On the Bending and Buckling of Very Thin Beams

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Abstract: - Bending and buckling deformations of very thin beams with curved cross-sections are discussed. Nonconvex strain-energy density functions are considered, and the critical loads and corresponding critical conditions are defined. The Euler-Lagrange equation with Weierstrass-Erdmann corner conditions is acceptable for globally stable deformations. The deformations are defined in the context of globally stable equilibrium deformations. Regions of low and high strain are revealed, separated by a discontinuity in the curvature of the elastic curve. Since the curvature is almost zero in one section, when combined with the other section, the beam will be treated as two separate sections. The one remaining is undeformed, and the other is deformed with constant curvature. Those phenomena are easily observed in the metallic measuring bands.

Key-Words: - Beam bending, beam buckling, non-convex strain energy function, coexistence of phases, stability, local stability, global stability, non-smooth elastic curve.

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1 Introduction

Anelastic deformation features may be discussed in the context of non-linear elasticity. For example, necking, yield, and phase change may be considered in the context of a nonconvex strain energy function, and the equilibrium equation may be formulated as a global minimum of the total energy function, with a continuous displacement function but a discontinuous strain (displacement derivative). The idea of globally stable elastic deformations has been introduced in [1], [2]. Globally stable elastic deformations in bars have also been presented in [3].

Based on this idea, the present work discusses two problems concerning curved-section very thin beams. The first is the bending problem of a very thin cantilever beam governed by a nonconvex strain-energy density function, and the second is the branching problem of an axially compressed bar. In fact, the total energy function is minimized by applying the Variational theory of broken extremals, as it has been shown in [4]. Following [4], minimization of the total energy requires satisfaction of the Euler-Lagrange equation under the Weierstrass-Erdmann corner conditions, since the strain energy is nonconvex.

Conventional buckling problems have been discussed in the classic books [5] and also [6]. Critical states are determined via the linear eigenvalue problem, and post-critical states are defined by a formal post-buckling analysis following conventional branching analysis. Mathematically, the various buckling problems are treated as bifurcation problems using Sturm-Liouville eigenvalue theory [7]. The critical conditions are identified, and the post-critical states are defined using formal branching theory. The definition of the post-critical states is based on the Fredholm Alternative theorem [8]. The present problem is based on the branching theory with continuous but nondifferentiable branches. Nonconvex elasticity problems have already been presented, assuming continuous but not differentiable deformations [1], [2], [9], [10], [11]. It is proven that the non-differentiability of the post-critical buckling branches introduces multiple critical lengths.

The proposed theory can be easily experimentally evaluated by observing the behavior of metallic measuring tapes under bending or buckling loading. The procedure will be presented in the following chapters, along with the corresponding applications.

2 The Globally Stable Bending Problem of a Beam

First, the bending problem with nonconvex strain energy density functions will be discussed, adopting the globally stable criterion for bending deformations. The theory will be applied to the bending phenomena of thin beams with curved cross-sections. The proposed theory may be easily observed in metallic measuring bands with a curved cross-section of small lateral curvature. The problem we have in mind is the bending of a very thin beam with a curved cross-section, (Figure 1), where under some bending state, the cross-section becomes a thin rectangular one (Figure 1).

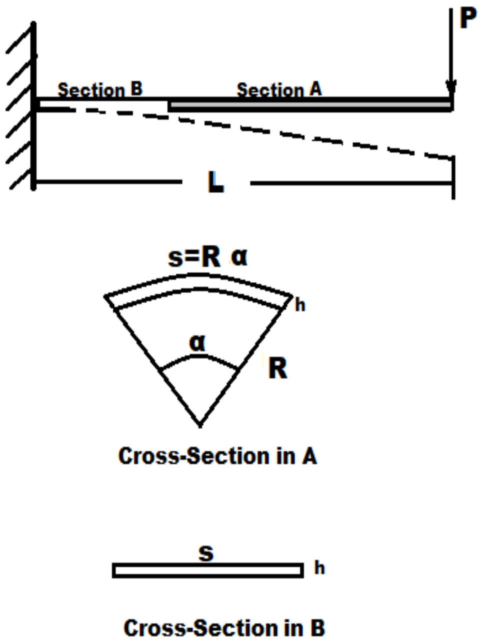

Fig. 1: Snap bending of thin beams
Source: created by the authors.

The stored energy of the strain energy function of the elastic beam is a function of the curvature κ of the plane elastic curve W(κ). This function is considered nonconvex, since in some sub-domains:

$$\frac{d^2W}{d\kappa^2}(\kappa) < 0, \tag{1}$$

The Weierstrass-Erdmann corner conditions transform a nonconvex variational problem into a convex one with a continuous rotation angle, but with a non-continuous curvature of the elastic curve. Figure 1 indicates the deformed elastic curve y(x) with,

$$\tan(\theta(x)) = \frac{dy(x)}{dx}. \tag{2}$$

The total energy of the beam is defined by:

$$V = \int_0^L (W(\Theta'(x)) - M(\Theta'(x)))dx \tag{3}$$

with $W(\Theta'(x))$ denoting the strain energy function. The minimum of the total energy V, defining the equilibrium equation, is expressed by,

$$\frac{d}{dx}\frac{\partial W(\Theta'(x))}{\partial \Theta'(x)} = 0 \tag{4}$$

Considering Eq.(3) in the form:
with,

$$F = W(\Theta'(x)) - M\Theta'(x) \tag{5}$$

Weierstrass-Erdmann corner conditions transform a nonconvex variational problem to a convex one with a continuous rotation angle, but with a non-continuous curvature of the elastic curve. The Weierstrass-Erdman corner conditions, Gelfand and Fomin [4] should additionally be satisfied with,

$$(F_{\Theta'})_{[x=c-0]} - (F_{\Theta'})_{[x=c+0]} = 0 \tag{6}$$

$$(F - \Theta' F_{\Theta'})_{[x=c-0]} - (F - \Theta' F_{\Theta'})_{[x=c+0]} = 0 \tag{7}$$

The Weierstrass-Erdman conditions, (Eqs. (6,7)), are pictured in Figure 2, with the coexistence of two phases, defined through the coexisting curvatures θ'_1 and θ'_2 at the same beam cross-section.

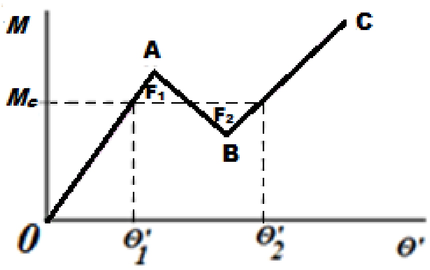

Fig. 2: The bending moment versus the curvature diagram, indicating the coexistence of the phase phenomenon
Source: created by the authors.

The two areas, F1 and F2, in Figure 2 are equal at those curvatures. Indeed,

$$M(\theta'(x)) = \frac{\partial F}{\partial \theta'(x)} = \frac{\partial W(\theta'(x))}{\partial \theta'(x)}. \tag{8}$$

Let us point out that,

$$\int_{\theta'_1}^{\theta'_2} M(\theta'(x))\, d\theta'_x = \int_{\theta'_1}^{\theta'_2} W'^{(\theta'(x))} d\theta'_x \ , \tag{9}$$

$$W(\theta'_2) - W(\theta'_1) - M_c(\theta'_2 - \theta'_1) = 0 \quad , \tag{10}$$

Let us point out that the beam deformation is completely characterized through the moment M_c and the coexisting curvatures θ_1' and θ_2'. For the convex parallel branches of the diagram, Figure 2,

$$M(X)=EI\theta'(x) \text{ for } 0<\theta'(x)<\theta_1' \tag{11}$$

$$M(X)=M_c+EI(\theta'(x)-\theta_2') \text{ for } \theta_2'<\theta'(x). \tag{12}$$

It is evident that the high values of the bending moment are located at the left fixed end of the beam, whereas the low values of the bending moment are located near the free end. The procedure is conventional when the bending moment is less than the critical value M_c. However, when the bending moment exceeds the critical bending moment M_c, a distribution of low cross-sectional rotation is observed in the region $0 < x_c < l$ of the beam. Let us point out that x_c defines the cross-section of the beam, where the jumping of the two curvatures θ_1' and θ_2' coexist.

The moments are quite low near the supports. The largest bending moments occur near the beam's middle. If the bending moments are less than M_c, the procedure is conventional. Nevertheless, when the bending moment at the beam's midpoint exceeds M_c, three bending regions form. The one in the middle of the beam with bending moments $M>M_c$ and the region near the supports with $M<M_c$. At the point of the bending moment diagram change x=c, there exists a curvature discontinuity $\theta_2' - \theta_1'$, see Figure 2.

3 The Globally Stable Buckling of a Simply Supported Beam under Axial Loading

A simply supported, inextensible, and axially compressed beam in its undeformed state is shown in Figure 3, along with the elastic curve of the buckled state.

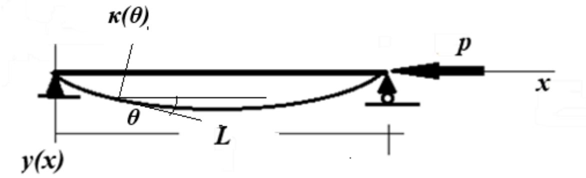

Fig. 3: Buckling of a simply supported beam
Source: created by the authors.

The elastic line of the beam is initially straight. The axial compression of the beam does not deform it, due to the inextensibility of the elastic curve. Increasing the axially applied load p, the critical value p_c is defined where the beam deforms. Figure 4 indicates the deformed elastic curve y(x) with,

$$\tan(\theta(x))=\frac{dy(x)}{dx}. \tag{13}$$

The total energy of the bar is defined by,

$$V = \int_0^L \left(W\left(\Theta'(x)\right) - p(1-\cos(\Theta(x))\right)dx, \tag{14}$$

with $W\left(\Theta'(x)\right)$ denoting the strain energy function.

The minimum of the total energy V, defining the equilibrium equation, is expressed by,

$$\frac{d}{dx}\frac{\partial W(\Theta'(x))}{\partial \Theta'(x)} + p\,sin(\Theta(x)) = 0, \tag{15}$$

with the boundary conditions,

$$\theta'(0)=\Theta'(L)=0. \tag{16}$$

For small rotations with $0<\theta(x)<<1$,

$$\sin(\theta(X)) \cong \Theta(X) + \frac{\Theta^3(X)}{6} + o(\Theta^3(X)) \tag{17}$$

Considering $p=p_o(1+\lambda)$, $\lambda<<1$,

$$\tag{18}$$

and

$$W(\theta'(x))=\frac{EI\,\theta'(x)^2}{2}, \tag{19}$$

Where, EI denotes the stiffness of the beam. Expanding Eq.(15) in powers of the θ(x),

$$EI\Theta''(x) + p_o\,\Theta(x) = -\lambda p_o \Theta(x) + p_0 \frac{\Theta^3(x)}{6} + o(\Theta^3), \tag{20}$$

Considering also the b.cs., Eqs. (16). It is evident that the zero (trivial) solution is always a solution to Eq.(15). However, the non-zero solutions are developed through the eigenvalue boundary problem. Formal branching solutions demand the definition of the kernel of the Eqs (15,16) defined by:

$$EI\Theta''(x) + p_o\,\Theta(x) = 0, \tag{21a}$$

$$\Theta'(0)=\Theta'(L)=0. \tag{21b}$$

The system Eqs.(21a,b) accepts the solution,

$$\varphi(x) = cos(\frac{\pi x}{L}), \tag{22}$$
with
$$p_o = \frac{EI\pi^2}{(L)^2}. \tag{23}$$

Following formal branching theory [8], [10], the solution of the non-linear problem is defined by,
$$\Theta(x) = \xi\varphi(x), \tag{24}$$

if, (recall Fredholm Alternative theorem [8]),
$$\int_0^L (-\lambda p_o \Theta(x) + p_0 \frac{\Theta^3(x)}{6})\varphi(x)dx = 0. \tag{25}$$

The ξ-solutions of Eq.(25) are defined with,
$$\xi = \mp\frac{2}{\sqrt{3}}\sqrt{\frac{P-P_o}{P_o}}. \tag{26}$$

Hence,
$$\Theta(\mathrm{x}) = \mp\frac{2}{\sqrt{3}}\sqrt{\frac{p-p_o}{p_o}}\, cos\left(\frac{\pi x}{L}\right) \tag{27}$$

4 Bending of a Very Thin Beam with a Curved Cross-Section. The Cantilever Beam.

The globally stable beam bending model will be approximated in the present section. In fact, the curvature of the bending beam is initially very small, due to the comparatively large moment of inertia of the curved cross-section. Nevertheless, due to the curvature discontinuity and the coexistence of phases, the curvature jumps to large values. In fact, the bending moment-versus-curvature diagram in that region is almost a vertical line parallel to the bending moment axis, indicating a constant curvature (Figure 4).

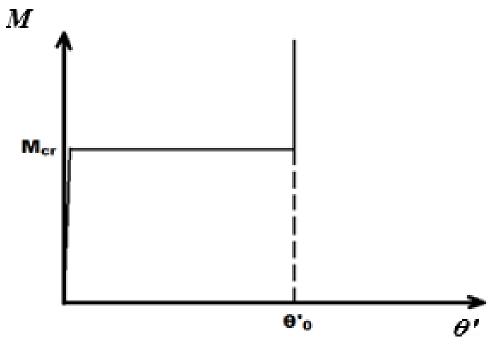

Fig. 4: The bending moment versus the curvature of a very thin beam
Source: created by the authors.

Therefore, in the region of high strains, where the moment of inertia is comparatively very small, constant curvature θ'_o is present with the loss of the transverse curvature of the very thin. That phenomenon is easily observed in the well-known very thin metallic measuring tapes. Indeed, the total potential energy V is defined by,
$$V = \int_0^{\theta_0} \frac{1}{2} EI\rho^{-1} d\theta - p[(\rho - \rho cos\theta_o) + (L - \rho\theta_o)sin\theta_o]. \tag{28}$$

where the moment of inertia I is taken to be that corresponding to zero lateral curvature.

The equilibrium equation is expressed by:
$$\frac{dV}{d\theta_o} = 0. \tag{29}$$

Eq.(29) yields,
$$\frac{EI}{2}\rho^{-1} - p(L - \rho\theta_o)cos\theta_o = 0. \tag{30}$$

Eq.(30) yields the value θ_o defining the deformation.

5 The Coexistence of the Curvature of Globally Stable Buckling of a Metallic Very Thin Beam under Axial Loading

The buckling problem of a metallic beam, with a thin curved cross-section under axial loading, will be discussed in the present paragraph. The problem presented in the previous paragraph will be applied with the specific diagram of bending moment versus curvature shown in Figure 4. In fact, the stiff bar may support a bending moment of critical value M_c. Due to the globally stable nonconvex system, the curvature jumps to the value θ'_o. For bending moment $M > M_c$, the curvature is almost constant. Therefore, the critical value for the axial buckling is given by the value p_o, see Eq.(23). Increasing the value of the axial loading ($p>p_o$), the postcritical deformation is shown in Figure 5. In fact, there exist two almost rigid external parts with curved cross-section, near the supports, as it was at its initial cross-section.

Fig. 5: Buckling of a two-phase metallic very thin beam
Source: created by the authors.

Due to snap buckling of the middle part of the metallic very thin beam, the beam's lateral curvature becomes zero. Hence, the moment of inertia of the cross-section is extremely small. In fact, there exist two parts of the rod that remain almost rigid with a moment of inertia Io, and the central part with a moment of inertia $I_i << I_o$. Considering a constant radius of curvature,

$$\rho=(\theta_o')^{-1} \tag{31}$$

At the end parts of length b, the deformed arc is defined by the angle θ, (Figure 6). The total energy of the system V is defined by:

$$V=\frac{1}{2}\int_{-\theta}^{\theta}\frac{EI}{\rho^2}ds-p\delta l \tag{32}$$

with

$$\delta l=L-2\rho\sin\theta-(L-2\rho\theta)\cos\theta. \tag{33}$$

Therefore,

$$V=\frac{EI\theta}{\rho}-p(L-2\rho\sin\theta-(L-2\rho\theta)\cos\theta \tag{34}$$

and the equilibrium equation,

$$\frac{\partial V}{\partial\theta}=0 \tag{35}$$

yields,

$$\frac{EI}{\rho}-p(L-2\rho\theta)\sin\theta=0. \tag{36}$$

6 First Application (The Two-Phase Bending)

Let us consider a cantilever beam of Figure 1, with length l=0.50 m and thickness $h=5*10^{-4}$ m and cross-section of circular arc shape of radius R=0.012m, corresponding to a focus angle of 90 ° (Figure 1). A load P is applied upon the free end B of the cantilever. The elastic (Young) modulus is $E=200\ 10^9$ Pa. The nonconvex bending moment – curvature diagram is shown in Figure 3: For small values of the applied load P, the elastic line of the beam is governed by Eq.(11). Nevertheless, that behavior of the beam deformation is valid up to the maximum $M=M_A=M_c=0.1$ Ntm. As soon as the maximum bending moment reaches that value, two different beam sections coexist. The one near the clamped end A follows the right path of the bending moment versus curvature diagram, Figure 3, and the other follows the conventional left path of the diagram.

Two curvatures exist at the moment M_c= 0.1 mtn. The curvature $\theta'\approx 0$ and the curvature $\theta_0'=1$. The beam elastic curve is shown in Figure 4 . Initially, for the small values of the load P<0.49 Nt, the beam follows the conventional deformation with,

$$y=\frac{0.05*x^2}{EI}. \tag{37}$$

Nevertheless, when P>0.49Nt, the axial curvature of the elastic line jumps from the value of 0.01 to the value $\theta_o'=1$.

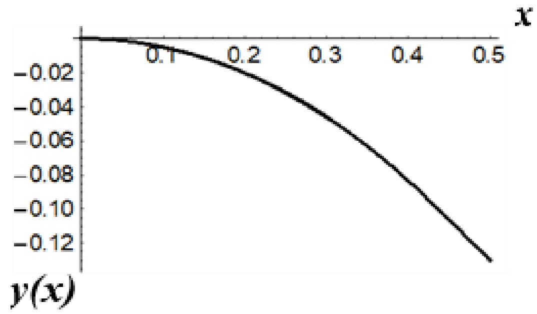

Fig. 6: The elastic curve of the two-phase cantilever beam
Source: created by the authors.

It is evident that for arclengths s<0.425, the elastic curve accepts a constant curvature equal to one. However, for the arc length s > 0.425 (the free end), the curvature is almost zero.

7 The Coexistence of Curvature of Globally Stable Buckling of a Metallic Very Thin Beam under Axial Loading. The simply Supported Beam.

The critical buckling load of a simply supported beam is equal to:

$$p_o=\frac{EI\pi^2}{L^2}. \tag{38}$$

Further, the elastic curve is defined by:

$$\Theta(x)=\mp\frac{2}{\sqrt{3}}\sqrt{\frac{p-p_o}{p_o}}\,cos(\frac{\pi x}{L}) \tag{39}$$

where θ(x) is the angle of the tangent to the elastic curve. Nevertheless, as soon as buckling deformation occurs, the cross-section geometry in the middle of the beam changes, and its lateral curvature is lost. Therefore, the moment of inertia of the rod from the initial value of the curved rod cross-section:

$$I_l=6.33\ x10^{-12}\ m^4 \tag{40}$$

reduces to:
$I_o=9.93 \times 10^{-14}\ m^4$. (41)

Therefore, for
$E=210\times 10^9\ Ntm^{-2}$ (42)

Eq.(36) yields for the beam length L=0.30m, the critical load,
$p_o=14.58$ N. (43)

Through Eq. (36), with $\rho=\theta_o'^{-1}=0.05$ m, the angle $\theta=17.19^o$.

Then, the moving support is displaced by δL,
$\delta L=L-(L-2\rho\theta)\cos\theta=0.0031$m. (44)

Figure 5 indicates the beam buckling deformation under axial loading

8 Conclusion

Beam bending deformations of metallic very thin beams with lateral snap deformations are discussed. Bending and buckling problems of beams with curved cross-sections and thin thickness are discussed. Due to their very small thickness, they exhibit a lateral snap-through deformation. Hence, the beam bending analysis is divided into two parts: one with an initial laterally curved cross-section and the other with a zero-curvature beam cross-section.

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