Novel Approach to Finite Element Simulation of the Timoshenko-Ehrenfest Beam under Static and Dynamic Loads

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Abstract: - The problem of Timoshenko-Ehrenfest beam bending under static and dynamic loading is considered in the finite element formulation. In the static formulation, the shear stiffness matrix is derived from the condition of stationarity of the shear deformation energy, in so doing linear shape functions and the corresponding rows of the bending stiffness matrix are used. As a numerical example, the bending of a simply supported beam made of fiber-reinforced concrete is considered. When constructing a dynamic finite element model, the shear damping matrix is derived from the requirement for the stationary state of its shear deformation energy dissipation. The results of the beam vibrations modeling are presented in comparison with the solution of a similar problem for the Euler-Bernoulli beam.

Key-Words: - Timoshenko-Ehrenfest beam, shear strain, finite element method, stiffness matrix, damping matrix, beam vibrations.

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1 Introduction

A huge variety of constructive forms and physical and mechanical properties of advanced materials utilized in modern engineering practice require the development of adequate and stable calculation models applicable to solving a wide class of problems.

The Euler-Bernoulli beam bending theory is original in its idea and is amazingly simple. It is widely used in solving static and dynamic problems of structural analysis. However, the area of applicability of this theory is limited by the hypothesis of small deformations. Another limitation is correlated to a certain ratio of the geometric parameters of the structure: its length must significantly exceed the characteristic size of the cross-section, so the shear effects can be neglected. However, there is a significant amount of applied problems where shear has to be taken into account, for example, in the analysis of thick plates and sandwich panels, [1]. In [2], it has been shown that when calibrating a nonlocal in time damping model of the Euler-Bernoulli bending elements, it is possible to achieve satisfactory agreement with the results of a numerical experiment only for frame structures with an extended span of the top girder. Calibration of the model for a frame with a relatively short girder does not allow one to achieve the required accuracy of the results.

To expand the limits of applicability of the Euler-Bernoulli beam model, S.P. Timoshenko developed the theory, [3], [4], which makes it possible to consider shear deformations that occur in a prismatic uniform beam during flexural vibrations. The obtained equation of motion was an extension of the Rayleigh equation [5] by considering the rotary inertia. It could be also applied to the calculation of short beam elements. Further, this theory was extended to the analysis of thin-walled beams of open profiles, [6], [7]. The detailed review of the methods based on the Timoshenko theory for beams of solid cross-section and thin-walled beams of open cross-section is presented in [8], [9], [10], [11], [12], [13], [14] and [15], respectively.

In 2019-2021, in order to celebrate the 100th anniversary of the beam theory involving the shear strain and rotary inertia, several monographs [9],

[10], and state-of-the-art articles [16], [17] have been published. It has been emphasized in [17] that "the Russian account of the refined theories of beams, plates and shells was given in [8], and this comprehensive book is still awaiting the English translation", since it involved practically all publications in the field appeared before 1974. The paper [16] contains a wide review of the Timoshenko beam theory history, paying a lot of attention to the effect of the second frequency spectrum. This question is not considered in our paper and will be a subject of further investigations.

Recently a set of papers [17], [18], [19], [20] and the monograph [9] were published as a result of the tremendous work on the history of the refined theories of beams and plates. It has been found historical evidence on Stepan P. Timoshenko collaborator, Austrian-born Dutch physicist Paul Ehrenfest, who at a time temporarily lived in St. Petersburg. It is written in [17], [18] that "for unknown reasons, they did not publish the work on the incorporation of rotary inertia and shear deformation in straight beams in a journal. Rather, Timoshenko included it in his book on the theory of elasticity [21]" (which was first published in Russian in Petrograd in 1916) indicating the contribution of Ehrenfest as a footnote number 2 on page 206: "By us, jointly with Prof. Ehrenfest, also an exact solution was also obtained for the beam with rectangular cross-section".

The authors of this paper completely agree with the statement proposed in [9], [17], [18], [19], [20] that "we are glad to be able to combine these names again, for the sake of truth and historic justice". As a conclusion it has been also emphasized in [17] that "we are sure that the Timoshenko beam theory is not exhausted, and many papers will appear in the future if the previous publication rate can serve as a guide". And it is really the case!

Formalization of computational approaches is a necessary step if these methods are supposed to be used in applied calculations for solving real practical problems. For this purpose, attempts were made to construct finite element models of the dynamic behavior of structural elements based on the Timoshenko-Ehrenfest theory. Thus, various finite element models of the Timoshenko-Ehrenfest beam theory for static analysis were reviewed in [13], wherein dynamic versions of various finite element models were also discussed. It has been suggested in [22] that the simultaneous consideration of damping from bending and shear would make it possible to obtain a picture of oscillations of beam elements that is closer to the real one.

In [23], the shear stiffness matrix is derived by the energy method, in so doing only elements directly related to the angle of rotation of the cross-section at the point of the beam are non-zero in the shear stiffness matrix. This model is free from shear locking and shows a good agreement between the obtained results and the classical Timoshenko theory. It has been shown that accounting for the shear rigidity influences significantly the deflections of beams, the ratio of the length to the height of which does not exceed the magnitude of 10.

In this article, static and dynamic finite element (FE) models of beam bending are developed considering shear strain. The shear stiffness matrix and the damping matrix due to variations in the velocities of shear strains are derived from the requirement for the stationary state of the total mechanical deformation energy of the system in motion.

In the majority of papers devoted to the FE analysis, the damping matrix is presented according to the Rayleigh hypothesis as the linear combination of the mass and stiffness matrices with the corresponding coefficients. The novelty of the research presented below in this paper lies in the fact that the damping matrix is derived from the condition of a minimum of the total energy of the system. As this takes place, the shear angle of the cross-section is determined using a shape function corresponding to the shear force in the elements.

2 Derivation of the Finite-Element Stiffness Matrix of the Timoshenko Beam

When considering shear strains, the static equilibrium equation of a bending beam in a finite element formulation has the following form:

$$(\boldsymbol{K}^b - \boldsymbol{K}^{sh})\,\boldsymbol{V} = \boldsymbol{F}, \tag{1}$$

where $\boldsymbol{V}$ is the nodal displacements vector, $\boldsymbol{K^b}$ is the global bending stiffness matrix of the FE model, $\boldsymbol{K^{sh}}$ is the global shear stiffness matrix of the FE model, and $\boldsymbol{F}$ is the load vector.

Suppose, following to [24], [25], that the bending stiffness matrix of a beam finite element is represented as:

$$\boldsymbol{K_i^b} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}, \tag{2}$$

where E is the Young's modulus, A, I and l are the cross-sectional area, moment of inertia of the cross section, and the length of the finite element, respectively.

Shear stiffness matrix of any i-th finite element of the computational model is obtained from the requirement for the stationary state of its shear deformation energy E_i^{sh}:

$$E_i^{sh} = \frac{1}{2}\int_l Q\gamma dz = \frac{\alpha}{2GA}\int_l Q^2 dz, \quad (3)$$

where Q is the shear force within the finite element, γ is the shear strain of a FE cross section, α is the shear coefficient, G is the shear modulus. i=1,2…n, and n is the number of finite elements of the computational model.

Stiffness of the beam finite element in the direction of transverse displacements of its nodes V_{in} and V_t is characterized by the elements of the first and third rows of the matrix (2). Thus, the vector of shear forces in the beam element is:

$$\begin{bmatrix} Q_{in} \\ Q_t \end{bmatrix} = \frac{6EI}{l^3}\begin{bmatrix} 2 & l & -2 & l \\ -2 & -l & 2 & -l \end{bmatrix}\begin{bmatrix} V_{in} \\ \varphi_{in} \\ V_t \\ \varphi_t \end{bmatrix}, \quad (4)$$

where Q_{in} and Q_t are, respectively, the initial and terminal shear forces in the finite element, φ_{in} and φ_t are the angles of rotation of the nodal cross-sections.

Relationship (4) could be represented in a more compact matrix form:

$$\boldsymbol{Q}_i = \boldsymbol{K}_{sh}^b \, \boldsymbol{V}_i. \quad (5)$$

The finite element method reduces the solution of the problem at the nodal points of the calculation model. When solving the problem in a finite-dimensional space in order to determine the shear force in any section of the beam element, the linear shape functions $[\boldsymbol{N_l}]$ could be used. Then the function of the shear force along the length of the beam element $\widetilde{\boldsymbol{Q}}_i(\xi)$ takes the form:

$$\widetilde{\boldsymbol{Q}}_i(\xi) = [\boldsymbol{N_l}]\,\boldsymbol{Q}_i = [1-\xi \quad \xi]\begin{bmatrix} Q_{in} \\ Q_t \end{bmatrix}, \quad (6)$$

where $\xi \in [0,1]$ is the dimensionless local coordinate measured along the bar axis.

Considering (5) and (6), relationship (3) could be reduced to:

$$E_i^{sh} = \frac{\alpha}{2GA}\int_l \left([\boldsymbol{N_l}]\boldsymbol{K}_{sh}^b \boldsymbol{V}_i\right)^2 dz, \quad (7)$$

Account for (7) in fulfilling the requirement of the stationary state of the potential energy of the bending beam element $\frac{\partial E_i^{sh}}{\partial \boldsymbol{V}_i} = 0,$ adds the second component $\boldsymbol{K}_i^{sh}\boldsymbol{V_i}$ to the left part of the matrix equilibrium equation of the i-th beam finite element

$$\boldsymbol{K_i^b}\,\boldsymbol{V_i} - \boldsymbol{K}_i^{sh}\,\boldsymbol{V_i} = \boldsymbol{F_i}\,, \quad (8)$$

where $\boldsymbol{K}_i^{sh}$ is the shear stiffness matrix of the finite element:

$$\boldsymbol{K}_i^{sh} = \frac{\alpha}{GA}\int_0^1 \left([\boldsymbol{N_l}]\boldsymbol{K}_{sh}^b\right)^T \left([\boldsymbol{N_l}]\boldsymbol{K}_{sh}^b\right) l \, d\xi, \quad (9)$$

or

$$\boldsymbol{K}_i^{sh} = \frac{\alpha}{GA}\int_0^1 \boldsymbol{B}^T\boldsymbol{B} \; l \, d\xi, \quad (10)$$

where

$$\boldsymbol{B} = [\boldsymbol{N_l}]\boldsymbol{K}_{sh}^b =$$

$$= \frac{6EI}{l^3}[1-\xi \quad \xi]\begin{bmatrix} 2 & l & -2 & l \\ -2 & -l & 2 & -l \end{bmatrix} \quad (11)$$

$$= [2-4\xi \quad l-2\xi l \quad -2+4\xi \quad l-2\xi l].$$

Substituting (11) in (10), and therefore in (9) yields:

$$\boldsymbol{K}_i^{sh} = \frac{\alpha l}{GA}\left(\frac{6EI}{l^3}\right)^2 \begin{bmatrix} 4 & 2l & -4 & 2l \\ 2l & l^2 & -2l & l^2 \\ -4 & -2l & 4 & -2l \\ 2l & l^2 & -2l & l^2 \end{bmatrix}. \quad (12)$$

It could be noted that according to the suggested approach, all elements of shear stiffness matrix are non-zero, or, in other words, it is considered that shear strains affects both bending and rotation components, which is different from [23].

3 Comparison of the Results Obtained using the Finite Element Model with the Analytical Solution

The reliability of any numerical model is determined by its compliance with the analytical solutions or experimental studies. With this purpose in mind, let us compare the numerical results provided by the developed Timoshenko-Ehrenfest beam FE model with the analytical solution.

Consider a simply supported beam (Figure 1) of the length L loaded by the uniformly distributed load $q = -10$ kN/m. The beam is made of the material with the following characteristics: $E = 32000$ MPa, $\nu = 0.18$ and has a solid rectangular cross section 0.1x0.2 m.

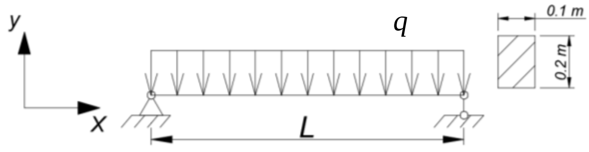

Fig. 1: Scheme of a simply-supported beam

The analytical value of the deflection in the middle of the span of such a beam with due account for shear strains could be determined following [26] as:

$$w_c = \frac{5qL^4}{384EI}\left(1+\frac{48\alpha EI}{5GAL^2}\right) = \frac{5qL^4}{384EI}\left(1+\frac{8\alpha(1+\nu)h^2}{5L^2}\right). \quad (13)$$

The shear coefficient α could be determined in several ways, [19], [26], [27], [28], [29], [30]. It could be calculated as the ratio of the value of the shear stress on the neutral axis to the average value of the shear stress in the cross section, [25]. For a solid rectangular section, the value of the shear coefficient calculated by this method is $\alpha = 1.5$.

It should be noted that in contemporary literature the shear coefficient is placed in the denominator of Eq. (3), and therefore it is considered that it takes on magnitudes less than unit. However, S.P. Timoshenko himself used the different notation [26], as is in expression (13), and considered shear coefficient larger than unit. In our research we used the second variant of notation.

The second way is to determine the shear coefficient by integrating the equations of the elasticity theory, [27]. At the same time, it is noted that the obtained solution is valid in cases where the shear force diagram of the structure does not have sharp jumps. With this approach, for solid rectangular sections, the shear coefficient α is:

$$\alpha = \frac{12+11\nu}{10(1+\nu)}. \quad (14)$$

For the considered example, relationship (14) provides $\alpha = 1.185$.

It is noted in [26] that in the general case when determining beam deflections with due account for shear strains accordingly to the principle of virtual work, it is preferable to use the shape factor under shear f_{sh} instead of α, where f_{sh} is determined by the expression:

$$f_{sh} = \frac{A}{l^2}\int_A \frac{S^2}{b^2}dA, \quad (15)$$

where S is the statical moment of the cross-sectional area, and b is the cross-section width.

For a solid rectangular cross section f_{sh} is 1.2, [26].

In [27], the reliability of the shear coefficients determined via various approaches was studied experimentally. Simply supported steel beams of rectangular and circular cross sections with different ratios of the characteristic size of the cross section to the span length were tested. The ratio of experimentally determined natural frequencies to frequencies calculated theoretically using the Euler-Bernoulli and Timoshenko hypotheses was used as a reliability criterion. It is shown that for beams of rectangular cross section, the relations "experiment - Euler-Bernoulli beam" and "Timoshenko beam - Euler-Bernoulli beam" are the best match for the coefficient proposed in [29]:

$$\alpha = \frac{6+5\nu}{5+5\nu}. \quad (16)$$

In this case, the shear coefficient for the beam under consideration is $\alpha = 1.17$. This value was used in further calculations, since the finite element model proposed would be used below for dynamic modeling.

The values of deflections in the middle of the beam under consideration are given in Table 1 for various lengths of its span. The table compares the values obtained analytically for the Euler-Bernoulli beam and the Timoshenko beam according to formula (13) with the deflection calculated by the finite element method using the considered model.

From Table 1 it is seen that with an increase in the beam span, the influence of shear strains on the calculated deflection value decreases. When the ratio of the beam span to its height exceeds 10, account for shear strains is almost ineffective on the calculated deflection value, what is consistent with the results obtained in [23].

Further, when modeling the dynamic behavior of the Timoshenko-Ehrenfest beam, a beam with a span 2 m would be considered.

Table 1. Comparison of the results obtained using a considered model with the analytical solutions

Beam Span, m	Midspan deflection w_c, mm			Deflection increment when considering shear, %
	Analytical solution		FE model considering shear	
	Euler-Bernoulli beam	Timoshenko beam		
2	-0.976	-1	-1	2.46
4	-15.62	-15.71	-15.72	0.64
8	-250	-250.3	-250.4	0.12

4 Simulation of Forced Vibrations of a Timoshenko-Ehrenfest Beam

The equation of motion of a deformable system with due account for shear strains in determining elastic and damping forces within the algorithm of the finite element method takes the form:

$$\boldsymbol{M}\ddot{\boldsymbol{V}}_{i+1} + (\boldsymbol{D^b} + \boldsymbol{D^{sh}})\dot{\boldsymbol{V}}_{i+1} + (\boldsymbol{K^b} - \boldsymbol{K^{sh}})\boldsymbol{V_{i+1}} = \boldsymbol{F_{i+1}}, \quad (17)$$

where i is the number of the current time step, $\boldsymbol{D^b}$ is the bending damping matrix [30], $\boldsymbol{D^{sh}}$ is the shear damping matrix obtained from the requirement of the stationary state of the total strain energy of the vibrating structure, $\boldsymbol{M}$ is the mass matrix, and $\boldsymbol{F}(t)$ is the load vector. The dots denote the time derivatives of the deflection vector $\boldsymbol{V}(t)$.

Frequently, in the finite element approach the Rayleigh damping is used, and the damping matrix is presented as a linear combination of mass and stiffness matrices, [31]. In this paper, to take damping due to shear deformation into account, the damping matrix is divided to the bending damping matrix $\boldsymbol{D^b}$ [32] and the shear damping matrix $\boldsymbol{D^{sh}}$ obtained from the requirement of the stationary state of the total strain energy of the vibrating structure,

The damping matrix of any (i-th) finite element of the computational model is obtained from the stationarity requirement of the dissipative function Δ_i^{sh}:

$$\Delta_i^{sh} = \frac{1}{2}\int_l \chi^{sh}\dot{\gamma}^2 A dz, \quad (18)$$

where $\dot{\gamma}$ is the shear strain rate, and χ^{sh} is the shear viscosity coefficient (in Pa · s).

By analogy with the derivation of the stiffness matrix, it could be obtained:

$$\Delta_i^{sh} = \frac{1}{2}\chi^{sh} A\alpha \int_l \left([\boldsymbol{N_l}]\boldsymbol{K}_{sh}^b \dot{V}_i\right)^2 dz. \quad (19)$$

Then the shear damping matrix takes the form:

$$\boldsymbol{D}_i^{sh} = \chi^{sh} A\alpha \int_0^1 \left([\boldsymbol{N_l}]\boldsymbol{K}_{sh}^b\right)^T \left([\boldsymbol{N_l}]\boldsymbol{K}_{sh}^b\right) l d\xi, \quad (20)$$

or

$$\boldsymbol{D}_i^{sh} = \frac{\chi^{sh} k_d^2}{Al}\begin{bmatrix} 4 & 2l & -4 & 2l \\ 2l & l^2 & -2l & l^2 \\ -4 & -2l & 4 & -2l \\ 2l & l^2 & -2l & l^2 \end{bmatrix}, \quad (21)$$

where k_d is the coefficient defined by the following formula:

$$k_d = \frac{6\alpha(1+\nu)I}{l^2}. \quad (22)$$

The solution of the equation of motion was carried out according to the implicit scheme, [24]. The equation for step-by-step calculation of the deflection vector has the following form:

$$\boldsymbol{Z}\boldsymbol{V_{i+1}} = \boldsymbol{F_{i+1}} + \boldsymbol{M}\ddot{\boldsymbol{V}}_{\boldsymbol{i}} + \boldsymbol{Q_1}\dot{\boldsymbol{V}}_{\boldsymbol{i}} + \boldsymbol{Q_2}\boldsymbol{V_i}, \quad (23)$$

where

$$\boldsymbol{Z} = \frac{2}{\Delta t^2}\boldsymbol{M} + \frac{1}{\Delta t}\left(\boldsymbol{D^b} + \boldsymbol{D^{sh}}\right) + \left(\boldsymbol{K^b} - \boldsymbol{K^{sh}}\right),$$
$$\boldsymbol{Q_1} = \frac{2}{\Delta t}\boldsymbol{M},$$
$$\boldsymbol{Q_2} = \frac{1}{\Delta t}\left(\frac{2}{\Delta t}\boldsymbol{M} + (\boldsymbol{D^b} + \boldsymbol{D^{sh}})\right). \quad (24)$$

Velocities and accelerations at the (i +1) time step are determined as:

$$\dot{\boldsymbol{V}}_{\boldsymbol{i+1}} = \frac{\boldsymbol{V_{i+1}} - \boldsymbol{V_i}}{\Delta t},$$
$$\ddot{\boldsymbol{V}}_{\boldsymbol{i+1}} = \frac{2}{\Delta t^2}\left(\boldsymbol{V_{i+1}} - \boldsymbol{V_i} - \dot{\boldsymbol{V}}_{\boldsymbol{i}}\Delta t\right) - \ddot{\boldsymbol{V}}_{\boldsymbol{i}}. \quad (25)$$

The initial conditions are $\boldsymbol{V_1} = \dot{\boldsymbol{V}}_{\boldsymbol{1}} = \ddot{\boldsymbol{V}}_{\boldsymbol{1}} = 0$.

Figure 2 shows the time histories of deflections of the beam middle section obtained using a finite element model neglecting shear deformations and the Timoshenko-Ehrenfest beam model proposed above.

The results have been obtained for beams with a span of 2m (Figure 2a), 4m (Figure 2b), and 8m (Figure 2c), with the same cross section and the same material as considered above (Figure 1). The beams are loaded with the instantly applied uniformly distributed load $q = -10$ kN/m. The relative damping coefficient of the material ξ was accepted as 3%, and the density of the material is $\rho = 2550$ kg/m^3.

In this research, bending and shear viscosity coefficients χ^b and χ^{sh} have been taken equal to each other, what is acceptable to evaluate a consideration of the influence of shear deformations on the dynamic simulation results. Those coefficients could be calculated as:

$$\chi^b = \chi^{sh} = E\frac{2\xi}{\omega_0}, \quad (26)$$

where ω_0 is the lowest natural frequency of the beam.

However, for solution of applied problems the damping capacity for bending and shear dynamic deformation has to be determined experimentally.

It can be seen from Figure 2 that the damping of oscillatory processes results in the decrease of dynamic deflections to their quasi-static values which are equal to those presented in Table 1. Account for shear results in the changes of the beam amplitudes and vibration frequency obtained via the numerical simulation, which is consistent with the experimental data from [29], in so doing with the increase in the beam span, the effect of considering the shear deformations on the results of the analysis of the oscillatory process decreases.

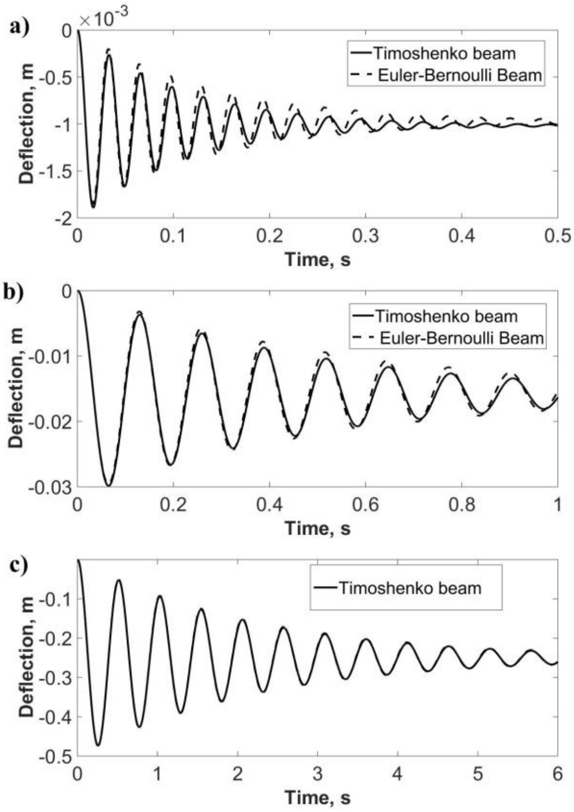

Fig. 2: Deflection time histories for the different beam models a) 2 m beam b) 4 m beam c) 8 m beam

5 Conclusion

In the majority of papers devoted to the FE analysis, the damping matrix is presented according to the Rayleigh hypothesis as the linear combination of the mass and stiffness matrices with the corresponding coefficients. The novelty of the research presented in this paper lies in the fact that the damping matrix is derived from the condition of a minimum of the total energy of the system. As this takes place, the shear angle of the cross-section is determined using a shape function corresponding to the shear force in the elements.

The application of the developed finite element models of a bending beam, considering the shear deformations, is preferable for static and dynamic calculation of the short and thick beams, as well as for the frame structures and trusses, containing short elements. Comparison of the numerical results via the proposed numerical model obtained from the requirement of the stationary state of the total deformation energy with those from analytical solutions according to the Timoshenko-Ehrenfest theory showed its acceptable accuracy. The results of the dynamic analysis of beam structures confirm the expediency of considering the internal friction in the structural material due to shear deformations. The implementation of a model that takes shear deformations into account as part of the finite element method algorithm makes it possible to use it in applied calculations of relatively complex structural systems. So further research is supposed to be focused on the expanding the proposed model for the frame structures, and also on the improvement of stability and efficiency of the numerical algorithm.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Author 1 planned the scheme and developed the mathematical modelling; Author 2 initiated and supervised the project; Author 3 examined the theory validation and analysed the empirical results. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

The authors have no conflicts of interest to declare.