.9049 & 0.001 & -0.012 & 0.01 \\ 0.5890 & 0.6557 & -0.589 & 0.2024 & 0.029 & 3.1142 & 2.2860 & 2.700 \\ -0.0003 & -0.0003 & 0.0003 & -0.9187 & -0.0002 & -0.002 & -0.05 & -0.06 \\ 0.001 & 0.0011 & -0.001 & 0.0485 & -0.9051 & 0.005 & 0.019 & 0.0000 \end{bmatrix} \begin{bmatrix} x_{01}(k) \\ x_{02}(k) \\ x_{03}(k) \\ x_{04}(k) \\ x_{05}(k) \\ e_1 x_{11}(k) \\ e_1 e_2 x_{21}(k) \\ e_1 e_2 x_{22}(k) \end{bmatrix} + \begin{bmatrix} 0.0071 \\ 0.4904 \\ 0.0247 \\ -0.0000 \\ 0.0000 \\ 0.1963 \\ -0.0001 \\ 0.0003 \end{bmatrix} u(k) \quad (10b)$$

The eigenspectrum of this system is of three-time scale nature from (10a) as shown below:

$$(1.0000; 0.9996; 0.9048; 0.9039; 0.8557; 0.3831; 0.0000; -0.0000)$$

The two-parameter system of three time scale nature is as follows: $x=[x_0, x_1, x_2]$

Three time scales are $x_0=[x_{01}, x_{02}, x_{03}, x_{04}, x_{05}]'$, $x_1=[x_{11}]'$, $x_2=[x_{21}, x_{22}]'$. Where u is step input, x_0 is the rotor velocity, x_1 is the angular displacement, and x_2 is the indicating that the delivered torque is proportional to the input amplifier voltage. This IVP is solved using the SPM. The zero- to second order illustration of IVP with exact solution is presented in Figure 3, Figure 4, Figure 5, Figure 6, Figure 7, Figure 8, Figure 9 and Figure 10 (Appendix) and also in Table 3 (Appendix), we observe that the IVP traces the solution in near approximation from the degenerate solution to zero order, zero order to first, first to second, and second to the exact solution.

ii. One parameter SPM for Initial Value Problem (IVP):

The second-order plant with sampling time of 30s [16] is considered for evaluation. The control system is there by resulted as follows:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.1380 & 0.0360 \\ 0.1260 & 1.2070 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0990 & 0.0000 \\ 0.0000 & 0.0840 \end{bmatrix} u(k) \quad (11a)$$

The eigenspectrum of this system(11a) is of two-time scale nature as shown below:
(1.0968; 1.2482)

The one-parameter system of time scale nature is as follows: $x = [x_1, x_2]$

Where u is step input, x_1 is the thermal capacitance voltage from the heat sinks (V_A), and x_2 is Thermal capacitance voltage from the battery compartment (V_B) is proportional to the input Peltier-Effect Heat Source (e_1) and Ambient Temperature (e_0).

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.1380 & 0.0036 \\ 0.1260 & 0.1207 \end{bmatrix} \begin{bmatrix} x_1(k) \\ \varepsilon x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0990 & 0.0000 \\ 0.0000 & 0.0840 \end{bmatrix} u(k) \quad (11b)$$

where $\varepsilon = 0.1$

The satisfying initial conditions of the respective system are: $x_1(0) = 1$; $x_2(0) = 1$;

This IVP is solved using the SPM. The zero- to second order illustration of IVP with exact solution is presented in the Figure 11 and Figure 12 (Appendix), we observe that the IVP traces the solution in near approximation from the degenerate solution to zero order, zero order to first, first to second, and second to the exact solution.

5 Discussion

The degenerate solution, which will not satisfy the initial conditions of fast variables, is obtained by making parameters to zero in (9). The BLCs incorporated zero-order solution has the capability of recovering initial conditions $x_{11}(0)$, $x_{21}(0)$, $x_{22}(0)$ and $x_2(0)$. It is nearly equal to degenerate value at $k \geq 1$. Further, the nearness traverses from order zero to first order, first order to second order and finally second order to exact solution [5].

6 Conclusions

To obtain the solution of higher-order stiff continuous and discrete-time systems requires special numerical methods, unlike conventional methodologies. By order reduction and stiffness removal process, SPM can be applied to discrete control systems and solve these real time problems. Here three time-scale discrete flexible joint manipulator model of eighth order is considered for

evaluation and thermal management system for electric vehicles model of second order is considered for evaluation. IVP with one and two parameters are studied and hence is modeled as a one and two parameters singularly perturbed system. The BLC solution, which is an apt demonstration of SPM, recovers the lost initial conditions of fast variables to the best of its nature with increase in the order of approximation too. In spite of the first-order approximation, in this paper, SPM with second-order approximation is also presented. Boundary value and optimal control problems, which are of practical importance and difficult can also be addressed by SPM.

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The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

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APPENDIX

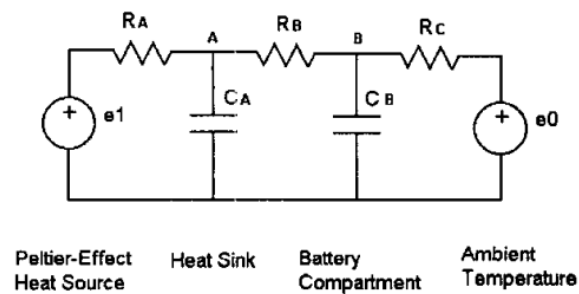


Fig. 1: Model for the Thermoelectric Units

Source: [16]

With:

- C_A : Thermal capacitance from the heat sinks
- C_B : Thermal capacitance from the battery compartment
- R_A : Thermal resistance of the heat sink (Conductive heat flow)
- R_B and R_C : Thermal resistance through forced convection
- e_1 : Peltier-Effect Heat Source
- e_0 : Ambient Temperature

This system is described by the following state variable:

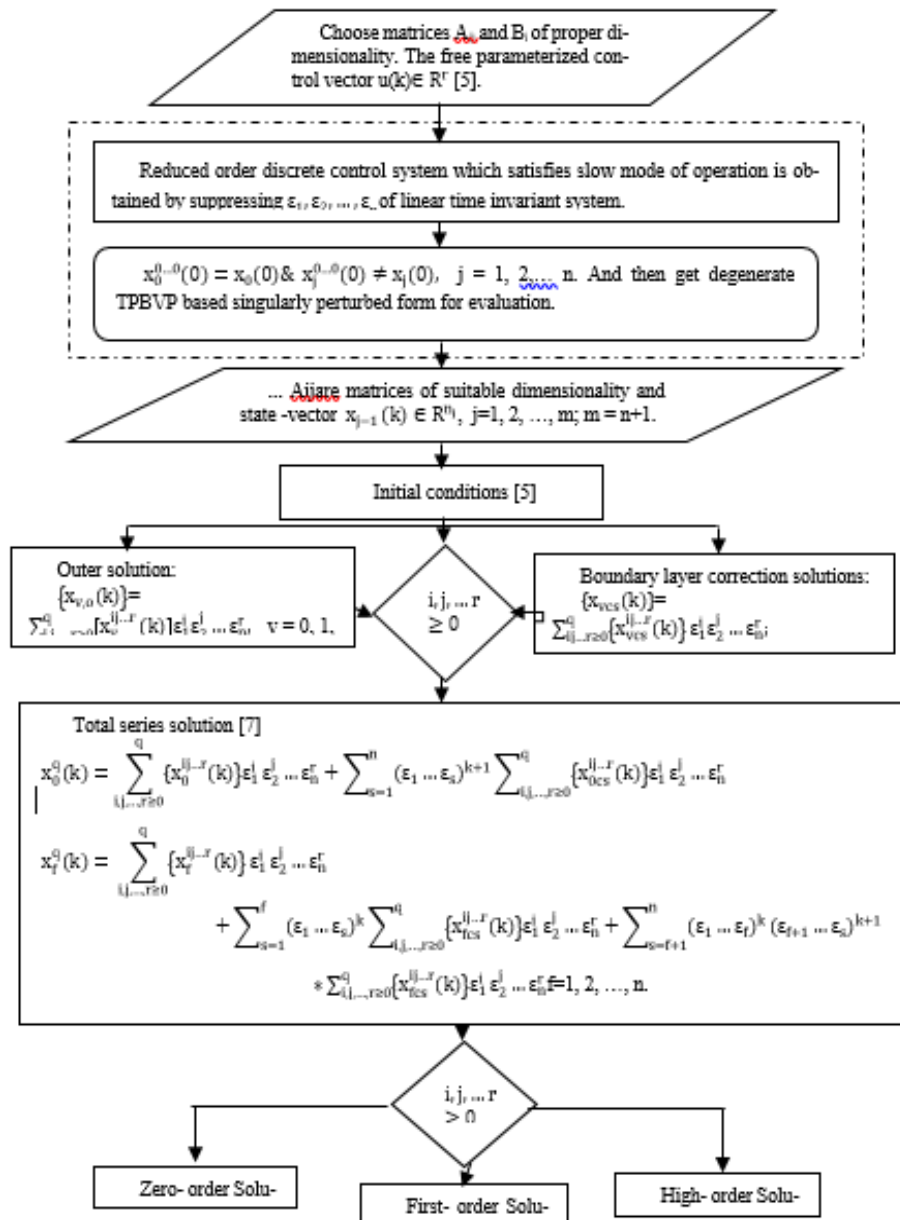


Fig. 2: Flow Chart for Singular Perturbation Methods

Source: created by the authors

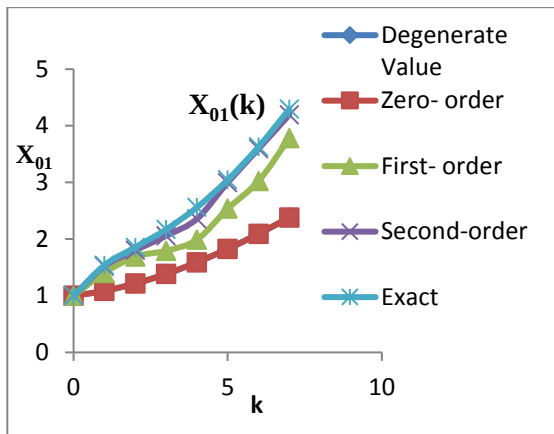


Fig. 3: Comparisons of various series solution of X_{01} with exact solution for IVP
Source: created by the authors from Table-3

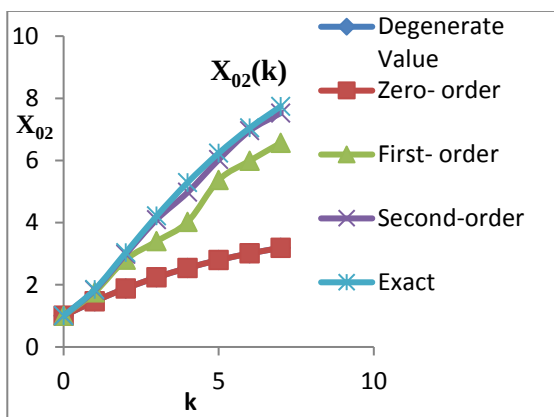


Fig. 4: Comparisons of various series solution of X_{02} with exact solution for IVP
Source: created by the authors from Table-3

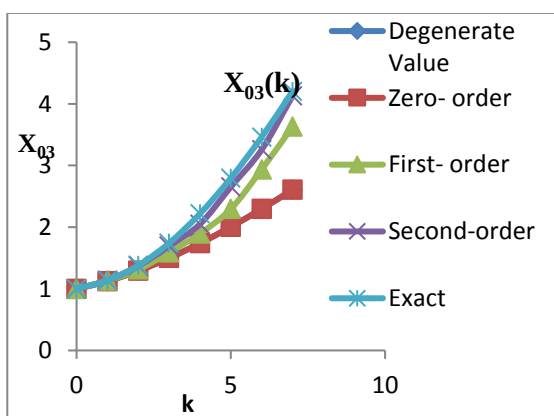


Fig. 5: Comparisons of various series solutions of X_{03} with exact solution for IVP
Source: created by the authors from Table-3

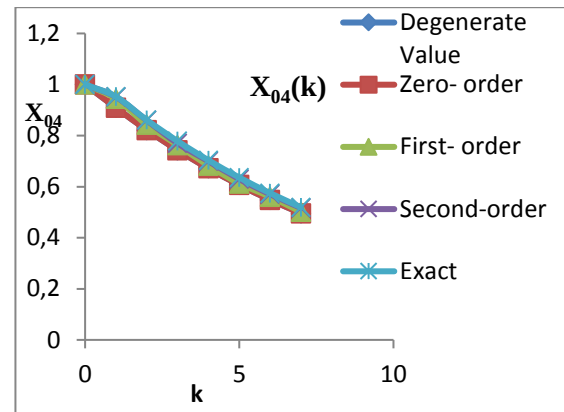


Fig. 6: Comparisons of various series solutions of X_{04} with the exact solution for IVP
Source: created by the authors from Table-3

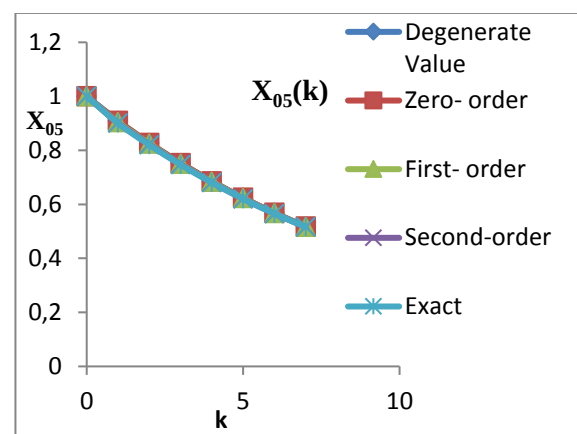


Fig. 7: Comparisons of various series solution of X_{05} with the exact solution for IVP
Source: created by the authors from Table-3

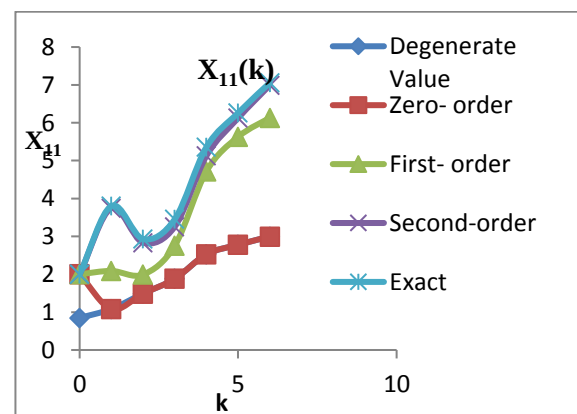


Fig. 8: Comparisons of various series solutions of X_{11} with the exact solution for IVP
Source: created by the authors from Table-3

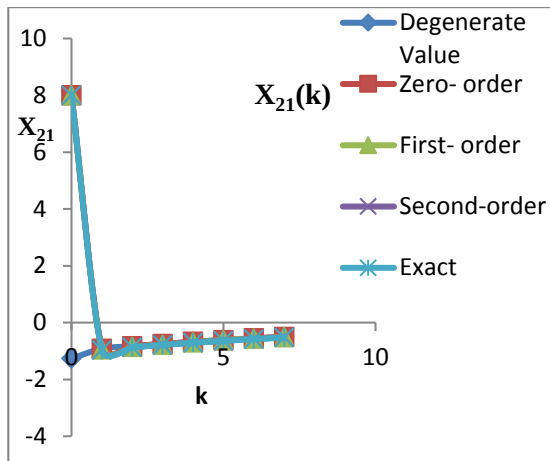


Fig. 9: Comparisons of various series solution of X_{21} with the exact solution for IVP
Source: created by the authors from Table-3

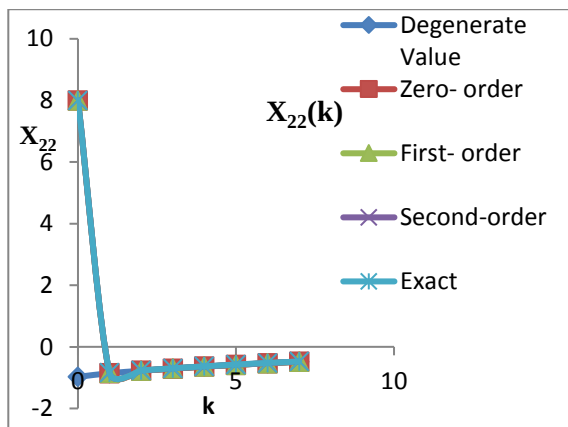


Fig. 10: Comparisons of various series solutions of X_{22} with exact solution for IVP
Source: created by the authors from Table-3

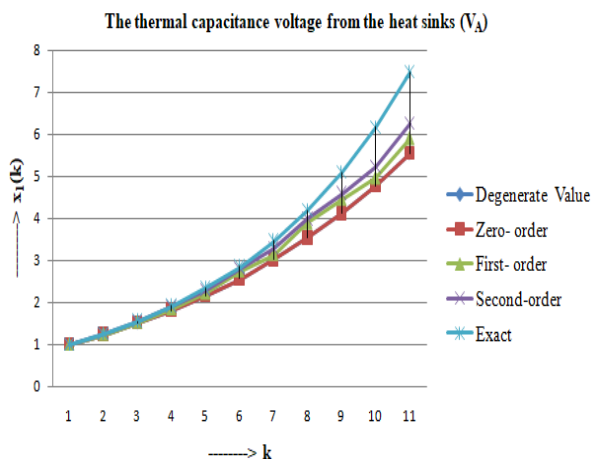


Fig. 11: Comparisons of various series solution of x_1 with the exact solution for IVP
Source: created by the authors from one parameter SPM

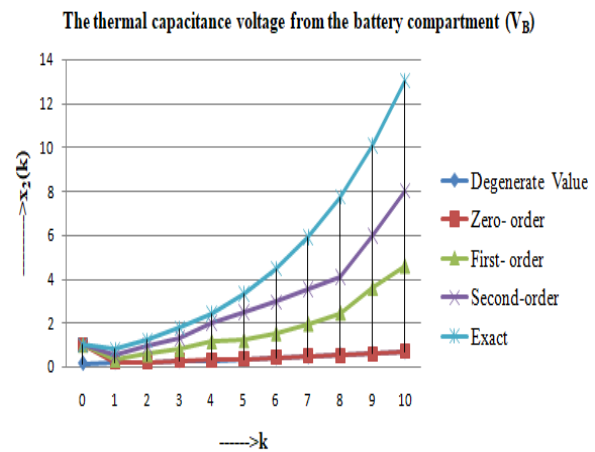


Fig. 12: Comparisons of various series solution of x_2 with the exact solution for IVP
Source: created by the authors from one parameter SPM

Table 3. Illustration of IVP exact solution versus different series solutions

$x(k)$	Solution				
	Degenerate Value	Zero-order	First-order	Second-order	Exact
$x(0)$	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000
	0.8452	2.0000	2.0000	2.0000	2.0000
	-1.2640	8.0000	8.0000	8.0000	8.0000
	-0.9752	8.0000	8.0000	8.0000	8.0000
$x(1)$	1.0864	1.0864	1.4035	1.5194	1.5382
	1.4708	1.4708	1.7598	1.8247	1.8476
	1.1239	1.1239	1.1339	1.1365	1.1365
	0.9093	0.9093	0.9439	0.9512	0.9525
	0.9089	0.9089	0.9042	0.9017	0.9003
	1.0834	1.0834	2.0874	3.7458	3.8020
	-0.9193	-0.9193	-0.9428	-0.9528	-0.9645
	-0.8552	-0.8552	-0.8498	-0.8405	-0.8390
$x(2)$	1.2149	1.2149	1.6915	1.8015	1.8559
	1.8848	1.8848	2.8104	2.9816	3.0314
	1.2922	1.2922	1.3124	1.3699	1.3795
	0.8219	0.8219	0.8425	0.8596	0.8609
	0.8273	0.8273	0.8252	0.8221	0.8200
	1.4902	1.4902	1.9912	2.8346	2.9301
	-0.8312	-0.8312	-0.8513	-0.8693	-0.8715
	-0.7779	-0.7779	-0.7710	-0.7685	-0.7659
$x(3)$	1.3837	1.3837	1.7935	2.0563	2.1689
	2.2401	2.2401	3.4010	4.0985	4.2166
	1.4989	1.4989	1.5898	1.6923	1.7425
	0.7429	0.7429	0.7628	0.7695	0.7782
	0.7531	0.7531	0.7501	0.7485	0.7470
	1.8803	1.8803	2.7528	3.2569	3.4525
	-0.7516	-0.7516	-0.7758	-0.7801	-0.7880
	-0.7074	-0.7074	-0.7005	-0.6989	-0.6965
$x(4)$	1.5895	1.5895	1.9899	2.3574	2.5592
	2.5415	2.5415	4.0265	4.9746	5.2933
	1.7383	1.7383	1.8956	2.0492	2.2190
	0.6715	0.6715	0.6824	0.6995	0.7034
	0.6856	0.6856	0.6836	0.6812	0.6806
	2.2270	2.2270	3.8591	4.1954	4.3819
	-0.6796	-0.6796	-0.6936	-0.7058	-0.7129
	-0.6434	-0.6434	-0.6401	-0.6375	-0.6327
$x(5)$	1.8274	1.8274	2.5369	2.9985	3.0466
	2.7957	2.7957	5.3691	6.0256	6.2355
	2.0055	2.0055	2.2956	2.6529	2.7965
	0.6069	0.6069	0.6125	0.6296	0.6358
	0.6242	0.6242	0.6229	0.6210	0.6204

	2.5257 -0.6145 -0.5852	2.5257 -0.6145 -0.5852	4.7196 -0.6143 -0.5850	5.1283 -0.6392 -0.5792	5.3587 -0.6451 -0.5746
x(6)	2.0930 3.0093 2.2961 0.5486 0.5685 2.7794 -0.5557 -0.5323	2.0930 3.0093 2.2961 0.5486 0.5685 2.7794 -0.5557 -0.5323	3.0249 5.9899 2.9365 0.5605 0.5671 5.6289 -0.5605 -0.5308	3.5936 6.9569 3.2567 0.5704 0.5659 6.1256 -0.5753 -0.5278	3.6284 7.0469 3.4617 0.5747 0.5657 6.2610 -0.5838 -0.5218
x(7)	2.3819 3.1884 2.6062 0.4959 0.5177 2.9934 -0.5025 -0.4842	2.3819 3.1884 2.6062 0.4959 0.5177 2.9934 -0.5025 -0.4842	3.7825 6.5634 3.6295 0.5012 0.5171 6.1285 -0.5095 -0.4818	4.1953 7.5236 4.1289 0.5167 0.5166 6.9936 -0.5208 -0.4755	4.2951 7.7406 4.2019 0.5194 0.5162 7.0558 -0.5283 -0.4739

Source: created by the authors