When the Future Perfectly Reflects the Past in Celestial Mechanics

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Abstract: Newton's equations of celestial mechanics for the N-body problem possess a continuum of solutions in which the future trajectories are a perfect reflection of their past. These solutions evolve from zero initial velocities of the N bodies. Consequently, the future gravitational forces acting on the N bodies are also a perfect reflection of their past. The proof is carried out via Taylor series expansions.

Key-Words: Celestial Mechanics; Even and odd solutions; Taylor Series; Holomorphic functions

Received: May 8, 2024. Revised: November 14, 2024. Accepted: March 7, 2025. Published: April 14, 2025.

1 Introduction

Symmetry is an important property for solution trajectories in celestial mechanics. A good example of symmetry is periodicity. For $1 \leq i \leq N$, let $f_i(t) \in \mathbb{R}^3$ be the orbit of the ith point mass m_i. Let $f(t) = [f_1(t), f_2(t), \ldots, f_N(t)]^T \in \mathbb{R}^{3N}$, where T stands for the transpose of the 1×3 vector. Let $\mathbb{N}$ denote the set of positive integers. We say $f(t)$ is a periodic function of with a period $\delta > 0$ if there exists $n \in \mathbb{N}$ such that

$$f(t - n\delta) \equiv f(t + n\delta). \tag{1}$$

Solutions trajectories such as circles and ellipses satisfy (1) and demonstrate the predictive powers Newtons' equations of celestial mechanics. See for example the Kepler two body problem, [1]. Moreover, circle and ellipse are even functions of t. Even functions and odd functions are also a manifestation of symmetry, i.e.

$$f(t) = f(-t), \; -f(t) = f(-t). \tag{2}$$

This work is devoted to study and analysis of celestial mechanics solutions that have symmetries. The universe has an obscure past, present, and future that can not be predicted. Are there astronomically remote subsystems of point masses that approximately possess symmetries that Newton's equations predict? We are interesting in discovering the way that past, present, and future can be studied via mathematical models. Compare with [2]. Hence we proposed the following question. Does the N body problem have solutions such that the future is the optimal picture of the past? The answer to this question is yes. However, this affirmative answer is conditional in that conditions must be imposed on the initial conditions to get such result. Indeed, this is correct if and only if the initial velocities of all N bodies are initialized from the zero vector. Our proof uses Taylor series expansions in a manner tangible to scientists and guarantees the existence of real analytic even solutions to the N body problem. For analytic solutions of differential systems compare e.g. with [3], [4], [5]. The Taylor series proof is written for a general second-order nonlinear vectorial autonomous differential equations $w''(t) = L(w)$ and is applicable to to central force problems, like the Manev problem, and the Pendulum equation. Compare e.g. with [6], [7], [8], [9], [10], [11], [12], [13].

The plan of work is as follows. In Section 2 we introduce preliminary notations and conventions and provide motivations for the N body problem. Section 3 contains the the first main theorem regarding the existence of even solutions to the celestial mechanics equations of the N. It is proven in the context of general second-order nonlinear vectorial autonomous differential systems $w''(t) = L(w)$. In Section 4 we provide a discussion about the equality $-L(w) = L(-w)$ and prove a lemma on the symmetry of the partial derivatives of $L(w)$. In Section 5 provides two successive even-order derivatives of the components of $w(t)$. Section 6 contains the second main theorem of this paper in which, under which conditions, we show that $w''(t) = L(w)$ possesses a continuum of odd solutions as well.

2 Preliminary Notations and an Approximation Model

Throughout this paper the following notation is used. Let $m, r, s, j, k, l, \lambda, s_l, z_l, k_l, N \in \mathbb{N} \cup \{0\}$. For $1 \leq j \leq N$, let m_j be the point mass of the jth body. Let $t \in \mathbb{C}$ be the time variable. Then $f_j(t) \in \mathbb{R}^3$, where $1 \leq j \leq N$, is the position vector of the jth body and $f^T(t) := [f_1^T(t), f_2^T(t), \ldots, f_N^T(t)] \in \mathbb{R}^{3N}$. (Recall that T is the transpose operator.) For $1 \leq j \leq N$, let $\overline{f}_j(t)$ be the complex conjugate of $f_j(t)$. Let $t_0 \in \mathbb{R}$, with

$$f_k(t_0), f_j(t_0) \in \mathbb{R}^3, \; f_j(t_0) \neq f_k(t_0) \text{ for } j \neq k, \quad \text{where } 1 \leq j, k \leq N. \tag{3}$$

Let

$$D_\epsilon(t_0) := \{t \,\big|\, |t - t_0| \leq \epsilon, \; \epsilon > 0, \; t_0 \in \mathbb{R}, \; t \in \mathbb{C}\}. \tag{4}$$

The notation

$$\|f_j(t)\| \equiv \|f_j\| = [f_j(t)^T \overline{f}_j(t)]^{\frac{1}{2}}, \; 1 \leq j \leq N \tag{5}$$

denotes the Euclidean norm of $f_j(t) \in \mathbb{C}^3$. We will also need the definition of an algebraic norm for a vector in $\mathbb{C}^n$.

Definition 1. *Let $M_{m\times n}(\mathbb{C})$ denote the set of $m \times n$ matrices with complex entries. Given $A^\star \in M_{m\times n}(\mathbb{C})$ and given $w \in M_{n\times 1}(\mathbb{C}) \equiv \mathbb{C}^n$, for any matrix norm $\|\cdot\|_M$, we define an algebraic norm $|\cdot|$ on $\mathbb{C}^n$ via the relation*

$$\big|A^\star w\big| \leq \|A^\star\|_M \big|w\big|. \tag{6}$$

Newton's equations of celestial mechanics imply that $f_k''(t) \in \mathbb{R}^3$, where

$$f_k''(t) := \sum_{j=1, j\neq k}^{N} \frac{Gm_j(f_j - f_k)}{\|f_j - j_k\|^3}, \; 1 \leq k \leq N. \tag{7}$$

For ease of exposition we have suppressed the independent variable t in the right side summands. Define $\hat{F}(f(t)) \in \mathbb{R}^{3N}$, where

$$\hat{F}(f(t)) \equiv \hat{F}(f) = \begin{bmatrix} \hat{F}_1(f) \\ \hat{F}_2(f) \\ \vdots \\ \hat{F}_N(f) \end{bmatrix} = f''(t) = \begin{bmatrix} f_1''(t) \\ f_2''(t) \\ \vdots \\ f_N''(t) \end{bmatrix} = \begin{bmatrix} \sum_{j=1,j\neq 1}^N \frac{Gm_j(f_j-f_2)}{\|f_j-f_1\|^3} \\ \sum_{j=1,j\neq 2}^N \frac{Gm_j(f_j-f_2)}{\|f_j-f_2\|^3} \\ \vdots \\ \sum_{j=1,j\neq N}^N \frac{Gm_j(f_j-f_N)}{\|f_j-f_N\|^3} \end{bmatrix}. \tag{8}$$

The initial value problem for N bodies is given by

$$f''(t) = \hat{F}(f), \; f(t_0) = f_0, \quad f'(t_0) = \eta, \; f_k(t_0) \neq f_j(t_0), \quad k \neq j, \, k, j = 1, 2, \ldots, N. \tag{9}$$

The N body problem then satisfies

$$-\hat{F}_k(-f) = -\sum_{j\neq k} \frac{Gm_j(-f_j + f_k)}{\|f_j - f_k\|^3} = \sum_{j\neq k} \frac{Gm_j(f_j - f_k)}{\|f_j - f_k\|^3} = \hat{F}_k(f) \implies -\hat{F}(f) = \hat{F}(-f). \tag{10}$$

The condition $f(t_0) = 0$ yields that all of the N bodies collide with each other. Then some of the terms in the right side of the initial value problem (9) will be undetermined and unbounded which makes the right side of the initial value problem (9) invalid. Consequently, it is impossible to seek for odd solutions for the initial value problem (9) with the constraint $f(t_0) = 0$. We propose a modified celestial mechanics system which allows for such an initial condition, namely

$$\hat{F}_k(f) = f_k''(t) := \sum_{j=1, j\neq k}^{N} \frac{Gm_j(f_j - f_k)}{[\|f_k - f_j\| + \epsilon(j,k)]^3}, \quad 1 \leq j, k \leq N, \; \epsilon(j,k) = \epsilon(k,j) > 0. \tag{11}$$

The definition of $\hat{F}_k(f)$ provided by (11) also satisfies $-\hat{F}(f) = \hat{F}(-f)$. Consequently, with

$$-\hat{F}^T(f) := -[\hat{F}_1^T(f), \hat{F}_2^T(f), \ldots, \hat{F}_N^T(f)] = \hat{F}^T(-f), \tag{12}$$

an initial value problem for an approximated equation is then obtained:

$$f''(t) = \hat{F}(f),\ f(t_0) = f_0,$$
$$f'(t_0) = \eta,\ f_0, \eta \in \mathbb{R}^{3N},$$
$$\text{'}f''(t) \text{ defined by (11).} \quad (13)$$

Formally, the limits of all $\hat{f}_k(f)$, with all $\epsilon(j,k) \to 0^+$, are respectively given by (7). The initial value problem (13) can be solved for any $f_0, \eta \in \mathbb{R}^{3N}$. Notice that

$$f_0, \eta \in \mathbb{R}^{3N},$$
$$t, t_0 \in \mathbb{R} \Longrightarrow f(t) \in \mathbb{R}^{3N}.$$

Note that for $f(t) \in \mathbb{R}^{3N}$, for any solution to the initial value problem (13), and for any algebraic norm, we find that

$$\|f''(t)\| = \left\|\hat{F}(f)\right\| \leq$$
$$\left\| \sum_{j=1, j\neq k}^{N} \frac{Gm_j(f_j - f_k)}{[\|f_k - f_j\| + \epsilon(j,k)]^3} \right\| \leq$$
$$\leq NG[\max(m_j)] \max\left(\frac{1}{[\epsilon(j,k)]^2}\right), \quad (14)$$

where $\max(m_j)$ and $\max([\epsilon(j,k)]^{-2})$ are taken over all $j = 1, 2, \cdots, N$. This asserts that all solutions of (13) exist on $(-\infty, \infty)$. Consequently, the system of equations $f'' = \widehat{F}(f)$ given by (13) is (so-called) complete. However, solutions of (13) as analytic functions of t could develop singularities in the complex plane.

The initial value problems of (9) and (13) are special instances of the autonomous vector system $w''(t) = L(w(t)) \equiv L(w)$, where $w(t), L(w) \in \mathbb{C}^n$ with

$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_n(t) \end{bmatrix},\ w_j(t) \in \mathbb{C},\ 1 \leq j \leq n, \quad (15)$$

and where

$$L(w) = \begin{bmatrix} L_1(w(t)) \\ L_2(w(t)) \\ \vdots \\ L_n(w(t)) \end{bmatrix},$$
$$L_j(w(t)) \equiv L_j(w) \in \mathbb{C},\ 1 \leq j \leq n. \quad (16)$$

Observe that $w''(t) = L(w)$ encapsulates the following system of n nonlinear autonomous second order differential equations

$$w_j''(t) = L_j(w(t)) \equiv L_j(w),\ 1 \leq j \leq n. \quad (17)$$

The main theorems of this paper discuss the existence of symmetric solutions to $w''(t) = L(w(t)) \equiv L(w)$, of which the celestial mechanics initial value problems are special cases (with $n = 3N$).

3 When $w''(t) = L(w)$ has Even Solutions

In this theorem, and throughout the remainder of this paper, we let, for $1 \leq k \leq n$,

$$\frac{dw_k}{dt} = w_k'(t) \equiv w_k'$$
$$\frac{d^2 w_k}{dt^2} = w_k''(t) \equiv w_k''$$
$$\frac{d^\ell w_k}{dt^\ell} = w_k^\ell(t) \equiv w_k^\ell,\ \ell \in \mathbb{N}. \quad (18)$$

Theorem 1. *Let $t, t_0 \in \mathbb{C}$ and $w_0, \eta \in \mathbb{R}^n$. Assume vector valued function $L(w)$, where $w(t), L(w) \in \mathbb{C}^n$, is analytic with respect to the vector variable w in a disk D with*

$$D := \{w \big| \|w - w_0\| \leq b^\star\} \quad \Longrightarrow \|L(w)\| \leq M^\star. \quad (19)$$

Then the initial value problem

$$w''(t) = L(w),\ w(t_0) = w_0,$$
$$w'(t_0) = \overrightarrow{0},\ \overrightarrow{0}^T := [0, 0, \cdots, 0] \quad (20)$$

possesses a unique analytic solution $w(t)$ for $|t - t_0| \leq \sqrt{\frac{2b^\star}{M^\star}}$ that satisfies $w(t - t_0) \equiv w(-(t - t_0))$. Namely, $w^{(m)}(t_0) = \overrightarrow{0}$ for all odd numbers m.

Proof: As our differential system is autonomous, we may set $t_0 = 0$. Two successive odd order derivatives are introduced below. Compare also with [14], [15], [16].

$$\frac{d^3 w_j(t)}{dt^3} = \sum_{k_1=1}^{n} \frac{\partial L_j(w(t))}{\partial w_{k_1}} \frac{dw_{k_1}(t)}{dt},$$
$$j = 1, 2, \ldots, n; \quad (21)$$

$$\frac{d^5 w_j(t)}{dt^5} = \sum_{k_1=1}^{n} \frac{\partial L_j(w(t))}{\partial w_{k_1}} \frac{d^3 w_{k_1}(t)}{dt^3}$$
$$+ \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \frac{\partial^2 L_j(w(t))}{\partial w_{k_2} \partial w_{k_1}}$$
$$\left[2 \frac{dw_{k_2}(t)}{dt} \frac{d^2 w_{k_1}(t)}{dt^2} + \frac{d^2 w_{k_2}(t)}{dt^2} \frac{dw_{k_1}(t)}{dt}\right]$$
$$+ \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} \frac{\partial^3 L_j(w(t))}{\partial w_{k_3} \partial w_{k_2} \partial w_{k_1}}$$
$$\frac{dw_{k_3}(t)}{dt} \frac{dw_{k_2}(t)}{dt} \frac{dw_{k_1}(t)}{dt}. \quad (22)$$

Equations (21) and (22) show that

$$w'(0) = \overrightarrow{0} \Longrightarrow w^{(3)}(0) = w^{(5)}(0) = \overrightarrow{0}. \quad (23)$$

The two special cases (21) and (22) give a clear idea of what should be the general form of a typical term in the higher odd order derivatives of $w_j(t)$. Furthermore, the identities (21) and (22) suggest why the property of zero initial velocities $w_j^{(1)}(0) = 0$ is inherited by subsequent derivatives of odd order.

In what follows we may suppress the notation (t) in $w(t), w_j^{(\lambda)}(t)$ etc., when clarity is not compromised. Assume that for $1 \leq j \leq n$, each component $w_j^{(2+m)}(t)$, m odd, is a finite sum of (not necessarily distinct) products of the form

$$S_m := \frac{\partial^l L_j(w(t))}{\partial w_{k_l} \dots \partial w_{k_2} \partial w_{k_1}} J_{ODD} J_{EVEN}, \quad (24)$$

where

$$J_{ODD} = w_{s_1}^{(2e_1+1)}(t) w_{s_2}^{(2e_2+1)}(t) \dots w_{s_r}^{(2e_r+1)}(t),$$
$$e_1, e_2, \dots, e_r \in \mathbb{N} \cup \{0\},\ r \geq 1,\ r \text{ odd}, \quad (25)$$

and where

$$J_{EVEN} = w_{z_1}^{(2c_1)}(t) w_{z_2}^{(2c_2)}(t) \dots w_{z_l}^{(2c_l)}(t),$$
$$l, c_1, c_2, \dots, c_l \in \mathbb{N} \cup \{0\} \quad (26)$$

If $l = 0$, then we choose $J_{EVEN} \equiv 1$. Note that J_{ODD} is a finite product of an odd number of odd order derivatives of components of $w(t)$ while J_{EVEN} is a finite product of even order derivatives of components of $w(t)$. The goal is to show that $S_m^{(2)}$ is the finite sum of terms of the form $\widehat{S_m^{(2)}}$, where

$$\widehat{S_m^{(2)}} := \frac{\partial^s l_j(w(t))}{\partial w_{k_s} \dots \partial w_{k_2} \partial w_{k_1}} \widehat{J}_{ODD} \widehat{J}_{EVEN}, \quad (27)$$

with

$$\widehat{J}_{ODD} = w_{s_1}^{(2g_1+1)}(t) w_{s_2}^{(2g_2+1)}(t) \dots w_{s_p}^{(2g_p+1)}(t),$$
$$g_1, g_2, \dots, g_p \in \mathbb{N} \cup \{0\},\ p \geq 1,\ p \text{ odd}, \quad (28)$$

and

$$\widehat{J}_{EVEN} = w_{z_1}^{(2b_1)}(t) w_{z_2}^{(2b_2)}(t) \dots w_{z_q}^{(2b_q)}(t),$$
$$q, b_1, b_2, \dots, b_q \in \mathbb{N} \cup \{0\}. \quad (29)$$

If $t = 0$, we have shown that $w^{(1)}(0) = w^{(3)}(0) = w^{(5)}(0) = \overrightarrow{0}$. It is not difficult to ensure that (21) and (22) are sums of products of the form (24). The main part of the induction is to prove that $\widehat{J}_{ODD}$ and $\widehat{J}_{EVEN}$ are of the desired form (28) and (29) for any m odd. To do that we differentiate twice (with respect to t) both sides of the formula (24). This yields the relation

$$S_m^{(2)} = Q_1^\star + Q_2^\star + Q_3^\star,$$

where

$$Q_1^\star := \sum_{k_{l+2}=1}^{n} \sum_{k_{l+1}=1}^{n} \frac{\partial^{l+2} l_j(w(t))}{\partial w_{k_{l+2}} \partial w_{k_{l+1}} \partial w_{k_l} \dots \partial w_{k_2} \partial w_{k_1}} w_{k_{l+2}}^{(1)} w_{k_{l+1}}^{(1)} J_{ODD} J_{EVEN} + \sum_{k_{l+1}=1}^{n} \frac{\partial^{l+1} L_j(w(t))}{\partial w_{k_{l+1}} \partial w_{k_l} \dots \partial w_{k_2} \partial w_{k_1}} w_{k_{l+1}}^{(2)} J_{ODD} J_{EVEN}, \quad (30)$$

$$Q_2^\star := 2 \sum_{k_{l+1}=1}^{n} \frac{\partial^{l+1} l_j(w(t))}{\partial w_{k_{l+1}} \partial w_{k_l} \dots \partial w_{k_2} \partial w_{k_1}} w_{k_{l+1}}^{(1)} [J_{ODD}^{(1)} J_{EVEN} + J_{ODD} J_{EVEN}^{(1)}], \quad (31)$$

$$Q_3^\star := \frac{\partial^l L_j(w(t))}{\partial w_{k_l} \dots \partial w_{k_2} \partial w_{k_1}} [J_{ODD} J_{EVEN}]^{(2)} = \frac{\partial^l L_j(w(t))}{\partial w_{k_l} \dots \partial w_{k_2} \partial w_{k_1}} [J_{ODD}^{(2)} J_{EVEN} + 2 J_{ODD}^{(1)} J_{EVEN}^{(1)} + J_{ODD} J_{EVEN}^{(2)}]. \quad (32)$$

Note that $Q_1^\star, Q_2^\star$ and $Q_3^\star$ are summations of certain products and we show that they are of the form (27) subject to (28) and (29). It is beneficial to remember that J_{ODD} is a finite product of an odd number $r \geq 1$ of odd order derivatives. The following terms should be introduced:

$$w_{k_{l+2}}^{(1)} w_{k_{l+1}}^{(1)} J_{ODD} J_{EVEN},$$
$$w_{k_{l+1}}^{(2)} J_{ODD} J_{EVEN},\ w_{k_{l+1}}^{(1)} J_{ODD}^{(1)} J_{EVEN},$$
$$w_{k_{l+1}}^{(1)} J_{ODD} J_{EVEN}^{(1)},\ J_{ODD}^{(1)} J_{EVEN}^{(1)},$$
$$J_{ODD}^{(2)} J_{EVEN},\ J_{ODD} J_{EVEN}^{(2)}. \quad (33)$$

For the products of the form $w_{k_{l+2}}^{(1)} w_{k_{l+1}}^{(1)} J_{ODD} J_{EVEN}$ we choose

$$\widehat{J}_{ODD} = w_{k_{l+2}}^{(1)} w_{k_{l+1}}^{(1)} J_{ODD},$$
$$\widehat{J}_{EVEN} = J_{EVEN}.$$

For products originating from $w_{k_{l+1}}^{(2)} J_{ODD} J_{EVEN}$ put

$$\widehat{J}_O = J_{ODD},\ \widehat{J}_{EVEN} = w_{k_{l+1}}^{(2)} J_{EVEN}.$$

Consider $w_{k_{l+1}}^{(1)} J_{ODD}^{(1)} J_{EVEN}$. Evidently, $J_{ODD}^{(1)}$ is a finite summation of r products

$$J_{ODD}^{(1)} = \sum_{j=1}^{r} w_{s_j}^{(2e_j+2)} \prod_{l \neq j, l=1}^{r} w_{s_l}^{(2e_l+1)}. \quad (34)$$

From (34) notice that each summand has $(r-1)$ odd order derivatives factors and exactly one factor that is an even order derivative of a certain component of w_{s_j}. Each such summand of $w^{(1)}_{k_{l+1}} J^{(1)}_{ODD} J_{EVEN}$ can be written as

$$\begin{aligned} \widehat{J}_{EVEN} &= w^{(2e_j+2)}_{s_j} J_{EVEN}, \\ r = 1 &\Rightarrow \widehat{J}_{ODD} = w^{(1)}_{k_{l+1}} \\ r \geq 2 &\Rightarrow \widehat{J}_{ODD} = w^{(1)}_{k_{l+1}} \prod_{l\neq j, l=1}^{r} w^{(2e_l+1)}_{s_l}. \end{aligned} \tag{35}$$

We now analyze products originating from $w^{(1)}_{k_{l+1}} J_{ODD} J^{(1)}_{EVEN}$. If $l = 0$, namely $J_{EVEN} \equiv 1$, then $J^{(1)}_{EVEN} \equiv 0$ and $w^{(1)}_{k_{l+1}} J_{ODD} J^{(1)}_{EVEN} \equiv 0$. If $l = 1$, let

$$\widehat{J}_{ODD} = w^{(1)}_{k_{l+1}} w^{(2c_1+1)}_{z_1} J_{ODD}, \;\; \widehat{J}_{EVEN} \equiv 1$$

If $l \geq 2$, we see that $J^{(1)}_{EVEN}$ can be written as a sum of l products of the form

$$J^{(1)}_{EVEN} = \sum_{j=1}^{l} w^{(2c_j+1)}_{z_j} \prod_{l\neq j, l=1}^{l} w^{(2c_l)}_{z_l}. \tag{36}$$

Each such summand of $w^{(1)}_{k_{l+1}} J_{ODD} J^{(1)}_{EVEN}$ can be written as

$$\begin{aligned} \widehat{J}_{ODD} &= w^{(1)}_{k_{l+1}} w^{(2c_j+1)}_{z_j} J_{ODD}, \\ \widehat{J}_{EVEN} &= \prod_{l\neq j, l=1}^{l} w^{(2c_l)}_{z_l}. \end{aligned} \tag{37}$$

Consider the products emanating from $J^{(1)}_{ODD} J^{(1)}_{EVEN}$. If $J_{EVEN} \equiv 1$, namely $l = 0$, then $J^{(1)}_{ODD} J^{(1)}_{EVEN} \equiv 0$. If $r = 1$ and $l = 1$, put

$$\widehat{J}_{ODD} = w^{(2c_1+1)}_{z_1}, \;\; \widehat{J}_{EVEN} = w^{(2e_1+2)}_{s_1}.$$

If $r \geq 2$ and $l = 1$, put

$$\begin{aligned} \widehat{J}_{ODD} &= w^{(2c_1+1)}_{z_1} \prod_{l\neq j, l=1}^{r} w^{(2e_l+1)}_{s_l}, \\ \widehat{J}_{EVEN} &\equiv 1. \end{aligned} \tag{38}$$

If $r \geq 2$, $l \geq 2$, we have by (34) and (36) that rl products in $J^{(1)}_{ODD} J^{(1)}_{EVEN}$. For a typical product put

$$\begin{aligned} \widehat{J}_{ODD} &= w^{(2c_j+1)}_{z_j} \prod_{l\neq j, l=1}^{r} w^{(2e_l+1)}_{s_l}, \\ \widehat{J}_{EVEN} &= w^{(2e_j+2)}_{s_j} \prod_{l\neq j, l=1}^{l} w^{(2c_l)}_{z_l}. \end{aligned}$$

It remains to consider $J^{(2)}_{ODD} J_{EVEN}$ and $J_{ODD} J^{(2)}_{EVEN}$. We first calculate the sum of products emanating from $J^{(2)}_{ODD}$ and multiply them with J_{EVEN}. If J_{ODD} has one factor, i.e. $r = 1$, this yields that $J^{(2)}_{ODD} = w^{(2e_1+3)}_{s_1}$. Thus $\widehat{J}_{ODD} = w^{(2e_1+3)}_{s_1}$ and $\widehat{J}_{EVEN} = J_{EVEN}$. If $r \geq 3$, then the products in $J^{(2)}_{ODD}$ can be classified into two kinds.The first type is

$$w^{(2e_j+2)}_{s_j} w^{(2e_k+2)}_{s_k} \prod_{l\neq j,k, l=1}^{r} w^{(2e_l+1)}_{s_l}, \;\; r \geq 3. \tag{39}$$

Then put

$$\begin{aligned} \widehat{J}_{ODD} &= \prod_{l\neq j,k, l=1}^{r} w^{(2e_l+1)}_{s_l}, \\ \widehat{J}_{EVEN} &= w^{(2e_j+2)}_{s_j} w^{(2e_k+2)}_{s_k} J_{EVEN}, \;\; r \geq 3. \end{aligned}$$

The second type of product in $J^{(2)}_{ODD}$ is $w^{(2e_j+3)}_{s_j} \prod_{l\neq j, l=1}^{r} w^{(2e_l+1)}_{s_l}$. Put

$$\begin{aligned} \widehat{J}_{ODD} &= w^{(2e_j+3)}_{s_j} \prod_{l\neq j, l=1}^{r} w^{(2e_l+1)}_{s_l}, \\ \widehat{J}_{EVEN} &= J_{EVEN}, \quad r \geq 3. \end{aligned}$$

It remains to analyze $J_{ODD} J^{(2)}_{EVEN}$. Hence we calculate the products in $J^{(2)}_{EVEN}$. If $l = 0$, then $J^{(2)}_{EVEN} \equiv 0$ and $J_{ODD} J^{(2)}_{EVEN} = 0$. If $l = 1$, then $J^{(2)}_{EVEN} \equiv w^{(2c_1+2)}_{z_1}$, and we put $\widehat{J}_{ODD} = J_{ODD}$ and $\widehat{J}_{EVEN} = w^{(2c_1+2)}_{z_1}$. Suppose $l \geq 2$. Then $J^{(2)}_{EVEN}$ is the sum of two types of products. This first kind is

$$w^{(2c_1+2)}_{z_j} \prod_{l\neq j, l=1}^{l} w^{(2c_l)}_{z_l}. \tag{40}$$

Then put

$$\begin{aligned} \widehat{J}_{ODD} &= J_{ODD}, \\ \widehat{J}_{EVEN} &= w^{(2c_1+2)}_{z_j} \prod_{l\neq j, l=1}^{l} w^{(2c_l)}_{z_l}. \end{aligned} \tag{41}$$

If $l = 2$, the second type is $w^{(2c_j+1)}_{z_j} w^{(2c_k+1)}_{z_k}$. Then put

$$\begin{aligned} \widehat{J}_{ODD} &\equiv w^{(2c_j+1)}_{z_j} w^{(2c_k+1)}_{z_k} J_{ODD}, \\ \widehat{J}_{EVEN} &\equiv 1. \end{aligned} \tag{42}$$

If $l \geq 3$, then type two of the product resulting from the sum of products in $J^{(2)}_{EVEN}$ is

$$w^{(2c_j+1)}_{z_j} w^{(2c_k+1)}_{z_k} \prod_{l \neq j,k,l=1}^{l} w^{(2c_l)}_{z_l}. \tag{43}$$

Then put

$$\widehat{J}_{ODD} = w^{(2c_j+1)}_{z_j} w^{(2c_k+1)}_{z_k} J_{ODD},$$
$$\widehat{J}_{EVEN} = \prod_{l \neq j,k,l=1}^{l} w^{(2c_l)}_{z_l}. \tag{44}$$

Thus we have shown that each term in $S^{(2)}_m$ has the desired form of (27) subject to (28) and (29). Furthermore, all $S_m(0) = 0$ implies that all $\widehat{S^{(2)}_m}(0) = 0$. Consequently, $w^{(m)}(0) = \overrightarrow{0}$ for odd $m \in \mathbb{N}$. □

Remark 1. *If independent variable t represents times, the expression $w(t) \equiv w(-t)$ shows that the future is the optimal image of the past. Theorem 1 shows that all odd order derivatives of $w(t)$ vanish at $t = 0$. The estimate $|t| \leq \sqrt{\frac{2b}{M}}$ can be obtained from the technicians in e.g., [5], Chapter 1, Page 20. Compare also with [3]. From Taylor series given by*

$$w(t) = w(0) + \sum_{l=1}^{n} [\frac{w^{(2l)}(0)}{(2l)!}] t^{2l},$$

we confirm that $w(t) \equiv w(-t)$. Since $w'' = L(w)$ is an autonomous system, then for any $t_0 \in \mathbb{C}$,

$$w(t) = w(t_0) + \sum_{l=1}^{n} \frac{w^{(2l)}(t_0)}{(2l)!} (t - t_0)^{2l}$$

is also a solution of $w'' = L(w)$. Not only that, the velocities are symmetric functions. This can be seen from $-w'(t) \equiv w'(-t)$. The future accelerations, which are second derivatives of the positions with respect to t, and hence the forces acting on the N bodies, are the optimal images of their past. Suppose that $t_c \in \mathbb{R}$ is a real-valued collision time, where $w_k(t_c) = w_j(t_c)$ for some $k \neq j$. If the variable t is allowed to be complex-valued, it would be possible to analytically continue a solution $w(t)$ from the real line into the complex plane, from time $t < t_c$ to $t > t_c$. Then $w(t) \equiv w(-t)$ holds for $t < t_c$ as well as for $t > t_c$, circumventing a collision at time $t = t_c$.

4 Even and Odd in Terms of w

We begin with the scalar function $K : \mathbb{R}^n \to \mathbb{R}$, where $K(w) \equiv K(w_1, w_2, \ldots, w_n) \in \mathbb{R}$. In what follows, the reader may choose to replace $\mathbb{R}$ (respectively $\mathbb{R}^n$) with $\mathbb{C}$ (respectively $\mathbb{C}^n$).

Definition 2. *Denote by OCS an open connected set in $\mathbb{R}^n$. Let $K : \mathbb{R}^n \to \mathbb{R}$. The function $K(w)$ is called an even function of w in OCS if the following condition holds.*

$$K(w) = K(-w), \; w \in OCS. \tag{45}$$

The function $K(w)$ is called an odd function of w in OCS if the following condition hold.

$$-K(w) = K(-w), \; w \in OCS. \tag{46}$$

The above definition is not as same as the definition requires that $K(w_1, w_2, \ldots, w_n) = K(w)$ be an even or an odd function in each individual coordinate w_j. To clarify the difference between the two concepts, it is necessary to introduce the following definition.

Definition 3. *Denote by OCS an open connected set in $\mathbb{R}^n$. Let $K : \mathbb{R}^n \to \mathbb{R}$. The function $K(w)$ is called an even function of w in OCS in the strict sense if*

$$K(w_1, w_2, \ldots, w_j, \ldots, w_n) = K(w_1, w_2, \ldots, -w_j, \ldots, w_n), \; w \in OCS, \; j = 1, 2, \ldots, n. \tag{47}$$

The function $K(w)$ is called an odd function of w in OCS in the stricter sense if

$$-K(w_1, w_2, \ldots, w_j, \ldots, w_n) = K(w_1, w_2, \ldots, -w_j, \ldots, w_n), \; w \in OCS, \; j = 1, 2, \ldots, n. \tag{48}$$

Example 1. *Consider the following two functions.*

$$K(w_1, w_2) := w_1^5 w_2^3, \; K^\star(w_1, w_2) = w_1^{10} w_2^6. \tag{49}$$

Note that $K(w_1, w_2) = w_1^5 w_2^3$ is an even function in $\mathbb{R}^2$; moreover, it is an odd function in the strict sense in $\mathbb{R}^2$. Moreover, $K^\star(w_1, w_2) = w_1^{10} w_2^6$ is an even function in $\mathbb{R}^2$; it is also an even function in the strict sense in $\mathbb{R}^2$.

Remark 2. *Consider a multinomial in $(r + l)$ independent variables $w_1, w_2, \ldots, w_j, \ldots, w_r, w_{r+1}, w_{r+2}, \ldots, w_{r+l}$.*

$$K(w) = w_1^{(2e_1+1)} w_2^{(2e_2+1)} \ldots w_j^{(2e_j+1)} \ldots w_r^{(2e_r+1)} * w_{r+1}^{(2c_1)} w_{r+2}^{(2c_2)} \ldots w_{r+l}^{(2c_l)}, \tag{50}$$

where $e_1, e_2, \ldots e_r, c_1, c_2, \ldots, c_l \in \mathbb{N} \cup \{0\}$ and $r, l \in \mathbb{N}$. If each component w_j is a function of an independent variable t, namely $w_j \equiv w_j(t)$, we may consider (50) as a finite multiplication of of r odd order derivatives and l even order derivatives. Formulation of necessary and sufficient conditions on the powers occurring in $K(w)$ such that

a) $K(w)$ is an even multivariate function,
b) $K(w)$ is an even multivariate function in the strict sense,
c) $K(w)$ is an odd multivariate function,
d) $K(w)$ is an odd multivariate function in the strict sense,

could further clarify the the difference between these two types of symmetry.

Definitions 2 and 3 can be generalized to the vector valued function $L : \mathbb{R}^n \to \mathbb{R}^n$, where

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \, w_j \in \mathbb{F}, \, 1 \le j \le n, \tag{51}$$

and where

$$L(w) = \begin{bmatrix} L_1(w_1, w_2, \ldots, w_n) \equiv L_1(w) \\ L_2(w_1, w_2, \ldots, w_n) \equiv L_2(w) \\ \vdots \\ L_n(w_1, w_2, \ldots, w_n) \equiv L_n(w) \end{bmatrix}, \quad L_j(w) \in \mathbb{R}, \, 1 \le j \le n. \tag{52}$$

Simply replace the $K : \mathbb{R}^n \to \mathbb{R}$ of Definitions 2 and 3 with the $L : \mathbb{R}^n \to \mathbb{R}^n$ as defined via (51) and (52).

Lemma 1. *Let $K(w) \in C^1(OCS)$, where $K : \mathbb{R}^n \to \mathbb{R}$.*

i) Suppose that $K(w) = K(-w)$ with $w \in OCS$. Then the partial derivatives

$$\Psi_j(w) := \frac{\partial K(w)}{\partial w_j}, \; j = 1, 2, \cdots, n,$$

are odd function in OCS.

ii) Suppose that $-K(w) = K(-w)$ with $w \in OCS$. Then the partial derivatives

$$\Psi_j(w) := \frac{\partial K(w)}{\partial w_j}, \; j = 1, 2, \cdots, n,$$

are even functions in OCS.

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be defined by (51) and (52) with $L_j(w) \in C^1(OCS)$ for $1 \le j \le n$.

iii) Assume that $L(w)$ is even, namely

$$L(w) = L(-w), w \in OCS.$$

Define

$$\hat{\Psi}(w) := \frac{\partial L(w)}{\partial w_j}, \; j = 1, 2, \ldots, n. \tag{53}$$

Then $\hat{\Psi}(w)$ is odd, i.e.

$$\hat{\Psi}(w) = -\hat{\Psi}(-w) := -\frac{\partial L(-w)}{\partial w_j}. \tag{54}$$

iv) Assume that $L(w)$ is odd, namely

$$-L(w) = L(-w), \; w \in OCS.$$

Then $\hat{\Psi}(w)$ is even, i.e.

$$\hat{\Psi}(w) = \hat{\Psi}(-w) := \frac{\partial L(-w)}{\partial w_j}. \tag{55}$$

Proof: We first prove i). For $w \in OCS$, with $h \neq 0$ and with σ arbitrarily small, define

$$Q_j^\star(w, \sigma) := \sigma^{-1} N_1, \tag{56}$$

where

$$N_1 := K(w_1, \ldots, w_{j-1}, w_j + \sigma, w_{j+1}, \ldots, w_n - K(w_1, \ldots, w_{j-1}, w_j, w_{j+1}, \ldots, w_n). \tag{57}$$

For a short hand notation put

$$\widehat{K}(w_j + \sigma) := K(w_1, \ldots, w_{j-1}, w_j + \sigma, w_{j+1}, \ldots, w_n), \tag{58}$$

and

$$K(-w_j - \sigma) := K(-w_1, \ldots, -w_{j-1}, -w_j - \sigma, -w_{j+1}, \ldots, -w_n). \tag{59}$$

Since $K(w)$ is an even function,

$$K(w_1, \ldots, w_{j-1}, w_j + \sigma, w_{j+1}, \ldots, w_n) = K(-w_1, \ldots, -w_{j-1}, -w_j - \sigma, -w_{j+1}, \ldots, -w_n). \tag{60}$$

and

$$K(w_1, \ldots, w_{j-1}, w_j, w_{j+1}, \ldots, w_n) = K(-w_1, \ldots, -w_{j-1}, -w_j, -w_{j+1}, \ldots, -w_n). \tag{61}$$

Substitute (58) and (59) into the right hand side of (56) to obtain

$$Q_j^\star(w, \sigma) = -Q_j^\star(-w, -\sigma) = -\frac{\widehat{K}(-w_j - \sigma) - K(-w_j)}{-\sigma}. \tag{62}$$

Then

$$\Psi(w) := \frac{\partial K(w)}{\partial w_j} = \lim_{\sigma \to 0} Q_j^\star(w, \sigma) = -\lim_{\sigma \to 0} Q_j^\star(-w, -\sigma) = -\Psi(-w).$$

Next we prove ii). Since $K(w)$ is an odd function,

$$K(w_1, \ldots, w_{j-1}, w_j + \sigma, w_{j+1}, \ldots, w_n) = -K(-w_1, \ldots, -w_{j-1}, -w_j - \sigma, -w_{j+1}, \ldots, -w_n), \tag{63}$$

and

$$K(w_1,\ldots,w_{j-1},w_j,w_{j+1},\ldots,w_n) = -K(-w_1,\ldots,-w_{j-1},-w_j,-w_{j+1},\ldots,-w_n). \tag{64}$$

Substitute (63) and (64) into (56) to obtain

$$Q_j^\star(w,\sigma) = Q_j^\star(-w,-\sigma). \tag{65}$$

Then

$$\Psi(w) := \frac{\partial K(w)}{\partial w_j} = \lim_{\sigma\to 0} Q_j(w,\sigma) = \lim_{\sigma\to 0} Q_j^\star(-w,-\sigma) = \Psi(-w).$$

The proofs of iii) and iv) follows from the proofs of i) and ii) and the definition of $L(w)$. □

5 Sample Two Successive Even Derivatives

A straightforward calculation yields

$$\frac{d^4 w_j(t)}{dt^4} = \sum_{k_1=1}^{n} \frac{\partial L_j(w(t))}{\partial w_{k_1}} \frac{d^2 w_{k_1}(t)}{dt^2} + \sum_{k_1=1}^{n}\sum_{k_2=1}^{n} \frac{\partial^2 L_j(w(t))}{\partial w_{k_2}\partial w_{k_1}} \frac{dw_{k_2}(t)}{dt}\frac{dw_{k_1}(t)}{dt}. \tag{66}$$

$$\begin{aligned}\frac{d^6 w_j(t)}{dt^6} &= \sum_{k_1=1}^{n} \frac{\partial L_j(w(t))}{\partial w_{k_1}} \frac{d^4 w_{k_1}(t)}{dt^4} \\ &+ \sum_{k_1=1}^{n}\sum_{k_2=1}^{n} \frac{\partial^2 L_j(w(t))}{\partial w_{k_2}\partial w_{k_1}} \Big[3\frac{d^3 w_{k_1}(t)}{dt^3}\frac{dw_{k_2}(t)}{dt} \\ &+ \frac{dw_{k_1}(t)}{dt}\frac{d^3 w_{k_2}(t)}{dt^3} 3\frac{d^2 w_{k_1}(t)}{dt^2}\frac{d^2 w_{k_2}(t)}{dt^2}\Big] \\ &+ \sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\sum_{k_3=1}^{n} \frac{\partial^3 L_j(w(t))}{\partial w_{k_3}\partial w_{k_2}\partial w_{k_1}} \\ &\Big[3\frac{d^2 w_{k_1}(t)}{dt^2}\frac{dw_{k_2}(t)}{dt}\frac{dw_{k_3}(t)}{dt} \\ &+ 2\frac{dw_{k_1}(t)}{dt}\frac{d^2 w_{k_2}(t)}{dt^2}\frac{dw_{k_3}(t)}{dt} \\ &+ \frac{dw_{k_1}(t)}{dt}\frac{dw_{k_2}(t)}{dt}\frac{d^2 w_{k_3}(t)}{dt^2}\Big] \\ &+ \sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\sum_{k_3=1}^{n}\sum_{k_4=1}^{n} \frac{\partial^4 L_j(w(t))}{\partial w_{k_4}\partial w_{k_3}\partial w_{k_2}\partial w_{k_1}} \\ &\frac{dw_{k_4}(t)}{dt}\frac{dw_{k_3}(t)}{dt}\frac{dw_{k_2}(t)}{dt}\frac{dw_{k_1}(t)}{dt}.\end{aligned} \tag{67}$$

These aid in formulating the induction proof of our second main theorem, Theorem 2.

6 When $w''(t) = L(w)$ has Odd Solutions

Theorem 2. *Let $t, t_0 \in \mathbb{C}$ and $w_0, \eta \in \mathbb{R}^n$. Assume vector valued function $L(w)$, where $w(t), L(w) \in \mathbb{C}^n$, is analytic with respect to the vector variable w in a disk D with*

$$D := \{w \big| \|w - w_0\| \le b\} \implies \|L(w)\| \le M. \tag{68}$$

Furthermore assume that $L(w)$ is odd, i.e.

$$L(-w) = -L(w), \quad w \in D. \tag{69}$$

Then the initial value problem

$$w''(t) = L(w),\ w(t_0) = \overrightarrow{0}, \quad w'(t_0) = \eta,\ \overrightarrow{0}^T := [0,0,\cdots,0]. \tag{70}$$

possesses a unique analytic solution $w(t)$ for $|t - t_0| \le \sqrt{\frac{2b}{M}}$ that satisfies $-w(t - t_0) \equiv -w(-(t - t_0))$. Namely, $w^{(m)}(t_0) = \overrightarrow{0}$ for all even numbers m.

Proof: As our differential system is autonomous, we will set $t_0 = 0$. We proceed by induction on m and follow a strategy analogous to that used in the proof of Theorem 1. The main difference is the additional assumption $-L(w) = L(-w)$. There are two ways to explain the reason for this requirement. First observe that Theorem 1 shows that L is even with respect to t, namely that $L(w(t)) = L(w(-t))$. Theorem 2 will show that L is odd with respect to t, namely that $-L(w(t)) = L(w(-t))$. But to guarantee that L is odd with respect to t, we not only need that $w(t)$ is odd, but that $L(w)$ is also odd since

$$\begin{aligned}-L(w(t)) &= -L(-w(-t)), \quad \text{when } w(t) \text{ odd} \\ &= L(w(-t)). \quad \text{when } -L(w) = L(w)\end{aligned}$$

Secondly, observe that if $L(w)$ is an odd function of w, then $L(\overrightarrow{0}) = \overrightarrow{0}$. Moreover, by Lemma 1, the even order partial derivatives of $L_j(w)$ with respect to the variables w_k (like $L_j^{(0)}(w) := L_j(w)$) are odd functions of w. Namely,

$$-\frac{\partial^i L_j(w(t))}{\partial w_{k_i}\ldots\partial w_{k_2}\partial w_{k_1}} = \frac{\partial^i L_j(-w(t))}{\partial w_{k_i}\ldots\partial w_{k_2}\partial w_{k_1}},$$

$$i = 0,2,4,\ldots, \implies \frac{\partial^i L_j(\overrightarrow{0})}{\partial w_{k_i}\ldots\partial w_{k_2}\partial w_{k_1}} = 0. \tag{71}$$

Per Lemma 1, the odd order partial derivatives of $L_j(w)$ with respect to w_j (unlike $L_j^{(0)}(w) = L_j(w)$) are even functions of w. Namely,

$$\frac{\partial^i L_j(w(t))}{\partial w_{k_i}\ldots\partial w_{k_2}\partial w_{k_1}} = \frac{\partial^i L_j(-w(t))}{\partial w_{k_i}\ldots\partial w_{k_2}\partial w_{k_1}},\ i \text{ odd}.$$

Since $w''(t) = L(w)$ and $L(0) = 0$, we conclude that $w_j^{(2)}(0) = 0$ for $1 \le j \le n$. We use (66) and (71) to deduce that $w_j^{(4)}(0) = 0$ for $1 \le j \le n$. Then (67) implies that $w_j^{(6)}(0) = 0$ for $1 \le j \le n$. For $m \ge 8$, with m even, assume that each component $w_j^{(m)}$, $j = 1, 2, \cdots, n$, is a finite sum of (not necessarily distinct) terms of the form

$$S_m := \frac{\partial^i L_j(w(t))}{\partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} J_{ODD} J_{EVEN}, \quad (72)$$

where

$$J_{EVEN} = w_{x_1}^{(2c_1)}(t) w_{x_2}^{(2c_2)}(t) \dots w_{x_l}^{(2c_l)}(t)$$
$$c_1, c_2, \dots, c_l \in \mathbb{N} \cup \{0\} \quad (73)$$
$$= \begin{cases} \Pi_{k=1}^{l=2p+1} w_{x_k}^{(2c_k)}(t), \ i \text{ is odd} \\ \\ \Pi_{k=1}^{l=2q} w_{x_k}^{(2c_k)}(t) \text{ or } J_{EVEN} \equiv 1, \ i \text{ is even,} \end{cases}$$

and where

$$J_{ODD} = w_{s_1}^{(2e_1+1)}(t) w_{s_2}^{(2e_2+1)}(t) \dots w_{s_r}^{(2e_r+1)}(t),$$
$$e_1, e_2, \dots, e_r \in \mathbb{N} \cup \{0\}, \ r \text{ even.} \quad (74)$$

Observe that J_{EVEN} is a finite product of even order derivatives of components of $w(t)$. But the number of factors depends on the parity of i. If i is odd, we have an odd number of factors, while if i is even, we have an even number factors. The definition of J_{EVEN}, when combined with (71) and the induction hypothesis of $w_{x_k}^{(2c_k)}(0) = 0$ for all $2c_k \le 2m - 2$, implies that $S_m(0) = 0$.

The goal is to show $w_j^{(m+2)}(t)$ is a finite sum of terms of the form

$$\widehat{S_m^{(2)}} := \frac{\partial^s L_j(w(t))}{\partial w_{k_s} \dots \partial w_{k_2} \partial w_{k_1}} \widehat{J}_{ODD} \widehat{J}_{EVEN}, \quad (75)$$

where

$$\widehat{J}_{EVEN} = w_{x_1}^{(2a_1)}(t) w_{x_2}^{(2a_2)}(t) \dots w_{x_p}^{(2a_p)}(t)$$
$$a_1, a_2, \dots, a_p \in \mathbb{N} \cup \{0\} \quad (76)$$
$$= \begin{cases} \Pi_{k=1}^{l=2b+1} w_{x_k}^{(2a_k)}(t), \ s \text{ is odd} \\ \\ \Pi_{k=1}^{l=2q} w_{x_k}^{(2a_k)}(t) \text{ or } \widehat{J}_{EVEN} \equiv 1, s \text{ is even,} \end{cases}$$

and where

$$\widehat{J}_{ODD} = w_{x_1}^{(2g_1+1)}(t) w_{x_2}^{(2g_2+1)}(t) \dots w_{x_q}^{(2g_q+1)}(t)$$
$$g_1, g_2, \dots, g_q \in \mathbb{N} \cup \{0\}, \ q \text{ even.} \quad (77)$$

Taking the second derivative of $S_m(t)$ with respect to t results in

$$S_m^{(2)}(t) = A_1 + A_2 + A_3, \quad (78)$$

where

$$A_1 := \sum_{k_{i+2}=1}^{n} \sum_{k_{i+1}=1}^{n} \frac{\partial^{i+2} L_j(w(t))}{\partial w_{k_{i+2}} \partial w_{k_{i+1}} \partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}}$$
$$w_{k_{i+2}}^{(1)} w_{k_{i+1}}^{(1)} J_{ODD} J_{EVEN}$$
$$+ \sum_{k_{i+1}=1}^{n} \frac{\partial^{i+1} L_j(w(t))}{\partial w_{k_{i+1}} \partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}}$$
$$w_{k_{i+1}}^{(2)} J_{ODD} J_{EVEN}, \quad (79)$$

where

$$A_2 := 2 \sum_{k_{i+1}=1}^{n} \frac{\partial^{i+1} L_j(w(t))}{\partial w_{k_{i+1}} \partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}}$$
$$w_{k_{i+1}}^{(1)} [J_{ODD}^{(1)} J_{EVEN} + J_{ODD} J_{EVEN}^{(1)}], \quad (80)$$

and where

$$A_3 := \frac{\partial^i L_j(w(t))}{\partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} [J_{ODD} J_{EVEN}]^{(2)}$$
$$= \frac{\partial^i L_j(w(t))}{\partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} \big[J_{ODD}^{(2)} J_{EVEN}$$
$$+ 2 J_{ODD}^{(1)} J_{EVEN}^{(1)} + J_{ODD} J_{EVEN}^{(2)} \big]. \quad (81)$$

Below we list the types of products in that occur in (78).

$$\frac{\partial^{i+2} L_j(w(t))}{\partial w_{k_{i+2}} \partial w_{k_{i+1}} \partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} w_{k_{i+2}}^{(1)} w_{k_{i+1}}^{(1)} J_{ODD} J_{EVEN};$$
$$\frac{\partial^{i+1} L_j(w(t))}{\partial w_{k_{i+1}} \partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} w_{k_{i+1}}^{(2)} J_{ODD} J_{EVEN}; \quad (82)$$

$$\frac{\partial^{i+1} L_j(w(t))}{\partial w_{k_{i+1}} \partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} w_{k_{i+1}}^{(1)} J_{ODD}^{(1)} J_{EVEN};$$
$$\frac{\partial^{i+1} L_j(w(t))}{\partial w_{k_{i+1}} \partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}}] w_{k_{i+1}}^{(1)} J_{ODD} J_{EVEN}^{(1)}; \quad (83)$$

$$\frac{\partial^i L_j(w(t))}{\partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} J_{ODD}^{(2)} J_{EVEN};$$
$$\frac{\partial^i L_j(w(t))}{\partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} J_{ODD}^{(1)} J_{EVEN}^{(1)};$$
$$\frac{\partial^i L_j(w(t))}{\partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} J_{ODD} J_{EVEN}^{(2)}. \quad (84)$$

We proceed to show that every product in (78) has the desired form. Consider the first term (82) and start with

$$B_{11} := \frac{\partial^{i+2} L_j(w(t))}{\partial w_{k_{i+2}} \partial w_{k_{i+1}} \partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}}$$
$$w_{k_{l+2}}^{(1)} w_{k_{l+1}}^{(1)} J_{ODD} J_{EVEN}. \quad (85)$$

Set

$$\widehat{J}_{ODD} := w_{k_{i+2}}^{(1)} w_{k_{i+1}}^{(1)} J_{ODD},$$
$$\widehat{J}_{EVEN} := J_{EVEN}. \quad (86)$$

Recall J_{ODD} has an even number of factors of odd order derivatives, say $r \geq 0$. As a result, $\widehat{J}_{ODD}$ has an even number of odd order derivatives of components of $w(t)$, namely $(r+2)$. Suppose that i is an odd number; consequently, $a = i+2$ is also an odd number. If i is an even number, then $a = i+2$ is also an even number. Then all the conditions in (75) hold. Then we have

$$\frac{\partial^i L_j(w(0))}{\partial w_{k_{i+2}} \partial w_{k_{i+1}} \dots \partial w_{k_2} \partial w_{k_1}} J_{ODD}(0) J_{EVEN}(0) = 0$$

implies

$$\frac{\partial^{i+2} L_j(w(0))}{\partial w_{k_{i+2}} \partial w_{k_{i+1}} \dots \partial w_{k_2} \partial w_{k_1}} \widehat{J}_{ODD}(0) \widehat{J}_{EVEN}(0) = 0.$$

Next we concentrate on the second representative product in (82), namely

$$B_{21} := \frac{\partial^{i+1} L_j(w(t))}{\partial w_{k_{i+1}} \partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} w^{(2)}_{k_{i+1}} J_{ODD} J_{EVEN}. \tag{87}$$

Put

$$\widehat{J}_{ODD} := J_{ODD}, \widehat{J}_{EVEN} := w^{(2)}_{k_{i+1}} J_{EVEN}. \tag{88}$$

If l is the number of factors of even order derivatives in J_{EVEN}, then the number of even order derivatives in $\widehat{J}_{EVEN}$ is $l+1$. Observe that $a = i+1$. Hence (75) holds.

Next we analyze the first representative product in (83), namely

$$B_{12} := \frac{\partial^{i+1} L_j(w(t))}{\partial w_{k_{i+1}} \partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} w^{(1)}_{k_{i+1}} J^{(1)}_{ODD} J_{EVEN}. \tag{89}$$

There is an even number r of factors of odd derivatives in J_{ODD} that make $J^{(1)}_{ODD}$ a sum of r products as follows.

$$\begin{aligned} J^{(1)}_{ODD} &= w^{(2e_1+2)}_{x_1} w^{(2e_2+1)}_{x_2} \dots w^{(2e_r+1)}_{x_r} \\ &+ w^{(2e_1+1)}_{x_1} w^{(2e_2+2)}_{x_2} \dots w^{(2e_r+1)}_{x_r} \\ &+ \dots + w^{(2e_1+1)}_{x_1} w^{(2e_2+1)}_{x_2} \dots w^{(2e_r+2)}_{x_r}. \end{aligned} \tag{90}$$

Every summand in (90) is a product of $(r-1)$ odd order derivatives and exactly one factor is an even order derivative of a certain component of w. Without loss of generality we re-name each product in (90) as:

$$J^{(1)}_{ODDS} := w^{(2u_1+1)}_{x_1} w^{(2u_2+1)}_{x_2} \dots w^{(2u_{r-1}+1)}_{x_{r-1}} w^{(2u_r+2)}_{x_r}. \tag{91}$$

Every summand in (90) is a product of $(r-1)$ odd order derivatives and exactly one factor is an even order derivative of a certain component of w. Without loss of generality we re-name each product in (90) as:

$$J^{(1)}_{ODDS} := w^{(2u_1+1)}_{x_1} w^{(2u_2+1)}_{x_2} \dots w^{(2u_{r-1}+1)}_{x_{r-1}} w^{(2u_r+2)}_{x_r}, \tag{92}$$

and put

$$\begin{aligned} \widehat{J}_{ODD} &= w^{(1)}_{k_{i+1}} w^{(2u_1+1)}_{x_1} w^{(2u_2+1)}_{x_2} \dots w^{(2u_{r-1}+1)}_{x_{r-1}}, \\ \widehat{J}_{EVEN} &= w^{(2u_r+2)}_{x_r} J_{EVEN}. \end{aligned} \tag{93}$$

Evidently, $\widehat{J}_{ODD}$, like J_{ODD}, has the same even number r of factors of odd order derivatives of components of w . Furthermore, $\widehat{J}_{EVEN}$ has $(l+1)$ number of factors of even order derivatives that is one more than J_{EVEN} has. However, $s = i+1$. Thus, B_{12} is of the desired form (75) which yields that $B_{12}(0) = 0$.

We now analyze the second representative product in (83), namely

$$B_{22} := \frac{\partial^{i+1} L_j(w(t))}{\partial w_{k_{i+1}} \partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} w^{(1)}_{k_{i+1}} J_{ODD} J^{(1)}_{EVEN}. \tag{94}$$

First we scrutinize the expression $J^{(1)}_{EVEN}$. If $J_{EVEN} \equiv 1$, then $B_{22} \equiv 0$ and it is obvious that $B_{22}(0) = 0$. If $l \geq 1$, then $J^{(1)}_{EVEN}$ is a summation of l products as follows.

$$\begin{aligned} J^{(1)}_{EVEN} &= w^{(2c_1+1)}_{x_1}(t) w^{(2c_2)}_{x_2}(t) \dots w^{(2c_l)}_{x_l}(t) \\ &+ w^{(2c_1)}_{x_1}(t) w^{(2c_2+1)}_{x_2}(t) \dots w^{(2c_l)}_{x_l}(t) \\ &+ \dots + w^{(2c_1)}_{x_1}(t) w^{(2c_2)}_{x_2}(t) \dots w^{(2c_l+1)}_{x_l}(t), \\ &\quad c_1, c_2, \dots, c_l, l \in \mathbb{N}. \end{aligned} \tag{95}$$

Without loss of generality assume that a representative product in (95) has the form

$$G_{22} := w^{(2c_1+1)}_{x_1}(t) w^{(2c_2)}_{x_2}(t) \dots w^{(2c_l)}_{x_l}(t). \tag{96}$$

Combine $w^{(1)}_{k_{i+1}}$ from (94) with the term $w^{(2c_1+1)}_{x_1}$ in (96) and put

$$\begin{aligned} \widehat{J}_{ODD} &:= w^{(1)}_{k_{i+1}} w^{(2c_1+1)}_{x_1} J_{ODD}, \\ \widehat{J}_{EVEN} &:= w^{(2c_2)}_{x_2}(t) \dots w^{(2c_l)}_{x_l}(t). \end{aligned} \tag{97}$$

Evidently, $\widehat{J}_{ODD}$ has an even number $(r+2)$ of odd order derivatives. Moreover, the number of factors in $\widehat{J}_{EVEN}$ is now $p = l-1$. Again, if i is an even number then $a = i+1$ is an odd number. If i is an odd number then, $a = i+1$ is an even number. Therefore, (75) hold. Hence (75) is of the desired form and $\widehat{S^2_m}(0) = 0$.

It is left now to deal with the terms in (84). We first concentrate a representative product given by the middle term of (84), namely

$$B_{23} := \frac{\partial^i L_j(w(t))}{\partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} J^{(1)}_{ODD} J^{(1)}_{EVEN}. \quad (98)$$

We recall the terms $J^{(1)}_{ODD}, ; J^{(1)}_{EVEN}$ in (90) and (95) respectively. There are rl terms in $J^{(1)}_{ODD} J^{(1)}_{EVEN}$. Each term is products of rl factors. It is necessary to analyze every product. Suppose that i is even. If $J_{EVEN} \equiv 1$ or $l = 0$, then

$$J^{(1)}_{ODD} J^{(1)}_{EVEN} \equiv 0 \Longrightarrow B_{23} \equiv 0 \Longrightarrow \frac{\partial^i L_j(w(0))}{\partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} J^{(1)}_{ODD}(0) J^{(1)}_{EVEN}(0) = 0. \quad (99)$$

It is important to note that $a = i$. Hence, assume that without loss of generality that a representative of one of these rl products has the form

$$J_{ODDS} J_{EVENS} := w^{(2v_1+1)}_{a_1} w^{(2v_2+1)}_{a_2} \dots w^{(2v_{r-1}+1)}_{a_{r-1}} w^{(2v_r+2)}_{a_r} w^{(2c_1+1)}_{x_1}(t) w^{(2c_2)}_{x_2}(t) \dots w^{(2c_l)}_{x_l}(t). \quad (100)$$

Put

$$\widehat{J}_{ODD} := w^{(2v_1+1)}_{a_1} w^{(2v_2+1)}_{a_2} \dots w^{(2v_{r-1}+1)}_{a_{r-1}} w^{(2c_1+1)}_{x_1},$$
$$\widehat{J}_{EVEN} := w^{(2v_r+2)}_{a_r}(t) w^{(2c_2)}_{x_2}(t) \dots w^{(2c_l)}_{z_l}(t). \quad (101)$$

It is not difficult to verify that $\widehat{J}_{ODD}$ and $\widehat{J}_{EVEN}$ have the same number of factors as J_{ODD} and J_{EVEN} respectively. As $a = i$, $\widehat{J}_{ODD}$ and $\widehat{J}_{EVEN}$ in (101) are of the desired form (75).

It is left to concentrate on the representative products in the first and third term in (84). They are

$$B_{13} := \frac{\partial^i L_j(w(t))}{\partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} J^{(2)}_{ODD} J_{EVEN},$$
$$B_{33} := \frac{\partial^i L_j(w(t))}{\partial w_{k_i} \dots \partial w_{k_2} \partial w_{k_1}} J_{ODD} J^{(2)}_{EVEN}. \quad (102)$$

We scrutinize the factors that make up the products in $J_{ODD} J^{(2)}_{EVEN}$ and in $J^{(2)}_{ODD} J_{EVEN}$. Work first with $J_{ODD} J^{(2)}_{EVEN}$. If $l = 0$, then $B_{33} \equiv 0$. If $l = 1$, then $a = i$, where a and i are both odd numbers, and put

$$\widehat{J}_{ODD} = J_{ODD},$$
$$\widehat{J}_{EVEN} = J^{(2)}_{EVEN} = w^{(2c_1+2)}_{x_1}, \quad (103)$$

and observe that (75) is satisfied. Suppose that $l \geq 2$. Then $J^{(2)}_{EVEN}$ is a sum of two types of products. The first type is

$$w^{(2c_j+2)}_{x_j} \prod_{i \neq j, i=1}^{l} w^{(2c_i)}_{x_i}. \quad (104)$$

The we put

$$\widehat{J}_{ODD} = J_{ODD},$$
$$\widehat{J}_{EVEN} = w^{(2c_j+2)}_{x_j} \prod_{i \neq j, i=1}^{l} w^{(2c_i)}_{x_i}. \quad (105)$$

Again we see that $\widehat{J}_{ODD}$ and $\widehat{J}_{EVEN}$ have the same number of factors as J_{ODD} and J_{EVEN} respectively. Not only that, but $a = i$. Hence, (75) is satisfied. The second type of product

$$w^{(2c_j+1)}_{x_j} w^{(2c_k+1)}_{x_k} \prod_{i \neq j,k, i=1}^{l} w^{(2c_i)}_{x_i},$$
$$\prod_{i \neq j,k, i=1}^{l} w^{(2c_i)}_{x_i} :\equiv 1, \quad \text{if } l = 2. \quad (106)$$

Put

$$\widehat{J}_{ODD} := w^{(2c_j+1)}_{x_j} w^{(2c_k+1)}_{x_k} J_{ODD},$$
$$\widehat{J}_{EVEN} := \prod_{i \neq j,k, i=1}^{l} w^{(2c_i)}_{x_i}. \quad (107)$$

Hence, with $a = i$, the parity of a, i, and the number of factors in $\widehat{J}_{EVEN}$ that is $(l-2)$ is the same. If i, l are both even numbers so are $a, l-2$. If i, l are both odd numbers so are $a, l-2$. The number of factors in J_{ODD} is r and the number of factors in $\widehat{J}_{ODD}$ is $(r+2)$ as needed. Then (75) is satisfied.

Finally we conclude the inductive process by concentrating B_{13} of (102). There are two types of products emanating from $J^{(2)}_{ODD}$. Since

$$J_{ODD} = w^{(2e_1+1)}_{a_1}(t) w^{(2e_2+1)}_{a_2}(t) \dots w^{(2e_r+1)}_{a_r}(t),$$
$$e_1, e_2, \dots, e_r \in \mathbb{N} \cup \{0\}, \quad (108)$$

the first type of second derivatives in $J^{(2)}_{ODD}$ involves the following r products.

$$w^{(2e_1+3)}_{a_1}(t) w^{(2e_2+1)}_{a_2}(t) \dots w^{(2e_r+1)}_{a_r}(t),$$
$$w^{(2e_1+1)}_{a_1}(t) w^{(2e_2+3)}_{a_2}(t) \dots w^{(2e_r+1)}_{a_r}(t), \dots,$$
$$w^{(2e_1+1)}_{a_1}(t) w^{(2e_2+1)}_{a_2}(t) \dots w^{(2e_r+3)}_{a_r}(t). \quad (109)$$

Thus, a representative product in $J^{(2)}_{ODD}J_{EVEN}$ will have the form

$$w^{(2e_1+1)}_{a_1}w^{(2e_2+1)}_{a_2}(t)\dots w^{(2e_j+3)}_{a_j}\dots w^{(2e_r+1)}_{a_r}(t)$$
$$w^{(2c_1)}_{k_1}(t)w^{(2c_2)}_{k_2}(t)\dots w^{(2c_l)}_{k_l}(t).$$

Put

$$\widehat{J}_{ODD} := w^{(2e_1+1)}_{a_1}\dots w^{(2e_j+3)}_{a_j}\dots w^{(2e_r+1)}_{a_r}(t),$$
$$\widehat{J}_{EVEN} = J_{EVEN}. \tag{110}$$

Evidently $\widehat{J}_{ODD}$ and $\widehat{J}_{EVEN}$ have the same number of factors as J_{ODD} and J_{EVEN} respectively. As $a = i$, $\widehat{J}_{ODD}$ and $\widehat{J}_{EVEN}$ are of the desired form (75). Without loss of generality, we assume that the second kind of product in $J^{(2)}_{ODD}$ has the form

$$w^{(2e_1+2)}_{a_1}(t)w^{(2e_2+2)}_{a_2}(t)w^{(2e_3+1)}_{a_3}\dots w^{(2e_r+1)}_{a_r}(t). \tag{111}$$

In general, we have the product

$$w^{(2e_j+2)}_{a_j}w^{(2e_k+2)}_{a_k}, \quad \text{if } r=2,$$
$$w^{(2e_j+2)}_{a_j}w^{(2e_k+2)}_{a_k}\prod_{i\neq j,k}^{r} w^{(2e_i+1)}_{a_i}, \quad i=1,2,\dots r, \quad r\geq 3. \tag{112}$$

Then put

$$\widehat{J}_{ODD} \equiv 1, \quad \text{if } r=2,$$
$$\widehat{J}_{EVEN} = w^{(2e_j+2)}_{a_j}w^{(2e_k+2)}_{a_k}J_{EVEN}, \tag{113}$$

else put

$$\widehat{J}_{ODD} := \prod_{i\neq j,k,i=1}^{r} w^{(2e_i+1)}_{a_i}, \ r\geq 3.$$
$$\widehat{J}_{EVEN} := w^{(2e_j+2)}_{a_j}w^{(2e_k+2)}_{a_k}J_{EVEN}. \tag{114}$$

Hence, $\widehat{J}_{ODD}, \widehat{J}_{EVEN}$ are of the desired form (75) and all $S_m(0) = 0$ yield that all $\widehat{S^{(2)}_m}(0) = 0$. Thus, $w^{(m)}(0) = \overrightarrow{0}$ for all even numbers $m \in \mathbb{N}\cup\{0\}$. □

7 Applications $w'' = L(w)$

This section presents applications of Theorem 1. These nonlinear differential systems and equations are autonomous with $L(w)$ independent of w'. Normally, such systems are either conservative non-dissipative systems or systems without damping. The first example was discussed Section 1, namely the N-body problem of celestial mechanics. Let $m_1, m_2, \dots, m_N$ be the masses of the N particles; let $t \in \mathbb{R}$ be the independent variable representing time. For $1 \leq j \leq N$, let $w_j \in \mathbb{R}^3$ be the the position vector of the jth particle and set $w^T = [w^T_1, w^T_2, \cdots, w^T_N] \in \mathbb{R}^{3N}$. Then $w'' = L(w)$, $L^T(w) = [L^T_1(w), L^T_2(w), \cdots, L^T_N(w)] \in \mathbb{R}^{3N}$, where, for each $1 \leq j \leq N$, $L_j(w) \in \mathbb{R}^3$ is defined via

$$w''_k = L_k(w) := \sum_{j\neq k}\frac{Gm_j(w_j - w_k)}{\|w_j - w_k\|^3}$$
$$= \frac{1}{m_k}\nabla_{w_k}V, \ G \text{ gravitational constant}, \tag{115}$$

and

$$V := \sum_{j<k}\frac{m_j m_k}{\|w_j - w_k\|} > 0. \tag{116}$$

Observe that V is the potential energy and $\nabla_{w_k}V$ is the gradient of V with respect to the components of w_k. The initial value problem for (115) is

$$w'' = L(w), \ w(t_0) = w_0, \ w'(t_0) = \eta,$$
$$w_k(t_0) \neq w_j(t_0), k \neq j, \ k, j = 1, 2, \cdots, N. \tag{117}$$

Compare with [1]. If the velocity vector η is chosen to be the zero vector, the initial value problem becomes

$$w'' = L(w), \ w(t_0) = w_0, \ w'(t_0) = \overrightarrow{0},$$
$$w_k(t_0) \neq w_j(t_0), k \neq j, \ k, j = 1, 2, \cdots, N. \tag{118}$$

Theorem 1 provides (118) a continuum of even solutions to (118), each parametrized by the position vector w_0. In particular these solutions satisfy $w[(t-t_0)] \equiv w[-(t-t_0)]$. Planetary motion consists of three time divisions; the past, the present, and the future. Compare with [2]. For $w'(t_0) = \eta = \overrightarrow{0}$, Theorem 1 implies that the positions of the N point masses in the future $t > t_0$ are a perfect reflection of the past $t < t_0$.

Several applications of Theorem 1 are found in [10]. See also, [7]. Many of them are scalar second order differential equations which model model conservative motion via

$$\frac{d^2w}{dt^2} = -f(w), \quad w, \ f(w) \in \mathbb{R}, \tag{119}$$

where $f(0) = 0$, $f(w)$ is strictly increasing for all w, and

$$\int_0^w f(w)dw \to \infty \text{ as } w \to \pm\infty. \tag{120}$$

For instance, $f(w) = b + \exp(w)$, where $b \in \mathbb{R}$ is a constant independent of t.

Another instance is

$$\frac{d^2w}{dt^2} = \begin{cases} -w + c\,\text{sgn}(w) \text{ if } |w| > c \\ 0 \text{ if } |w| \leq c. \end{cases} \tag{121}$$

System (121), with $c \in \mathbb{R}$, models the movement of a particle attached to a fixed point Q on a smooth horizontal plane by an elastic string, since w will denote the displacement from Q.

8 Conclusion

In this paper we discovered conditions which ensure that the solution to the initial value problem $w''(t) = L(w)$, with $w(t_0) = w_0$ and $w'(t_0) = \eta$, is even with respect to $t - t_0$; see Theorem 1. We also found conditions which ensure that the initial value problem $w''(t) = L(w)$, with $w(t_0) = w_0$ and $w'(t_0) = \eta$, is odd with respect to $t - t_0$; see Theorem 2. In the process of formulating Theorem 2, we defined the concepts of even and odd vector functions; see Section4. The proofs of Theorems 1 and 2 are carried out via Taylor series expansions. By applying Theorem 1 to the N-body problem of celestial mechanics implies we find a a continuum of solutions in such that the upcoming positions of the N bodies are an optimal image of their ancient past.

References:

[1] Pollard, H., "Celestial Mechanics", Mathematical Association of America, (1976).

[2] Brumberg, V. A., "Celestial Mechanics: Past, Present, Future", Solar System Research 47, 376–389 (2013).

[3] Hille, E., "Lectures on Ordinary Differential Equations", Addison -Wesley Reading Massachusetts (1969).

[4] Krantz, S. G., Parks, H. R., "A Primer of Real Analytic Functions", Birkhäuser Advanced Texts - Basler Lehrbüecher (Second ed.) Boston: Birkhäuser Verlag, pp. xiv+205 (2002).

[5] Hsieh, Po-Fang., Sibuya, Y., "Basic Theory of Ordinary Differential Equations", Universitext Springer-Verlag New York (1999).

[6] Boyce, E. W., DiPrima, R. C., "Elementary Differential Equations and Boundary Value Problems", John Wiley & Sons N.Y. Last Edition, (1992).

[7] Brauer, F., Nohel, J. A., "The qualitative theory of ordinary differential equations: An introduction", Dover N.Y., (1989).

[8] Diacu, F., Pérez-Chavelab, E., and Santoprete, M., "The Kepler problem with anisotropic perturbations", Journal of Mathematical Physics 46, 072701.1-072701.21 (2005).

[9] Guckenheimer, J., Holmes, P., "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector fields", Springer-Verlag New York (1983).

[10] Jordan, D. W., Smith, P., "Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers", 4th edition Oxford Texts in Applied and Engineering Mathematics (2007).

[11] Perko, L., "Differential Equations and Dynamical Systems", Texts in Applied Mathematics 7 Springer-Verlag New York 3rd edition, (2001).

[12] Diacu, F., Mioc, V., Stoica, C., "Phase-space structure and regularization of Manev-type problems", Nonlinear Analysis 41, 1029–1055 (2000).

[13] Delgado, J., Diacu, F., Lacomba, E. A., Mingarelli, A., Mioc, V., Pérez-Chavela, E., and Stoica, C., "The Global Flow of the Manev Problem", J. Math. Phys 37, 2748–2761 (1996).

[14] Arbogast, L. F. A., "Du calcul des derivations", [On the calculus of derivatives] (in French) Strasbourg: Levrault, pp. xxiii+404 (1800). (Entirely freely available from Google books).

[15] Faà di Bruno, F., Sullo sviluppo delle funzioni, [On the development of the functions], Annali di Scienze Matematiche e Fisiche (in Italian) 6, 479–480 LCCN 06036680. Entirely freely available from Google books (1855).

[16] Encinas, L. H., Masque, J. M., "A Short Proof of the Generalized Faa di Bruno's Formula", Applied Mathematics Letters 16, 975-979 (2003).

Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Harry Gingold created the research methodology and wrote the initial draft manuscript. Ali Abdulhussein created the initial derivative calculations and edited the manuscript. All authors reviewed the mathematical calculations. Jocelyn Quaintance wrote the final draft of the manuscript.

Sources of funding for research presented in a scientific article or scientific article itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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