Hydrodynamic Pressure during Operation of Oil Wells with Deep Well Pumps

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Abstract: - The article examines the determination of hydrodynamic pressure and the influence of the relaxation properties of oil on this pressure during the operation of oil wells with deep-well sucker rod pumps. It has been revealed that if at the initial moment the plunger speed changes abruptly (instantaneous increase in speed), then the hydrodynamic pressure drop can have a pulsed nature, and the relaxation properties of oil lead to a "delay" in reaching the maximum pressure. It has been established that the relaxation parameters of oil, along with viscosity and density, can serve as limiting factors in the calculation of hydrodynamic pressure.

Key-Words: - plunger, viscoelastic fluid, stress relaxation time, velocity lag time, hydrodynamic pressure, sucker rod pump.

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1 Introduction

Viscoelastic fluids, which possess both viscous and elastic properties, are widely used in various fields of science and technology, including oil production. Numerous studies have been devoted to the study of the motion of viscoelastic fluids, of which we note, [1], [2], [3]. Experiments have shown that oils with a high content of asphaltene-resinous substances have relaxation properties and are viscoelastic fluids. Important performance indicators of oil wells directly depend on the rheological properties of the produced oil, which significantly affect hydrodynamic processes.

Heavy oil exhibits very complex viscoelastic behavior due to its composition. It has a significant impact on oil production and transportation. In [4], the viscoelastic behavior of three different heavy oils was measured using a rotational rheometer. To describe the physical characteristics of viscoelastic material deformation, the generalized Maxwell model and the fractional Maxwell model were used to establish the constitutive relation for heavy oil. In [5], is presented a review of modern methods and technologies for producing high-viscosity and ultra-high-viscosity oil. The theoretical aspects of these technologies, their applicability limits, and practical implementation are considered. Mechanical oil production is accomplished using various types of pumping units. Currently, the most common method is the use of sucker rod pumping units. Operating experience with these units demonstrates their potential for effective use in medium- and high-flow rate wells at great depths. At fields containing viscoelastic oils, an important task is to improve well performance by taking into account the specific features and influence of rheological characteristics on the operating parameters of the downhole plunger pump.

When operating oil wells with sucker-rod pumps, the plunger experiences pressure generated by the weight of the liquid column in the riser pipes. As the plunger begins to move upward, it encounters the fluid load. This load causes the rods to stretch, which can lead to failure. Because the plunger's speed is variable, the fluid movement in the annular gap between the rods and the pump cylinder is unsteady. This also affects the hydrodynamic pressure exerted on the plunger.

Numerous works have been devoted to mathematical modeling of oil well operations using sucker-rod pumps, including, [6], [7], [8], [9]. A new strategy for mathematical modeling of oil pumping processes is presented in [6]. By integrating advanced modeling techniques and optimized design, the petroleum industry can achieve several significant benefits. In [7], a numerical model for calculating fluid-solid interaction was developed to study the pressure and velocity field dynamics of multistage self-compensating soft-piston plunger pumps. In [8], a model of unsteady fluid flow in pump pipes and a discharge pipeline equipped with pneumatic compensators was developed. In [9], an analysis of unsteady hydromagnetic viscoelastic fluid flow through an annular space bounded by two infinite coaxial cylinders was conducted. An Oldroyd fluid model was used to study viscoelastic effects. The describing differential equations were solved analytically using modified Bessel functions. The influence of relaxation and lag parameters on other flow parameters is demonstrated, showing that their influence is maximum in the central part of the annular region.

Of theoretical and practical interest is the determination of the hydrodynamic pressure on the plunger during the operation of oil wells with sucker rod pumps. Such problems were considered in [10], [11], [12]. In [10], when the condition $\frac{r_2}{R} > 0.2$, where R is the radius of the lifting pipe, r_2 is the radius of the rods, and the annular gap between the lifting pipe and the rods is modeled as a flat gap. Using this assumption, considering oil to be a viscous Newtonian fluid, an approximate solution to the problem is obtained. An exact solution for viscous and viscoelastic fluids was obtained in [11]. This article also considered the influence of the relaxation properties of oil on the hydrodynamic - pressure in a well.

However, modeling the annular gap as a flat slit limits the practical application of the obtained formulas, since the condition $\frac{r_2}{R} > 0.2$ is not always satisfied. A solution to the problem without using the mentioned assumption (i.e., for a round pipe) is given in [12]. Oil was considered a viscous Newtonian fluid; a comparison of the calculation results obtained for a flat slit and a round pipe is also given.

When operating wells producing viscoelastic oil, the relaxation properties of the oil will need to be taken into account in calculations.

This article discusses mathematical modeling of hydrodynamic pressure. oil in a production well and a study of the influence of stress relaxation time and velocity lag time on this pressure.

2 Formulation of the Problem

Let's consider the determination of the hydrodynamic pressure of viscoelastic oil on a plunger in the annular space between the lift pipe and the rod of a deep-well pump. Figure 1 shows the fluid velocity distribution relative to a stationary system.

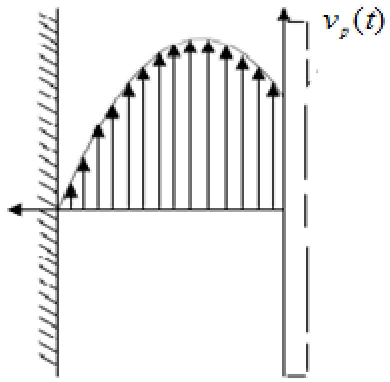

Fig. 1: Distribution of liquid velocity in the annular space between the column of pipes and rods relative to the stationary system.
Source: [10]

For mathematical modeling of the process, we use generally accepted assumptions: the relative velocity of the fluid on the surface of the lifting pipe is zero, and on the surface of the moving pipe(s), it is equal to the pipe velocity; a smooth flow is maintained; the fluid is assumed to be incompressible; and end effects are neglected.

The total pressure $p(t)$ of the liquid on the plunger is

$$p(t) = \Delta p(t) + \rho g(L - h) + p_0, \quad (1)$$

where t is time; $\Delta p(t)$is the hydrodynamic pressure (pressure drop) during non-stationary fluid flow in the lifting pipe; $p_0$is the pressure at the wellhead; L is the height of the raised column of liquid; h is the immersion depth of the pump; ρ is the density of oil; g -acceleration of gravity, $g = 9.81\ m/s^2$.

The velocity of a fluid $v(r,t)$ in a non-stationary laminar flow, taking into account circular symmetry, can be determined from the equation

$$\rho \frac{\partial v}{\partial t} = q(t) + \frac{1}{r}\frac{\partial}{\partial r}(r\tau), (r_2 < r < R), \quad (2)$$

where r is the radial coordinate; R is the radius of the lifting pipe; r_2 is the radius of the rods; $q(t) = -\partial p/\partial z = \Delta p/L$ is the pressure drop per unit length L along the axis of the lifting pipe; τ is the shear stress. Then the total pressure on the plunger can be represented as

$$p(t) = Lq(t) + \rho g(L - h) + p_0.$$

For the rheological equation of a liquid, we take the equation that takes into account the relaxation time and the delay time, [1], [2], [3].

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\tau(r,t)=\mu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial r},\quad (3)$$

where μ is the dynamic viscosity; λ_1 and λ_2 are the stress relaxation time and the velocity lag time. Equation 3) at $\lambda_1>0$, $\lambda_2>0$ expresses the Oldroyd-B model of a two- parameter viscoelastic - fluid.

If we consider (3) as a differential equation with respect to the unknownτ:

$$\tau+\lambda_1\frac{\partial\tau}{\partial t}=\mu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial r},$$

Then its solution under zero initial conditions has the form:

$$\tau(r,t)=\mu\frac{\lambda_2}{\lambda_1}\frac{\partial v}{\partial r}+\frac{\mu}{\lambda_1}\int_0^t e^{-\frac{t-\xi}{\lambda_1}}\frac{\partial v(r,\xi)}{\partial r}d\xi.\quad (4)$$

Substituting (4) into (2), we obtain the equation for the fluid velocity:

$$\rho\frac{\partial v}{\partial t}=\frac{\lambda_2}{\lambda_1}\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\lambda_1-\lambda_2}{\lambda_1^2}\frac{\mu}{r}\cdot$$
$$\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\int_0^t e^{-\frac{t-\xi}{\lambda_1}}v(r,\xi)d\xi\right)+q(t).\quad (5)$$

Representing (3) in operator form:

$$\tau=\mu\frac{\left(1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial r}}{\left(1+\lambda_1\frac{\partial}{\partial t}\right)},$$

Substituting into the equation of motion (2) we obtain the differential equation for the velocity:

$$\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial t}=\left(1+\lambda_1\frac{\partial}{\partial t}\right)q(t)$$
$$+\mu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)\right].\quad (6)$$

At the initial moment the liquid is not moving, therefore the initial conditions for (6) have the form

$$v(r,0)=0,\quad v_t'(r,0)=0,\ r_2\le r\le R.\quad (7)$$

In accordance with the adopted assumptions , the boundary conditions will be as follows:

$$v(r_2,t)=v_p(t),\qquad v(R,t)=0,(t>0),\quad (8)$$

where $v_p(t)$is the plunger speed.

To determine the hydrodynamic pressure, as an additional condition, we use the fluid flow balance equation:

$$Q=\pi(r_1^2-r_2^2)v_p(t)=2\pi\int_{r_2}^R r\,v(r,t)dr,\qquad (9)$$

where $r_1$is the plunger radius, $Q$is the fluid flow rate. (1)-(9) express the mathematical model of the process under consideration.

3 Solution of the Problem

In (5)-(9) we move on to dimensionless variables. For this purpose, we introduce the following dimensionless quantities:

$$\bar t=\frac{U_0}{R}t,\qquad \bar\lambda_1=\frac{U_0}{R}\cdot\lambda_1,\qquad \bar\lambda_2=\frac{U_0}{R}\lambda_2,$$
$$\bar r=\frac{r}{R},\qquad \alpha=\frac{r_2}{R},\qquad \alpha_1=\frac{r_1}{R},$$
$$\alpha^*=\alpha_1^2-\alpha^2,\quad \bar v(\bar r,\bar t)=\frac{v(r,t)}{U_0},$$
$$\bar v_p(\bar t)=\frac{v_p(t)}{U_0},\ \ \bar q(\bar t)=\frac{R\cdot q(t)}{\rho U_0^2},$$
$$\mathrm{c}=\frac{1}{\bar\lambda_1},\qquad c_2=\frac{\bar\lambda_2}{\bar\lambda_1},\qquad c_1=c(1-c_2).$$

where U_0 is some characteristic value of the fluid velocity.

Then

$$t=\frac{R}{U_0}\bar t,\qquad \lambda_1=\frac{R}{U_0}\bar\lambda_1,\qquad \lambda_2=\frac{R}{U_0}\bar\lambda_2,$$
$$r=\bar r\cdot R,\qquad r_1=\alpha_1\cdot R,\qquad r_2=\alpha\cdot R,$$
$$v=U_0\cdot\bar v,\quad v_p=U_0\cdot\bar v_p,\qquad q=\frac{\rho\,U_0^2}{R}\bar q.$$

Substituting them into (5)-(9), we obtain

$$\frac{\rho\,U_0^2}{R}\cdot\frac{\partial\bar v}{\partial\bar t}=\frac{\bar\lambda_2}{\bar\lambda_1}\cdot\frac{\mu U_0^2}{R\bar r}\cdot\frac{\partial}{\partial\bar r}\left(\bar r\frac{\partial\bar v}{\partial\bar r}\right)+$$
$$+\frac{\bar\lambda_1-\bar\lambda_2}{\bar\lambda_1^2}\cdot\frac{\mu U_0^2}{R\bar r}\frac{\partial}{\partial\bar r}\left(\bar r\frac{\partial}{\partial\bar r}\int_0^{\bar t}e^{-\frac{\bar t-\xi}{\bar\lambda_1}}\cdot\bar v d\xi\right)$$
$$+\frac{\rho\,U_0^2}{R}\bar q(\bar t),\quad (\alpha<\bar r<1);$$

$$\frac{\rho\,U_0^2}{R}\left(1+\bar\lambda_1\frac{\partial}{\partial\bar t}\right)\frac{\partial\bar v}{\partial\bar t}$$
$$=\frac{\rho\,U_0^2}{R}\bar q(\bar t)+\frac{\mu U_0}{R^2}\left(1+\bar\lambda_2\frac{\partial}{\partial\bar t}\right)\left[\frac{1}{\bar r}\frac{\partial}{\partial\bar r}\left(\bar r\frac{\partial\bar v}{\partial\bar r}\right)\right];$$

$$U_0\bar v(\bar r,0)=0,\qquad \bar v_{\bar t}'(\bar r,0)=0,$$
$$U_0\bar v(\alpha,\bar t)=\bar v_p(\bar t),\qquad U_0\bar v(1,\bar t)=0,$$
$$\pi R^2\cdot(\alpha_1^2-\alpha^2)U_0\bar v_p(\bar t)=$$
$$=2\pi R^2U_0\cdot\int_\alpha^1\bar r\,\bar v(\bar r,\bar t)d\bar r$$

or

$$\frac{\partial\bar v}{\partial\bar t}=\frac{\mathrm{c}_2}{Re}\cdot\frac{1}{\bar r}\frac{\partial}{\partial\bar r}\left(\bar r\frac{\partial\bar v}{\partial\bar r}\right)+$$

$$+\frac{c_1}{Re}\frac{1}{\bar r}\frac{\partial}{\partial\bar r}\left(\bar r\frac{\partial}{\partial\bar r}\int_0^{\bar t}\bar v(\bar r,\xi)e^{-\frac{\bar t-\xi}{\bar\lambda_1}}d\xi\right)+\bar q(\bar t);$$

$$\left(1+\bar\lambda_1\frac{\partial}{\partial\bar t}\right)\frac{\partial\bar v}{\partial\bar t}=\frac{1}{Re}\left(1+\bar\lambda_2\frac{\partial}{\partial\bar t}\right)\left[\frac{1}{\bar r}\frac{\partial}{\partial\bar r}(\bar r)\frac{\partial\bar v}{\partial\bar r}\right]+\left(1+\bar\lambda_1\frac{\partial}{\partial\bar t}\right)\bar q(t);$$

$$\bar v(\bar r,0)=0,\qquad \bar v_{\bar t}'(\bar r,0)=0,\qquad (\alpha\le\bar r\le 1);$$
$$\bar v(\alpha,\bar t)=\bar v_p(\bar t),\qquad \bar v(1,\bar t)=0,\qquad (\bar t>0).$$
$$\int_\alpha^1\bar r\,\bar v(\bar r,\bar t)d\bar r=\frac{\alpha_*^2}{2}\bar v_p(\bar t).$$

For ease of notation, during the solution process the "-" signs above dimensionless quantities will be omitted.

Then we have the following equations and additional conditions:

$$\frac{\partial v}{\partial t}=q(t)+\frac{c_2}{Re}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{c_1}{Re}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\int_0^t v(r,\xi)e^{-\frac{t-\xi}{\lambda_1}}d\xi\right),\quad (10)$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial t}=\frac{1}{Re}\left(1+\lambda_2\frac{\partial}{\partial t}\right)\cdot\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)\right]+\left(1+\lambda_1\frac{\partial}{\partial t}\right)q(t),\quad (11)$$

$$v(r,0)=0,\ \ v_t'(r,0)=0,(\alpha\le r\le 1);$$
$$v(\alpha,t)=v_p(t),v(1,t)=0,(t>0).\quad (12)$$

$$\int_\alpha^1 r\,v(r,t)dr=\frac{\alpha_*^2}{2}v_p(t),\quad (13)$$

Multiplying (10) by r,integrating over the interval $[\alpha,1]$, taking into account equality (13), we obtain:

$$q(t)=\frac{\alpha_*^2}{1-\alpha^2}v_p'(t)-\frac{1}{Re}\frac{2c_2}{1-\alpha^2}\left[\frac{\partial v(1,t)}{\partial r}-\alpha\frac{\partial v(\alpha,t)}{\partial r}\right]-\int_0^t e^{-c(t-\xi)}\left[\frac{\partial v(1,\xi)}{\partial r}-\alpha\frac{\partial v(\alpha,\xi)}{\partial r}\right]d\xi.\quad (14)$$

Substituting (14) into (10), we obtain the equation for the fluid velocity:

$$\frac{\partial v}{\partial t}=\frac{c_2}{Re}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)-\frac{1}{Re}\frac{2c_2}{1-\alpha^2}\cdot\left[\frac{\partial v(1,t)}{\partial r}-\alpha\frac{\partial v(\alpha,t)}{\partial r}\right]+\frac{c_1}{Re}\frac{1}{r}\frac{\partial}{\partial r}\left[r\int_0^t e^{-c(t-\xi)}\frac{\partial v(r,\xi)}{\partial r}d\xi\right]-\frac{1}{Re}\frac{2c_1}{1-\alpha^2}\int_0^t e^{-c(t-\xi)}\left[\frac{\partial v(1,\xi)}{\partial r}-\alpha\frac{\partial v(\alpha,\xi)}{\partial r}\right]d\xi+\frac{\alpha_*^2}{1-\alpha^2}v_p'(t).\quad (15)$$

Assuming $v(r,t)=z(r)U(t)$, for a homogeneous equation corresponding to (15) with zero boundary conditions, we obtain the Sturm-Liouville problem for the Bessel equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dz}{dr}\right)+\beta^2 z=\frac{2}{1-\alpha^2}[z'(1)-\alpha z'(\alpha)],$$
$$z(\alpha)=0,z(1)=0.\quad (16)$$

$\beta=0$ is an eigenvalue of the problem. The corresponding eigenfunction $z_0(r)$, taking into account (13), can be found in the form:

$$z_0(r)=A_\alpha\left[1-r^2+(1-\alpha^2)\frac{\ln r}{\ln\alpha}\right],\quad (17)$$

where

$$A_\alpha=\frac{(2\alpha_*^2+\alpha^2-1)\ln\alpha}{(1-\alpha^2)[1-\alpha^2+(1+\alpha^2)\ln\alpha]}.$$

Let us represent the required functions in the form:

$$v(r,t)=v_0(r,t)+v_1(r,t),\quad (18)$$

$$q(t)=q_0(t)+q_1(t),\quad (19)$$

where $v_0(r,t)=z_0(r)v_p(t)$is the solution of the homogeneous equation corresponding to equation (15) with zero initial and boundary conditions.

$q_0(t)$ is determined by substituting into - $v_0(r,t)$ (14):

$$q_0(t)=\frac{\alpha_*^2}{1-\alpha^2}v_p'(t)+\frac{4A_\alpha}{Re}\left[c_2v_p'(t)+c_1\int_0^t e^{-c(t-\xi)}v_p'(\xi)d\xi\right].\quad (20)$$

Then $v_1(r,t)$, $q_1(t)$ we obtain an equation for the definition.

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial v_1}{\partial t}=\frac{1}{Re}\left(1+\lambda_2\frac{\partial}{\partial t}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_1}{\partial r}\right)\right]+\left(1+\lambda_1\frac{\partial}{\partial t}\right)q_1(t),\quad (21)$$

satisfying the conditions:

$$v_1(r,0)=0,v_{1t}'(r,0)=0,(\alpha\le r\le 1);\quad (22)$$

$$v_1(\alpha,t)=v_p(t),v_1(1,t)=0,(t>0).\quad (23)$$

$$\int_\alpha^1 r\,v_1(r,t)dr=\frac{1-\alpha^2}{4}v_p(t),\quad (24)$$

Equality (24) follows from (13). To solve equations (21) with conditions (22)-(24), we apply the Laplace integral transform [13]:

$$\tilde{v}_1(r,s) = \int_0^{+\infty} e^{-st}\, v_1(r,t)dt,$$

$$\tilde{v}_p(s) = \int_0^{+\infty} e^{-st}\, v_p(t)dt,$$

$$\tilde{q}_1(s) = \int_0^{+\infty} e^{-st}\, q_1(t)dt.$$

Taking into account conditions (22), in the Laplace image we obtain the equation

$$\frac{1}{Re}\frac{1}{r}\frac{d}{dr}\left(r\frac{d\tilde{v}_1}{dr}\right) - k^2\tilde{v}_1 = -\frac{k^2}{s}\tilde{q}_1(s), \qquad (25)$$

with boundary conditions

$$\tilde{v}_1(0,s) = \tilde{v}_p(s),\ \ \tilde{v}_1(1,s) = 0, \qquad (26)$$

where $k = \sqrt{Re\, s\,(1+\lambda_1 s)/(1+\lambda_2 s)}$.

Equality (24) in the Laplace representation takes the form

$$\tilde{v}_1(r,s)dr = \frac{1-\alpha^2}{4}\tilde{v}_p(s). \qquad (27)$$

The general solution of equation (25) is

$$\tilde{v}_1(r,s) = C_1 J_0(ikr) + C_2 Y_0(ikr) + \frac{\tilde{q}_1(s)}{s},$$

where $J_0(x)$, $Y_0(x)$ are the Bessel functions of the first and second kind of zero order.

Having defined the constant numbers C_1 and C_2 из boundary conditions (26), we obtain

$$\tilde{v}_1(r,s) = \frac{1}{D_\alpha(ik)}\cdot\{\,\tilde{q}_1(s)[D_\alpha(ik) + d_2(ik)J_0(ikr) - d_1(ik)Y_0(ikr)]/s + \tilde{v}_p(s)\cdot[Y_0(ik)J_0(ikr) - J_0(ikr)Y_0(ikr)]\}, \qquad (28)$$

where

$d_1(x) = J_0(\alpha x) - J_0(x)$; $d_2(x) = Y_0(\alpha x) - Y_0(x)$; $D_\alpha(x) = J_0(\alpha x)Y_0(x) - J_0(x)Y_0(\alpha x)$.

From (28), taking into account equality (27), we obtain

$$\tilde{q}_1(s) = s\tilde{v}_p(s)\cdot\tilde{f}(s), \qquad (29)$$

where

$$\tilde{f}(s) = \frac{\phi(ik)}{\psi(ik)};$$

$$\phi(x) = \frac{1-\alpha^2}{4}xD_\alpha(x) + Y_0(x)d_3(x) - J_0(x)d_4(x);$$

$$\psi(x) = \frac{1-\alpha^2}{2}xD_\alpha(x) - d_2(x)d_3(x) + d_1(x)d_4(x);$$

$$d_3(x) = \alpha J_1(\alpha x) - J_1(x);$$

$$d_4(x) = \alpha Y_1(\alpha x) - Y_1(x),$$

$J_1(x), Y_1(x)$ are Bessel functions of the first and second kind of the first order.

Let's find the original $f(t)$ image $\tilde{f}(s)$. β_n $(n = 1,2,3,\dots)$ are zeros of the function $y = \psi(\beta)$, that is, simple positive roots of the equation $\psi(\beta) = 0$, or

$$\frac{1-\alpha^2}{2}\beta D_\alpha(\beta) - d_2(\beta)d_3(\beta) + d_1(\beta)d_4(\beta) = 0. \qquad (30)$$

From here $\psi(\beta_n) = \psi(ik) = 0$,

$$\beta_n = ik = i\sqrt{\frac{Re\, s\,(1+\lambda_1 s)}{1+\lambda_2 s}}.$$

β_n are simple poles of a function of a complex variable $\tilde{f}(s)$, so the original corresponding $f(t)$, to the image $\tilde{f}(s)$ can be found using the formula, [14]:

$$f(t) = \sum_{n=1}^{\infty}\left[\frac{\phi(ik)e^{st}}{\psi'(ik)(ik)'_s}\bigg|_{s=s_{1n}} + \frac{\phi(ik)e^{st}}{\psi'(ik)(ik)'_s}\bigg|_{s=s_{2n}}\right]$$

Taking into account the adopted notations, after some identical transformations, we obtain

$$f(t) = \frac{4}{Re}\sum_{n=1}^{\infty}\frac{\beta_n\phi_n e^{-a_n t}}{C_n\psi_{1n}}(A_n ch b_n t + B_n b_n sh b_n t),$$

where

$a = 1/(2\lambda_1)$, $a_n = a(1+\lambda_2 x_n^2)$,
$b_n = \sqrt{a_n^2 - 2a x_n^2}$, $x_n = \frac{\beta_n}{\sqrt{Re}}$, $\psi_{1n} = (1-\alpha^2)D_\alpha(\beta_n) + \frac{1-\alpha^2}{2}\beta_n[\alpha d_7(\beta_n) + d_8(\beta_n)] + d_1(\beta_n)d_6(\beta_n) - d_2(\beta_n)d_5(\beta_n)$,

$\phi_n = \phi(\beta_n)$, $d_5(x) = \alpha^2 J_0(\alpha x) - J_0(x)$;
$d_6(x) = \alpha^2 Y_0(\alpha x) - Y_0(x)$;
$d_7(x) = J_0(x)Y_1(\alpha x) - J_1(\alpha x)Y_0(x)$;
$d_8(x) = J_1(x)Y_0(\alpha x) - J_0(\alpha x)Y_1(x)$;
$A_n = 1 + 2\lambda_2 x_n^2(2 + a\lambda_2^2 x_n^2) + (\lambda_1 + \lambda_2)(\lambda_2(a_n^2 + b_n^2) - 2a_n) - 4\lambda_2^2 a_n x_n^2$;
$B_n = 2(\lambda_2 - \lambda_1)(\lambda_2 a_n - 1)$; $C_n = 1 + 4\lambda_1 x_n^2 + \lambda_2 x_n^4 + 2\lambda_1\lambda_2(a_n^2 + b_n^2) - 4\lambda_1 a_n(1 + \lambda_2 x_n^2)$.

The original $q_1(t)$, corresponding to the image $\tilde{q}_1(s)$ is found from (26). Since k $s\tilde{v}_p(s)$ corresponds to the original $v_p'(t)$, according to the composition theorem, the image $s\tilde{v}_p(s)\cdot\tilde{f}(s)$corresponds to the original.

$$\int_0^t f(t-\xi)v_p'(\xi)d\xi.$$

In accordance with (29) we obtain:

$$q_1(t) = -\frac{4}{Re}\ \cdot$$

$$\cdot\sum_{n=1}^{\infty}\frac{\beta_n\phi_n}{C_n\psi_{1n}}\int_0^t v_p'(\xi)e^{-a_n(t-\xi)}(A_n chb_n(t-\xi) + B_n b_n shb_n(t-\xi))d\xi. \qquad (31)$$

Then we obtain the following formula:

$$q(t) = \frac{\alpha_*^2}{1-\alpha^2}v_p'(t) + \frac{4A_\alpha}{Re}\left[c_2 v_p(t) + c_1\int_0^t e^{-c(t-\xi)}\, v_p(\xi)d\xi\right] - \frac{4}{Re}\cdot$$

$$\cdot\sum_{n=1}^{\infty}\frac{\beta_n\phi_n}{C_n\psi_{1n}}\int_0^t v_p'(\xi)e^{-a_n(t-\xi)}(A_n chb_n(t-\xi) + B_n b_n shb_n(t-\xi))d\xi. \qquad (32)$$

Taking into account that in (32) the "-" signs above the dimensionless quantities are omitted, we have

$$\bar{q}(\bar{t}) = \frac{\alpha_*^2}{1-\alpha^2}\bar{v}_p'(\bar{t}) + \frac{4A_\alpha}{Re}\left[c_2\bar{v}_p(\bar{t}) + c_1\int_0^{\bar{t}} e^{-c(\bar{t}-\xi)}\,\bar{v}_p(\xi)d\xi\right] - \frac{4}{Re}\cdot$$

$$\cdot\sum_{n=1}^{\infty}\frac{\beta_n\phi_n}{C_n\psi_{1n}}\int_0^{\bar{t}}\bar{v}_p'(\xi)e^{-a_n(\bar{t}-\xi)}\big(A_n chb_n(\bar{t}-\xi) + B_n b_n shb_n(\bar{t}-\xi)\big)d\xi\ . \qquad (33)$$

In [12], a similar formula for a viscous - Newtonian fluid was obtained:

$$\bar{q}(\bar{t}) = \frac{\alpha_*^2}{1-\alpha^2}\bar{v}_p'(\bar{t}) + \frac{4A_\alpha}{Re}\bar{v}_p(\bar{t}) - \frac{2}{Re}\sum_{n=1}^{\infty}\frac{\beta_n\phi_n}{\psi_{1n}}\int_0^{\bar{t}}\bar{v}_p'(\xi)e^{-x_n^2(\bar{t}-\xi)}d\xi. \qquad (34)$$

Using these functions, the pressure drop is determined by the formula

$$\Delta p(t) = \frac{\rho U_0^2 L}{R^2}\bar{q}(\bar{t}), \qquad (35)$$

Total pressure according to formula (1).

4 Results and Discussion

The main result of the work is formula (33). It allows us to determine the hydrodynamic pressure of oil on the plunger during its rise and to investigate the influence of the stress relaxation time and the velocity lag time on this pressure.

From formulas (33) and (35), it is clear that the pressure drop on the plunger depends not only on its speed, but also on its acceleration. If the plunger speed changes abruptly (instantaneous acceleration): $v_p(t) = U_0\eta(t)$, then the influence of the terms containing acceleration can become dominant, since, $v'_p(t) = U_0\delta(t)$,where is $\eta(t)$ the unit Heaviside function, $\delta(t)$is the Dirac delta function.

To compare the formula for a viscoelastic fluid (33) with the formula (34) for a viscous fluid, we assume that in some time interval, the plunger speed is constant: $v_p(t) = v_m = const$, i.e. $\bar{v}_p(\bar{t}) = 1$. Then from (33) and (34), we obtain, respectively

$$\bar{q}(\bar{t}) = \frac{4A_\alpha}{Re}\left[c_2 + c_1\left(1 - e^{-c\bar{t}}\right)\right], \qquad \bar{q}(\bar{t}) = \frac{4A_\alpha}{Re},$$

It is clear that in the time interval where the plunger speed is constant, the pressure drop will be constant only for a viscous liquid; for a viscoelastic liquid, it will not be constant.

If we assume that the plunger speed corresponds to the speed of the rod suspension point [10], then the plunger speed can be determined by the formula

$$v_p(t) = \frac{48v_m}{\pi^3}\sum_{n=1}^{\infty}\frac{sin\frac{2\pi(2n-1)t}{T}}{(2n-1)^3}, \qquad (36)$$

where $0 \le t \le T, T$ is the period of one piston operating cycle; v_m is the average speed of the rod suspension point.

Numerical calculations were performed using the obtained formulas. The following initial data values were used in the calculations:

$L = 1000\,m,\ h = 100\,m,\ U_0 = v_m = 0.60\,m/s,$
$p_0 = 10^5\,Pa$, $\mathrm{T} = 20\,s,\ r_2 = 0.01\,m,$
$r_1 = 0{,}02988\,m,\ R = 0.030\,m.$

Figure 2 shows graphs of the dependence of the hydrodynamic pressure drop on time $\Delta p, MPa$for oil viscosity and density $\mu = 0.09\,Pa\cdot s,\ \rho = 920\,kg/m^3$. The values of the relaxation parameters are as follows: $1 - \lambda_1 = \lambda_2 = 0$ (Newtonian fluid); $2 - \lambda_1 = 1\,s;\ \lambda_2 = 0.5\,s;\ 3 - \lambda_1 = 2\,s;\ \lambda_2 = 1\,s;\ 4 - \lambda_1 = 3\,s;\ \lambda_2 = 1\,s.$

Note that here and below, for comparison, graphs for a viscous fluid are also provided (curve 1 in the figures).

As can be seen from Figure 2, as the plunger rises faster, the pressure drop for the viscoelastic fluid is smaller than for the viscous fluid. Conversely, as the plunger rises more slowly, the pressure for the viscoelastic fluid is greater than for the viscous fluid.

The graphs show that the relaxation properties of oil can lead to both a decrease (during acceleration) and an increase (during deceleration) in the pressure drop. The greatest difference in values is observed at the moment the lift stops. Relaxation properties lead to a "delay" in reaching maximum pressure. As these properties increase, the difference between the pressure profiles increases, especially as the ascent slows.

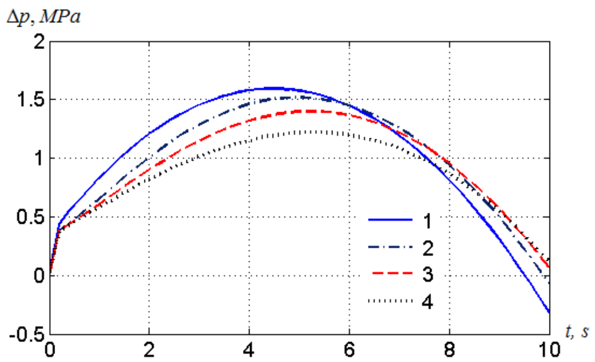

Fig. 2: Graphs of the change in pressure drop over time $\Delta p, MPa$ at $\mu = 0.09\ Pa \cdot s, \rho = 920\ kg/m^3$.

The curve numbers correspond to :
$1 - \lambda_1 = \lambda_2 = 0$ (Newtonian fluid) ;
$2 - \lambda_1 = 1\ s;\ \lambda_2 = 0.5\ s;\ 3 - \lambda_1 = 2\ s;\ \ \lambda_2 = 1\ s;$
$4 - \lambda_1 = 3\ s; \lambda_2 = 1\ s.$
Source: created by the authors

Figure 3 shows graphs of the total plunger pressure versus time. Calculations were performed for the following oil viscosity and density values: $\mu = 0.07\ Pa \cdot s, \rho = 800 \frac{kg}{m^3}$.

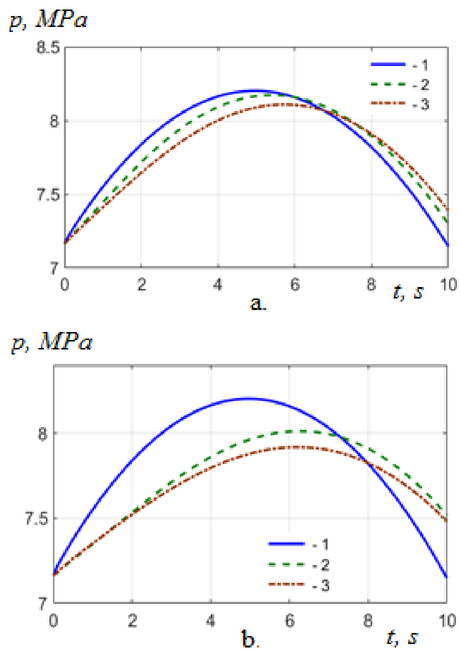

Fig. 3: Graphs of the dependence of the total pressure on the plunger on time as it rises upward $\mu = 0.07\ Pa \cdot s,\ \rho = 800\ kg/m^3$.
The curve numbers correspond to:
a. $1 - \lambda_1 = \lambda_2 = 0$(Newtonian fluid);
$2 - \lambda_1 = 1\ s,\ \lambda_2 = 0.5\ s;$
$3 - \ \lambda_1 = 2\ s,\ \lambda_2 = 1\ s;$
b. $1 - \lambda_1 = \lambda_2 = 0$(Newtonian fluid);
$2 - \lambda_1 = 3\ s, \lambda_2 = 1\ s;$
$3 - \lambda_1 = 5\ s,\ \lambda_2 = 2\ s.$
Source: created by the authors

It can be seen that as the relaxation properties of oil increase (Figure 3b), the difference between the pressure profiles for the viscous and viscoelastic fluids increases. The greatest difference in values is observed at the peak and at the moment of ascent. Relaxation properties lead to a shift of the graph maximum to the right along the time axis.

The effect of changing the plunger speed on the values of the total oil pressure on the plunger is considered. The calculation results are given $v_p =$ 0,40; 0,60; 0,80; 1,00 $\frac{m}{s}$ in Table 1. The calculations show that with an increase in the plunger speed, the total pressure increases up to a certain time and then decreases. Moreover, for a Newtonian fluid, it always remains greater than for a viscoelastic fluid.

Table 1. Time dependence of total pressure values at different plunger speeds

	$v_p, m/s$			
	0,40	0,60	0,80	1,00
t, s	p, MPa			
0	8,0461	8,0461	8,0461	8,0461
1,0	8,6333	8,9269	9,2205	9,5141
2,0	8,9182	9,3542	9,7903	10,2263
3,0	9,1092	9,6408	10,1723	10,7039
4,0	9,2063	9,7865	10,3666	10,9467
5,0	9,2098	9,7916	10,3735	10,9553
6,0	9,1201	9,6570	10,1940	10,7310
7,0	8,9374	9,3831	9,8288	10,2744
8,0	8,6618	8,9697	9,2775	9,5854
9,0	8,2928	8,4161	8,5395	8,6628
10,0	7,8342	7,7282	7,6223	7,5163

Source: created by the authors

Figure 4a and Figure 4b show graphs of the change in total pressure at $\mu = 0.10\ Pa \cdot s,\ \rho = 900\ kg/m^3$. Source: created by the authors

As can be seen, with increasing oil density and viscosity, the hydrodynamic pressure on the plunger increases. The shape of the pressure profiles and the nature of the influence of viscoelastic properties remain unchanged, but the degree of their influence increases. Increasing the values of the relaxation parameters (Figure 4b) leads to an increase in the difference in pressure values for the viscous and viscoelastic fluids, as well as a delay in reaching the maximum pressure.

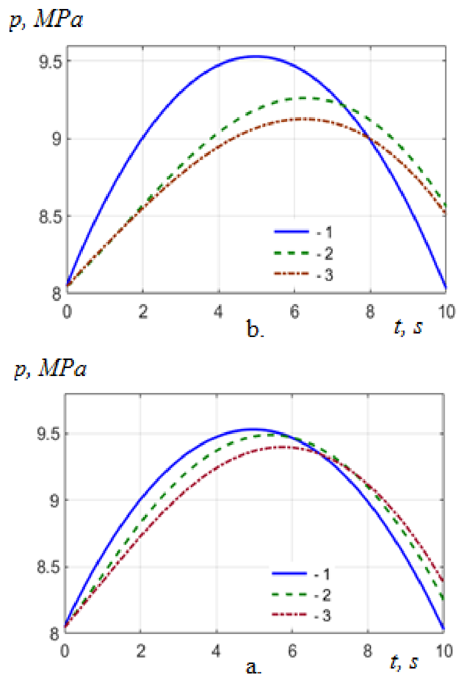

Fig. 4: Graphs of the dependence of the total pressure on the plunger on time as it rises upward $\mu = 0.10\ Pa \cdot s,\ \rho = 900\ kg/m^3$.
Correspondence of curve numbers as in Figure 3.
Source: created by the authors

5 Conclusion

An analytical solution to the problem of non-stationary motion of a viscoelastic fluid in an oil well is obtained. Using numerical calculations, the influence of relaxation parameters (stress relaxation time and velocity lag time) on the hydrodynamic pressure on the plunger was studied. It was found that if at the initial instant the plunger velocity changes abruptly (instantaneous increase in velocity), then the hydrodynamic pressure of the fluid on it can have a pulsed character. In the time interval where the plunger velocity is constant, the hydrodynamic pressure will be constant only for a Newtonian fluid. And for a viscoelastic fluid, it will not be constant. It has been shown that the relaxation properties of a fluid lead to a peculiar "delay" in reaching maximum pressure, i.e., to its shift to the right along the time axis. The relaxation properties of oil, along with viscosity and density, can serve as limiting factors in hydrodynamic pressure calculations. When operating wells with deep - -well pumps producing viscoelastic oils, it is necessary to take into account the relaxation properties of the fluid.

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- M.S. Dzhabbarov: methodology, data selection, writing original draft, formal analysis.
- Yu.Sh. Gaybulov: writing-review and editing, data selection, writing-review and editing, visualization.software, approval, writing-original draft preparation, editing, and visualization.
- Z.K. Shukurov: software, writing-original draft preparation, writing-review and editing.

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