A Fractional Differential Model for Elastoviscoplastic Filtration of Liquids in Porous Media

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Abstract: - In this work, the problem of anomalous filtration of liquid in a porous medium is studied, taking into account the threshold pressure gradient (TPG). The flow of a homogeneous liquid in a porous medium was modeled by a differential equation with fractional derivatives. Fractional derivatives are defined in the Caputo sense. The problem was solved numerically by the finite difference scheme method. The pressure gradient and filtration velocity were calculated with known values of the TPG. The profiles of pressure and filtration velocity are determined for different values of the orders of fractional derivatives α and β, with respect to the filtration velocity and pressure gradient, respectively. An increase in TPG leads to a decrease in the filtration velocity. Moreover, this decrease is greater as the time increases t. Simultaneously with the decrease in g_0, the zone of distribution of the filtration velocity decreases, which means an increase in the size of the zone with stationary liquid.

Key-Words: - anomalous filtration, fractional derivative, finite difference method, maximum pressure gradient, porous medium, relaxation time.

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1 Introduction

It is known that oils from many fields have anomalous rheological properties. The anomalous properties of oil are caused by the presence of various components in its composition, particularly asphaltenes, resins, paraffins, and other heavy organic compounds. Anomalous viscosity of oil affects its movement in the field collector systems and in-field preparation, pipeline transport, etc. Anomalous viscosity usually has a stronger effect on the process of oil flow in the reservoir [1], which affects the filtration law. The classical Darcy's law is violated in this case. One of the manifestations of anomalies in oil filtration in a reservoir is the dependence of its mobility and viscosity on the pressure gradient at low filtration velocity [1]. Anomalous properties of oil can be the cause of low efficiency of the field development system, in particular, low oil recovery [2].

The anomaly of the rheological properties of heavy oils can manifest itself in the presence of plastic properties along with viscous properties [1], [2], [3], [4], [5]. For highly resinous paraffinic oils, the rheological curve at low shear stresses is nonlinear, and at high shear stresses, it is linear. In most cases, the curves pass through the origin of coordinates. Various rheological models can be proposed to approximate the curves. Sometimes, nonlinear parts of the curves can be approximated by a linear function. Then the entire flow curve can be divided into two sections, in each of which the curves have the form of straight lines. The straight line corresponding to low shear stresses has a smaller slope on graphs $\tau-\dot{\gamma}$, where τ is shear stress and $\dot{\gamma}$ is the shear velocity. Based on dependence $\tau=\mu\cdot\dot{\gamma}$ we have $\mu=\frac{\tau}{\dot{\gamma}}=ctg\varphi$, where φ is the angle of inclination of the straight line. Since with small τ we have a smaller φ, therefore, in this area the oil has a higher viscosity. For relatively large τ we have a larger φ, which means a smaller value of μ. The straight line for small τ cuts off the value τ_0 on the τ axis, which is interpreted as the threshold shear stress. At $\tau > \tau_0$, oil behaves as a viscous-plastic medium, which, for the linear dependence $\tau - \dot{\gamma}$ can be considered as a Shvedov-Bingham medium

$$\tau = \tau_0 + \mu \cdot \dot{\gamma}, \ \ \tau \geq \tau_0 , \qquad (1)$$

One of the modifications of the Shvedov-Bingham model in the case of a nonlinear dependence of $\dot{\gamma}$ on τ is the Kesson model

$$\tau^{1/n} = \tau_0^{1/n} + (\mu \cdot \dot{\gamma})^{1/n} , \qquad (2)$$

where n is a real positive number.

The noted features of rheological curves fully apply to drilling and cement solutions, especially those treated with high-molecular water-soluble polymers [5], [6], [7].

In a more general case, the dependence $\dot{\gamma} = f(\tau)$ can be nonlinear with convexity to the axis τ, $f''(\tau) > 0$. Such fluids are called pseudoplastic. The Shvedov-Bingham model of viscous-plastic fluid is a special case of the pseudoplastic model [8].

The presence of the threshold shear stress τ_0 in rheological curves (1), (2) lead to similar phenomena during liquid filtration in a porous medium.

During the filtration of asphaltene-resinous and paraffinic oils in a porous medium, phenomena with characteristics similar to those of the flow for viscous-plastic oils were discovered. The threshold shear stress τ_0 in filtration processes leads to the appearance of a threshold pressure gradient ∇p_0. Consequently, Darcy's law should be modified, taking into account ∇p_0. In [5], the generalized Darcy's law is written as

$$\vec{v} = -\frac{k}{\mu}\left(1 - \frac{|\nabla p|_0}{|\nabla p|}\right)\nabla p \ at \ |\nabla p| > |\nabla p|_0 ,$$

$$\vec{v} = 0 \ at \ |\nabla p| \leq |\nabla p|_0 , \qquad (3)$$

where k is the permeability of the porous medium, $\vec{v}$ is the filtration velocity, p is the pressure of the liquid in the porous medium.

As was indicated above, the rheological curve $\tau - \dot{\gamma}$ can be represented as two straight lines. Similarly, in the region of small $|\nabla p|$, i.e. when $|\nabla p| \leq |\nabla p|_0$, it is possible to propose a linear Darcy's law with a very small k/μ. Generally speaking, the dependence $\vec{v} - \nabla p$ can be approximated not only by two straight sections, but also by larger ones.

There is a direct relation between τ_0 and $|\nabla p|_0$. It is established as [9]

$$|\nabla p|_0 = \alpha \frac{\tau_0}{\sqrt{k}} , \qquad (4)$$

where α is a constant coefficient.

Taking into account (4), the generalized filtration law with a threshold pressure gradient can be written in the form

$$\nabla p = -\Phi(\vec{v}) , \qquad (5)$$

where

$$\Phi(\vec{v}) = \frac{\mu}{k}\vec{v} + |\nabla p|_0 \frac{\vec{v}}{v} , \qquad (6)$$

α is the coefficient.

From (5), (6) we have

$$\vec{v} = -\frac{k}{\mu}\left(\nabla p + |\nabla p|_0 \frac{\vec{v}}{v}\right). \qquad (7)$$

The filtration law (7) for $v > 0$ is written as

$$\vec{v} = \begin{cases} -\dfrac{k}{\mu}\left(\nabla p + |\nabla p|_0\right) \ at \ |\nabla p| > |\nabla p|_0 \\ 0 \qquad at \ |\nabla p| \leq |\nabla p|_0 \end{cases} , \qquad (8)$$

at $v < 0$ we have

$$\vec{v} = \begin{cases} -\dfrac{k}{\mu}\left(\nabla p - |\nabla p|_0\right) \ at \ |\nabla p| > |\nabla p|_0 \\ 0 \qquad at \ |\nabla p| \leq |\nabla p|_0 . \end{cases} \qquad (9)$$

Note that the threshold shear stress τ_0 is not the only cause of the occurrence of $|\nabla p|_0$ [3], [4]. The interaction of the liquid with the surface of the skeleton of the porous medium can be the cause of anomalous phenomena during filtration, in particular, the occurrence of $|\nabla p|_0$. It is also known that when a Newtonian fluid is filtered through clay layers, $|\nabla p|_0$ occurs. Another reason for the occurrence of anomalous phenomena may be the low permeability and porosity of the deposit rock. Research related to low-speed flow that does not obey Darcy's law was carefully analyzed for the first time, and the existence of a TPG was discussed [10]. Typically, deviation from Darcy's law manifests itself as a nonlinear dependence of v on $|\nabla p|$, especially in the region of low $|\nabla p|$. Usually, the intersection with the $|\nabla p|$ axis of the linear dependence at relatively low $|\nabla p|$ is determined by the TPG [11]. TPG manifests itself in the filtration of both single-phase and multiphase liquids in low-permeability media.

Unlike our previous study [12], which considered only a single relaxation mechanism (in filtration velocity) without fractional derivatives and without TPG, the present work generalizes the model in three essential directions: 1) double relaxation – both pressure and velocity relaxation; 2) fractional derivatives (in the Caputo sense) of arbitrary orders α and β and 3) explicit inclusion of the threshold pressure gradient g_0 . Furthermore, we provide a detailed numerical algorithm with stability and convergence analysis, which was not performed in [12]. Thus, the current manuscript represents a significant extension rather than a repetition of previous work.

In this paper, in contrast to [13], we consider a generalized relaxation fractional-differential model. Based on this generalized model, filtration equations are derived. The problem of anomalous filtration of liquid in a porous medium is posed and solved numerically, taking into account the TPG. The influence of the orders of fractional derivatives on the pressure and the filtration velocity in the medium at different moments in time is estimated.

2 Derivation of Equations

The filtration model with double relaxation in the one-dimensional case, taking into account the TPG, is taken as

$$\vec{v}+\lambda_v\frac{\partial\vec{v}}{\partial t}=\begin{cases}0 \quad at \quad |\nabla p|\le g_0\,,\\ -\dfrac{k}{\mu}\cdot\left[\dfrac{|\nabla p|-g_0}{|\nabla p|}\nabla p+\lambda_p\dfrac{\partial}{\partial t}\left(\dfrac{|\nabla p|-g_0}{|\nabla p|}\nabla p\right)\right]\end{cases}$$

$$at \quad |\nabla p|>g_0. \tag{10}$$

where g_0 is the TPG, λ_v, λ_p are the relaxation times of filtration velocity $\vec{v}$, and pressure p, k is the permeability of the medium, μ is the viscosity of the liquid, $|\nabla p|$ is the absolute value of the pressure gradient, $g_0=|\nabla p|_0$.

The continuity equation is written as

$$\frac{\partial v}{\partial x}+\beta^*\frac{\partial p}{\partial t}=0, \tag{11}$$

where β^* is the coefficient of elastic capacity of the formation.

We write equation (10) in fractional differential form

$$v+\lambda_v D_t^\beta v=\begin{cases}0 \quad at \quad \left|\dfrac{\partial p}{\partial x}\right|\le g_0\,,\\ -\dfrac{k}{\mu}\left(\dfrac{\left|\frac{\partial p}{\partial x}\right|-g_0}{\left|\frac{\partial p}{\partial x}\right|}\dfrac{\partial p}{\partial x}+\lambda_p D_t^\alpha\left(\dfrac{\left|\frac{\partial p}{\partial x}\right|-g_0}{\left|\frac{\partial p}{\partial x}\right|}\dfrac{\partial p}{\partial x}\right)\right)\end{cases}$$

$$at \quad \left|\frac{\partial p}{\partial x}\right|>g_0\,, \tag{12}$$

where α, β are the orders of fractional derivatives, D_t^β, D_t^α are the fractional derivative operators according to Caputo [14], $0<\alpha\le 1$, $0<\beta\le 1$.

The operator D_t^α has dimension (time)$^\alpha$. Therefore, in the term $\lambda_p D_t^\alpha p$, the relaxation time λ_p must have dimension (time)$^\alpha$ to make the product dimensionless.

In (12) the relaxation times λ_v and λ_p have a fractional dimension: $\lambda_v=T^\beta$, $\lambda_p=T^\alpha$, where T is the dimension of time. This approach characterizes effects that are nonlocal in time. In particular, from (12) we have

$$v+\lambda_v D_t^\beta v=0 \quad at \quad g_0\le 0, \tag{13}$$

$$v+\lambda_v D_t^\beta v=-\frac{k}{\mu}\left[\frac{\partial p}{\partial x}-g_0+\lambda_p D_t^\alpha\frac{\partial p}{\partial x}\right] at \quad \frac{\partial p}{\partial x}>0\,, \tag{14}$$

$$v+\lambda_v D_t^\beta v=-\frac{k}{\mu}\left[\frac{\partial p}{\partial x}+g_0+\lambda_p D_t^\alpha\frac{\partial p}{\partial x}\right] at \quad \frac{\partial p}{\partial x}<0\,. \tag{15}$$

By differentiating of equation (12) with respect to the coordinate x we obtain

$$\frac{\partial v}{\partial x}+\lambda_v\cdot\frac{\partial}{\partial x}\cdot D_t^\beta v=-\frac{k}{\mu}\frac{\partial}{\partial x}\cdot$$

$$\cdot\left(\frac{\left|\frac{\partial p}{\partial x}\right|-g_0}{\left|\frac{\partial p}{\partial x}\right|}\frac{\partial p}{\partial x}+\lambda_p D_t^\alpha\left(\frac{\left|\frac{\partial p}{\partial x}\right|-g_0}{\left|\frac{\partial p}{\partial x}\right|}\frac{\partial p}{\partial x}\right)\right) at \quad \frac{\partial p}{\partial x}<0. \tag{16}$$

Taking from equation (11), the derivative of order β with respect to time, we obtain

$$D_t^\beta\frac{\partial v}{\partial x}+\beta^* D_t^{\beta+1}p=0. \tag{17}$$

Using equation (17), we write equation (16) in the following form:

$$-\beta^*\cdot\frac{\partial p}{\partial t}-\lambda_v\cdot\beta^*\cdot D_t^{\beta+1}p=$$

$$= -\frac{k}{\mu}\frac{\partial}{\partial x}\left[-\left(-\frac{\partial p}{\partial x} - g_0\right) - \lambda_p D_t^{\alpha}\left(-\frac{\partial p}{\partial x} - g_0\right)\right],$$

from which:

$$\frac{\partial p}{\partial t} + \lambda_v D_t^{\beta+1} p = \varkappa \left(\frac{\partial^2 p}{\partial x^2} + \lambda_p D_t^{\alpha}\left(\frac{\partial^2 p}{\partial x^2}\right)\right), \qquad (18)$$

where $\varkappa = \frac{k}{\mu \beta^*}$ is the coefficient of piezoconductivity.

At $\alpha = 1$, $\beta = 1$ from (15), we obtain the conventional equation of relaxation filtration [13], [15]

$$\frac{\partial p}{\partial t} + \lambda_v \cdot \frac{\partial^2 p}{\partial t^2} = \varkappa \left(\frac{\partial^2 p}{\partial x^2} + \lambda_p \frac{\partial^3 p}{\partial t \partial x^2}\right). \qquad (19)$$

Equation (18) does not contain g_0 because this term cancels out during the derivation. Specifically, substituting the fractional filtration law (12)–(15) into the continuity equation (11) and then applying the fractional time derivative leads to cancellation of all terms containing g_0.

3 Statement of the Problem and Its Numerical Solution

The initial and boundary conditions for equation (18) for filtration in a finite medium [0, L] are taken in the following form

$$p(0,x) = 0, \qquad (20)$$

$$p(t,0) = p_0, \quad p_0 = const, \quad p(t,L) = 0. \qquad (21)$$

For (18) at $\beta > 0$ the initial condition (20) is not sufficient. One more condition must be added, for example

$$\frac{\partial p(0,x)}{\partial t} = 0. \qquad (22)$$

Equation (18) is solved under conditions (20), (21), (22).

4 Numerical Solution Algorithm

To numerically solve problems (18), (20), (21), and (22), we use the finite difference method. In the domain $\Omega = \{0 \le x \le L, 0 \le t \le T_{\max}\}$ we introduce a uniform grid

$$\omega_{h\tau} = \left\{ \begin{matrix} (x_i, t_i),\ x_i = ih,\ i = \overline{0,N},\ h = L/N,\ t_i = j\tau, \\ i = \overline{0,M},\ \tau = T_{\max}/M \end{matrix} \right\},$$

where h is the grid step in the coordinate x, τ is the grid step in time. We denote the grid function at a point (x_i, t_j) by p_i^j.

The difference approximation of equation (18) has the form

$$\frac{p_i^{j+1} - p_i^j}{\tau} + \lambda_v \cdot \frac{\tau^{2-\beta}}{\Gamma(3-\beta)} \cdot \left[\sum_{k=0}^{j-2} \frac{p_i^{k+1} - 2\cdot p_i^k + p_i^{k-1}}{\tau^2} \cdot \right.$$

$$\left. \cdot\left((j-k+1)^{2-\beta} - (k-1)^{2-\beta}\right) + \frac{p_i^{j+1} - 2\cdot p_i^j + p_i^{j-1}}{\tau^2}\right] =$$

$$= \kappa\left(\frac{p_{i+1}^{j+1} - 2\cdot p_i^{j+1} + p_{i-1}^{j+1}}{h^2}\right) + \kappa\left(\frac{\lambda_p}{h^2}\cdot\left(S_1 - 2\cdot S_2 + S_3\right)\right),$$

$$S_1 = D_t^{\alpha} p_{i+1}^{j+1} = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \cdot \qquad (23)$$

$$\cdot\left[\sum_{k=0}^{j-2} \frac{p_{i+1}^{k+1} - p_{i+1}^k}{\tau}\cdot\left((j-k+1)^{1-\alpha} - (j-k)^{1-\alpha}\right) + \frac{p_{i+1}^{j+1} - p_{i+1}^j}{\tau}\right],$$

$$S_2 = D_t^{\alpha} p_i^{j+1} = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \cdot$$

$$\cdot\left[\sum_{k=0}^{j-2} \frac{p_i^{k+1} - p_i^k}{\tau}\cdot\left((j-k+1)^{1-\alpha} - (j-k)^{1-\alpha}\right) + \frac{p_i^{j+1} - p_i^j}{\tau}\right],$$

$$S_3 = D_t^{\alpha} p_{i-1}^{j-1} = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \cdot$$

$$\cdot\left[\sum_{k=0}^{j-2} \frac{p_{i-1}^{k+1} - p_{i-1}^k}{\tau}\cdot\left((j-k+1)^{1-\alpha} - (j-k)^{1-\alpha}\right) + \frac{p_{i-1}^{j+1} - p_{i-1}^j}{\tau}\right],$$

where $\Gamma(\cdot)$ is the gamma function.

When approximating fractional derivatives in (23), the methodology of [16], [17], [18] was used.

Some results of numerical calculations according to (23) are shown below. The following values of initial parameters were used in the calculations: $k = 10^{-14}\ m^2$; $\mu = 1.768\cdot 10^{-3}\ Pa\cdot s$; $p_0 = 5\cdot 10^5\ Pa$; $\beta^* = 3\cdot 10^{-8}\ Pa^{-1}$, $L = 5\ m$.

From (15), the filtration velocity field is determined. Since the elastic filtration regime is considered, i.e., the compressibility of the porous medium and liquid is taken into account, it is natural to expect a non-uniform distribution of the filtration velocity field over time.

With known $p(t,x)$, the filtration velocity is determined from (15), after discretization, which takes the form

$$v_i^{j+1} + \frac{\lambda_v}{\Gamma(2-\beta)\cdot\tau^\beta}\cdot\sum_{k=0}^{j-1}\left(v_i^{k+1}-v_i^k\right)\cdot$$
$$\cdot\left((j-k+1)^{1-\beta}-(j-k)^{1-\beta}\right)+\left(v_i^{j+1}-v_i^j\right)=$$
$$=-\frac{k}{\mu\cdot h}\left(\left(p_{i+1}^{j+1}-p_i^{j+1}\right)+g_0+\frac{\lambda_p}{\Gamma(2-\alpha)\cdot\tau^\alpha}\cdot\right. \tag{24}$$
$$\cdot\left(\sum_{k=0}^{j-1}\left(p_{i+1}^{k+1}-p_{i+1}^k\right)\cdot\left((j-k+1)^{1-\alpha}-(j-k)^{1-\alpha}\right)+p_{i+1}^{j+1}-p_{i+1}^j-\right.$$
$$\left.\left.-\sum_{k=0}^{j-1}\left(p_i^{k+1}-p_i^k\right)\cdot\left((j-k+1)^{1-\alpha}-(j-k)^{1-\alpha}\right)-p_i^{j+1}+p_i^j\right)\right).$$

The filtration velocity on the upper time layer from (24) is determined as

$$v_i^{j+1}=\frac{\tau}{\Gamma(2-\beta)\cdot\tau^\beta+\lambda_v}\cdot\left(-\frac{k}{\mu\cdot h}\left(\left(p_{i+1}^{j+1}-p_i^{j+1}\right)+g_0+\right.\right.$$
$$+\frac{\lambda_p}{\Gamma(2-\alpha)\cdot\tau^\alpha}\left(\sum_{k=0}^{j-1}\left(p_{i+1}^{k+1}-p_{i+1}^k\right)\cdot\left((j-k+1)^{1-\alpha}-\right.\right.$$
$$\left.-(j-k)^{1-\alpha}\right)+p_{i+1}^{j+1}-p_{i+1}^j-\sum_{k=0}^{j-1}\left(p_i^{k+1}-p_i^k\right)\cdot \tag{25}$$
$$\left.\left.\cdot\left((j-k+1)^{1-\alpha}-(j-k)^{1-\alpha}\right)-p_i^{j+1}+p_i^j\right)\right)-\frac{\lambda_v}{\Gamma(2-\beta)\cdot\tau^\beta}\cdot$$
$$\left.\cdot\sum_{k=0}^{j-1}\left(v_i^{k+1}-v_i^k\right)\cdot\left((j-k+1)^{1-\beta}-(j-k)^{1-\beta}\right)+\frac{\lambda_v\cdot v_i^j}{\Gamma(2-\beta)\cdot\tau^\beta}\right).$$

Some results of calculations according to (25) are given below. As we can see from (18), the pressure field does not depend on g_0. Our main goal is to investigate the influence of g_0 on filtration characteristics. Therefore, below we present only the filtration velocity profiles for different g_0. However, to determine the filtration velocity profiles, we use the pressure field obtained for the corresponding values of the parameters λ_v, λ_p, α, β etc.

For the linearized fractional diffusion equation, the finite difference scheme is conditionally stable. For integer-order derivatives, the stability criterion is:

$$\frac{k\tau}{h^2}\le\frac{1}{2}.$$

For fractional orders $0<\alpha\le1$, the same CFL-like condition holds due to the positivity of the Grünwald–Letnikov coefficients.

5 Discussion and Results

The results for the case of equal relaxation parameters $\lambda_p=\lambda_v$ are shown in Figure 1; Filtration velocity profiles at different β, and Figure 2; Filtration velocity profiles at different α. A decrease of β from 1 leads to a complex dynamics of change of v. In this case, up to a certain x the values of v become larger, then smaller. A decrease of α from 1 leads to a completely inverse relationship (Figure 2). The solid lines in Figure 1 and Figure 2 correspond to the solution of the classical equation of piezoconductivity, derived from the linear equilibrium Darcy's law.

Figure 3; Filtration velocity profiles at different β, and Figure 4 show the distribution of the filtration velocity with a decrease of β from 1 and α from 1 in the case of $\lambda_v>\lambda_p$, respectively. With a decrease in the order of the derivative of β from 1, a relatively slow distribution of the filtration velocity is observed.

An increasing λ_v leads to a slowdown in the filtration velocity (Compare Figure 1 with Figure 3 and Figure 2 with Figure 4).

The separate influence of relaxation parameters λ_v and λ_p on the distribution of filtration velocity is shown in Figure 5, shows the effect of pressure relaxation on filtration velocity, and Figure 6 illustrates the influence of velocity relaxation. In the absence of pressure relaxation (Figure 5), with a decrease in β from 1, the distribution zone v decreases. Considering that in this case we have a wave equation, this means a finite velocity of disturbance propagation in the medium, both pressure and filtration velocity. In the absence of relaxation in filtration velocity, a decrease in α from 1 leads to similar phenomena as in (Figure 6).

As can be seen from the piezoconductivity equation (18), g_0 does not affect the pressure distribution. In order to evaluate the influence of g_0 on the distribution of filtration velocity, calculations were carried out for different values of it. Some results are presented in Figure 7 and Figure 8 present the effect of TPG on velocity profiles at different times. In all cases, with an increase in g_0, a decrease in v is observed. The figures show the filtration velocity profiles for two values of time t. With an increase in t at small x, a decrease in v can be observed. This is due to a change in the pressure gradient in this zone. With an increase in time, the pressure increases in the entire medium, and the modulus of the pressure gradient decreases. Therefore, a decrease in v occurs. In the distant zone from $x=0$, an increase of v occurs. This is also explained by the change in the pressure gradient. With an increase of g_0, the magnitude of the decrease of v progresses in time, i.e., the decrease of v with relatively large t becomes more noticeable. It should also be noted that the propagation zone of v lags behind the decrease of g_0. This means that with an increase of g_0, the size of the zone of immobile liquid increases, which is quite natural. The influence of parameters λ_v, λ_p, α, β on the distribution of v is the same as in the absence of g_0.

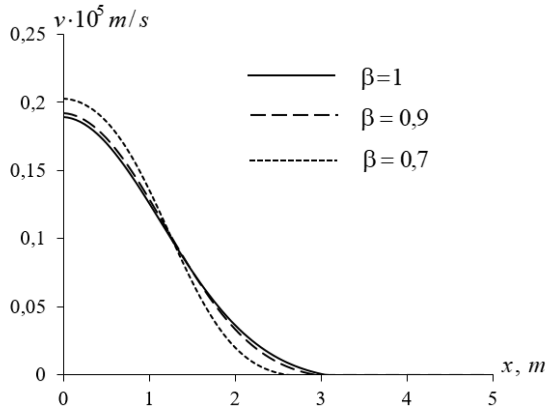

Fig. 1: Profiles of filtration velocity at various β and $t = 3600\,s$, $\lambda_v = 500\,s^{\beta}$, $\lambda_p = 500\ s^{\alpha}$, $\alpha = 1$, $g_0 = 10^4\ Pa/m$.

Source: created by the authors.

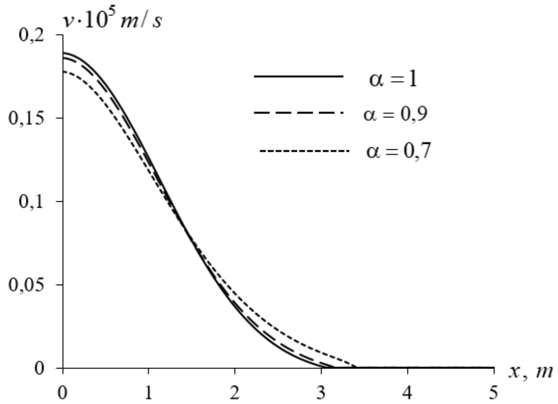

Fig. 2: Profiles of filtration velocity at various α and $t = 3600\,s$, $\lambda_v = 500\ s^{\beta}$, $\lambda_p = 500\ s^{\alpha}$, $\beta = 1$, $g_0 = 10^4\ Pa/m$.

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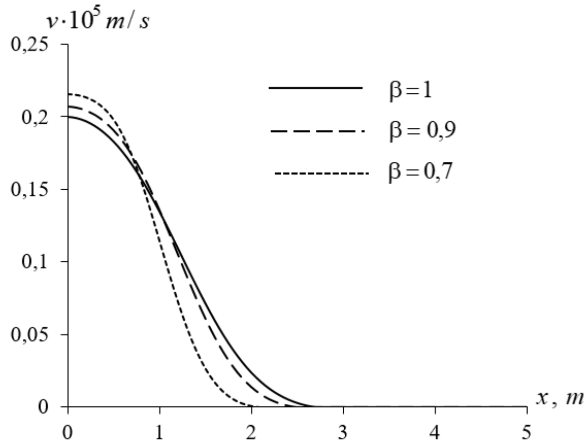

Fig. 3: Profiles of filtration velocity at various β and $t = 3600\,s$, $\lambda_v = 1000\,s^{\beta}$, $\lambda_p = 500\ s^{\alpha}$, $\alpha = 1$, $g_0 = 10^4\ Pa/m$.

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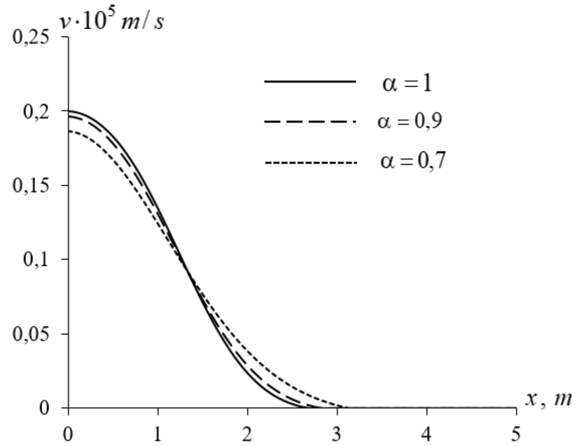

Fig. 4: Profiles of filtration velocity at various α and $t = 3600\,s$, $\lambda_v = 1000\ s^{\beta}$, $\lambda_p = 500\ s^{\alpha}$, $\beta = 1$, $g_0 = 10^4\ Pa/m$.

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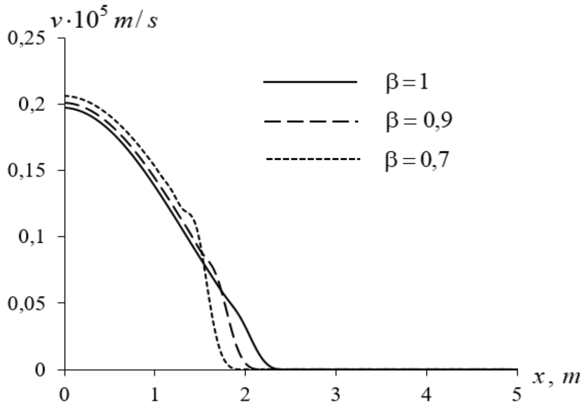

Fig. 5: Profiles of filtration velocity at various β and $t = 3600\,s$, $\lambda_v = 500\ s^{\beta}$, $\lambda_p = 0\ s^{\alpha}$, $\alpha = 1$, $g_0 = 10^4\ Pa/m$.

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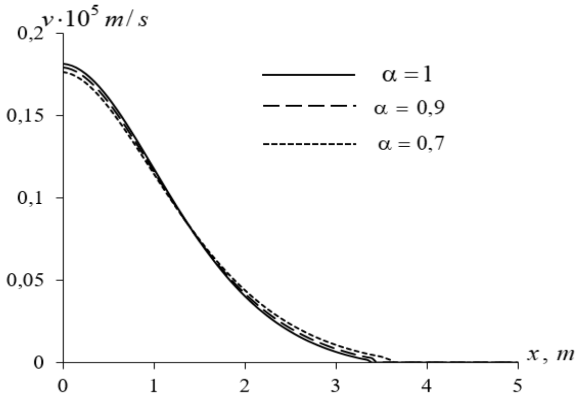

Fig. 6: Profiles of filtration velocity at various α and $t = 3600\,s$, $\lambda_v = 0\ s^{\beta}$, $\lambda_p = 500\ s^{\alpha}$, $\beta = 1$, $g_0 = 10^4\ Pa/m$.
Source: created by the authors.

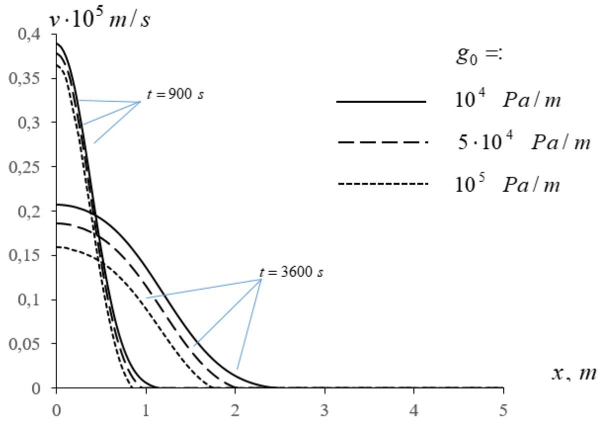

Fig. 7: Velocity change profiles at different g_0 and $\lambda_v = 1000 s^{\beta}$, $\lambda_p = 500 s^{\alpha}$, $\alpha = 1$, $\beta = 0.9$.
Source: created by the authors.

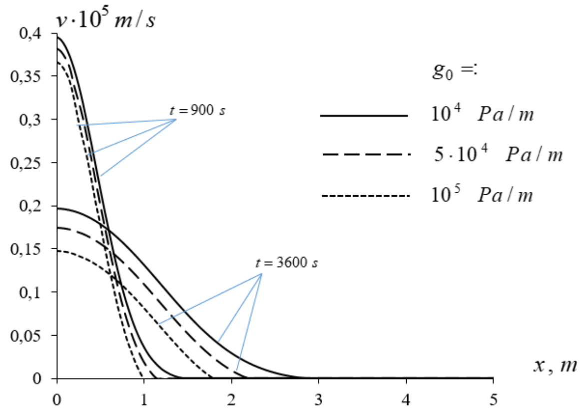

Fig. 8: Velocity change profiles at different g_0 and $\lambda_v = 1000 s^{\beta}$, $\lambda_p = 500 s^{\alpha}$, $\alpha = 0.9$, $\beta = 1$.
Source: created by the authors.

6 Conclusions

The paper considers the problem of anomalous non-steady filtration of a homogeneous liquid in a homogeneous porous medium, taking into account the TPG based on the relaxation fractional-differential filtration law, which takes into account relaxation phenomena both in pressure and in filtration velocity. Using this law, the equation of piezoconductivity is derived, for which the problem of filtration in a finite one-dimensional medium is set. The problem is solved numerically by the finite difference method. The filtration velocity fields are determined for different values of parameters g_0, α and β. Increasing g_0 leads to a decrease in the filtration velocity values. Moreover, the longer the time t, the greater this decrease. Simultaneously with the decrease g_0, the zone of propagation v decreases, which means an increase in the size of the zone with immobile liquid. Changing the values of g_0 does not affect the pressure distribution. The effect of g_0 is felt only in the filtration velocity field. Increasing the size of the zone with immobile liquid means expanding the zone from $\left|\frac{\partial p}{\partial x}\right| \le g_0$.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Khuzhayorov B has derived the mathematical model and interpreted the results.
Mamatov Sh approximated the model.

The results were obtained in Python using the model and its approximation, and the overall framework was structured by M. Zokirov., Dzhiyanov T.

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Conflict of Interest

The authors declare that there is no conflict of interest.