Mechanics of Indentation for an Elastic Half-Space by Punches

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Abstract: - The dynamic and static problems of finding stress components under four moving punches ($a \leq |X| \leq b, c \leq |X| \leq d$), located close to each other over an elastic half-plane ($Y = 0$), are solved. Employing the Fourier integral transform, the problem is reduced to a set of integral equations in both cases. Using the Hilbert transform technique, the integral equations are solved to obtain the stress and displacement components. Finally, exact expressions for the stress components under the punches and the normal displacement component in the region outside the punches have been derived. Numerical results showing the variations in stress intensity factors (SIF) at the punch ends, and the absolute value of torque applied over the contact regions with different values of the parameters used in the problems have been presented in the form of graphs.

Key-Words: - Punch, Fourier transform, Integral equations, Hilbert transform technique, Stress intensity factor, Elastic half-space, Moving punches.

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1 Introduction

Contact problems are common in engineering and material sciences. Several punch problems in elastodynamics have been discussed in detail in the books by [1], [2]. [3], solved the moving punch problems with the aid of the complex variable method. [4], considered the in-plane problem of indentation of an elastic layer over a rigid base by moving punches. [5], considered the problems of anti-plane indentation of an elastic layer by a pair of moving punches. [6], solved the same problems with two pairs of moving punches, [7], [8]. Structures formed as a solid foundation inserted under the ground, are examples of large-scale indentation. A problem of indentation of an elastic half-plane by a wedge shaped punch, taking into account the frictional and tangential-displacements effects has been solved by [9]. [10], [11], [12], [13], solved a few problems on symmetric, non-symmetric, and frictionless indentations employing the method of homogeneous function. [14], have reviewed recent works on the inclusions of infinite, and semi-infinite spaces under different forms of loading. Different methods of solving the problems of one or more inclusions have been presented in their work. [15], solved an axisymmetric problem of unilateral frictionless indentation of a semi-infinite elastic medium. [16], studied the problem of moving punch over a layer under plane strain conditions. The problem has been solved numerically after converting it into a Cauchy-type singular integral equation. [17], solved the case of indentation of an orthotropic layer on an isotropic half-plane by a steadily moving rigid cylindrical punch with the aid of the Fourier integral transform technique, Galilean transformation, and Gauss-Jacobi integral formula. [18], have solved the problem of a moving punch associated with the normal component of displacement as an even degree polynomial. They studied the effect of the degree of the polynomial function on SIF at the punch end and on the torque over the contact region. The residual stresses are identified by instrumented elliptical indentation, and inverse analysis by [19]. The simulation for JKR-type adhesive contact of rough elliptical punches is done by [20], using Boundary Element Method (BEM), and the Fast Fourier Transform. The problem of adhesion of a cylindrical punch with varied elastic properties is studied by [21]. [22], studied the Sevostianov-Kachanov approximation for incremental compliance of non-elliptical contacts. [23], has solved the Boussinesq and Cattaneo problems for an ellipsoidal power-law indenter analytically, [24]. The problem of axisymmetric contact of two different power-law graded elastic bodies has been solved by [25], after reducing it to an integral equation with two different kernels. Small contacts are also occurring in different engineering fields. Such cases of penetration are taken into account for studying the distribution of stress under the indenter. These studies have applications in designing geotechnical and footing engineering and in indentation tests for characterizing matters, [26], [27], [28], [29], [30], [31], [32].

In this paper, the integral transform technique has been utilized to solve both dynamic and static problems for finding stress components under four punches ($a \leq |X| \leq b, c \leq |X| \leq d$), located close to each other on an elastic half-space ($Y = 0$).The Fourier integral transformation has been employed to transform the problem into a set of five integral equations. The use of the Hilbert transform technique has been made for solving the integral equations, and the stress component under the punches and the normal displacement component in the region outside the punches have been derived. Finally, SIF at the punch ends and torque over the contact regions are calculated; and the variations in those with velocity of punch for different values of the contact region of the inner pair of the punches are presented graphically.

2 Formulation and Solution of Problem I

We consider an isotropic (Figure 1), homogeneous, and semi-infinite medium given by $Y \leq 0$, which is stress-free, and no displacement is prescribed on any part of the boundary $Y = 0$. Thus, the initial conditions are zero. Four punches, located at $Y = 0,\ a \leq |X| \leq b, c \leq |X| \leq d$ are assumed to be moving at a constant speed, V along positive direction of the X axis. The equations of motion (neglecting body force) in terms of displacements are

$$(\lambda + 2\mu)\left[u_{,XX} + v_{,XY}\right] + \mu\left[u_{,YY} - v_{,XY}\right] = \rho u_{,TT},$$
$$(\lambda + 2\mu)\left[u_{,XY} + v_{,YY}\right] + \mu\left[v_{,XX} - u_{,XY}\right] = \rho u_{,TT}, \quad \text{(1a,b)}$$

where u, v denote the displacement components along the X and Y axes respectively, λ, μ are Lame's constants, and $u_{,X}$ denotes the partial derivative with respect to X. We introduce the Galilean transformation

$$x = X - Vt, y = Y, z = Z \text{ and } t = T$$

with x, y and z as the moving coordinate system as shown in Figure 1. Therefore, the deformation about the y–axis will remain symmetric throughout the motion.

Following the detailed analysis of the method, [6], [7] and using the boundary conditions (due to symmetry about $x = 0$)

$$v(x, 0) = -p, \quad \text{for } a \leq x \leq b, c \leq x \leq d$$
$$\sigma_{yy}(x, 0) = 0, \quad \text{for } 0 < x < a, b < x < c, x > d$$
$$\sigma_{xy}(x, 0) = 0, \quad \text{for } x > 0, \quad \text{(2a-c)}$$

we obtain the following integral equations in $A(\xi)$

$$\int_0^\infty A(\xi)\cos(x\xi)d\xi = \frac{\pi(2 - M^2)}{2M^2}p,$$
$$\text{for } a \leq x \leq b, c \leq x \leq d \quad \text{(3a,b)}$$

$$\int_0^\infty \xi A(\xi)\cos(x\xi)d\xi = 0,$$
$$\text{for } 0 < x < a,\ b < x < c, x > d \quad \text{(4a-c)}$$

where p is a constant and $M = \frac{V}{c_2}$.

To solve these equations, we assume

$$A(\xi) = \frac{1}{\xi}\int_a^b \frac{h(u^2)}{u}(1 - \cos(u\xi))du + \frac{1}{\xi}\int_c^d \frac{g(s^2)}{s}(1 - \cos(s\xi))ds \quad (5)$$

Next, using (5) in the expressions given by (4a-c), we note that the expression of $A(\xi)$ is independent of the choice of the unknown functions

$h(u^2)$, $g(s^2)$. Using (5) in the equations given by (3a,b) , we get

$$\int_a^b \frac{h(u^2)}{u} \ln\left|1-\frac{u^2}{x^2}\right| du + \int_c^d \frac{g(s^2)}{s} \ln\left|1-\frac{s^2}{x^2}\right| ds = \frac{\pi(2-M^2)}{2M^2} p, \quad \text{for } a \le x \le b, c \le x \le d \qquad (6)$$

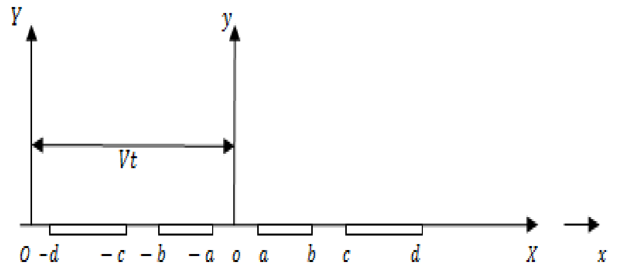

Fig. 1: Geometry and coordinate system

On differentiation with respect to x, this gives:

$$\int_a^b \frac{u h(u^2)}{u^2-x^2} du + \int_c^d \frac{s g(s^2)}{s^2-x^2} ds = 0, \text{ for } a \le x \le b, \; c \le x \le d. \qquad (7)$$

Using the method of solutions of the above integral equations, [6], [7], we get:

$$h(u^2) = \sqrt{\frac{d^2-a^2}{c^2-a^2}} \frac{C_1\sqrt{c^2-u^2}}{\sqrt{(u^2-a^2)(b^2-u^2)(d^2-u^2)}} - \frac{C_2\sqrt{u^2-a^2}}{\sqrt{(b^2-u^2)(c^2-u^2)(d^2-u^2)}},$$

$$g(u^2) = \sqrt{\frac{d^2-a^2}{c^2-a^2}} \frac{C_1\sqrt{u^2-c^2}}{\sqrt{(u^2-a^2)(u^2-b^2)(d^2-u^2)}} + \frac{C_2\sqrt{u^2-a^2}}{\sqrt{(u^2-b^2)(u^2-c^2)(d^2-u^2)}}. \qquad (8a,b)$$

Multiplying the equation (6) by $\frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}$, and integrating taking limits $x = a$ to $x = b$ and multiplying the same equation by $\frac{x}{\sqrt{(x^2-c^2)(d^2-x^2)}}$, and integrating taking limits $x = c$ to $x = d$, we get a system of linear equations involving C_1, C_2. Solving those, we get:

$$C_1 = \frac{\pi(2-M^2)p}{2M^2} \frac{L_2-L_4}{L_2L_3-L_1L_4} \sqrt{\frac{c^2-a^2}{d^2-a^2}},$$

$$C_2 = \frac{\pi(2-M^2)p}{2M^2} \frac{L_1-L_3}{L_2L_3-L_1L_4}. \qquad (9a,b)$$

where

$$L_1 = H(b)\int_a^b I(t)dt + \int_c^d H(t)I(t)dt,$$

$$L_2 = H(b)\int_a^b J(t)dt - \int_c^d H(t)J(t)dt,$$

$$L_3 = \int_a^b G(t)I(t)dt + G(c)\int_c^d I(t)dt,$$

$$L_4 = \int_a^b G(t)J(t)dt - G(c)\int_c^d J(t)dt \qquad (10a\text{-}d)$$

with

$$I(t) = \frac{\sqrt{c^2-t^2}}{t\sqrt{(t^2-a^2)(b^2-t^2)(d^2-t^2)}},$$

$$J(t) = \frac{\sqrt{t^2-a^2}}{t\sqrt{(b^2-t^2)(c^2-t^2)(d^2-t^2)}},$$

$$H(t) = \ln\left|\frac{\sqrt{t^2-a^2}-\sqrt{t^2-b^2}}{a+b}\right|,$$

and

$$G(t) = \ln\left|\frac{\sqrt{c^2-t^2}-\sqrt{d^2-t^2}}{c+d}\right|. \qquad (11a\text{-}d)$$

The normal component of stress in the plane of the punches and just below those are given as:

$$[\sigma_{yy}(x,0)]_{a<x<b} = \frac{\mu R(M) h(x^2)}{x(2-M^2)\sqrt{(1-M^2k^2)}},$$

$$[\sigma_{yy}(x,0)]_{c<x<d} = \frac{\mu R(M) g(x^2)}{x(2-M^2)\sqrt{(1-M^2k^2)}}. \qquad (12a,b)$$

with

$$R(M) = 4\sqrt{(1-M^2k^2)}\sqrt{(1-M^2)} - (M^2-2)^2,$$

and $k = \frac{c_2}{c_1}$ and the normal displacement component outside the contact regions can now shown to be given by:

$$v(x,0) = \frac{-p}{2(L_2L_3-L_1L_4)} \left[\left\{(L_2-L_4)\int_a^b I(t)\ln\left|1-\frac{t^2}{x^2}\right| dt + \int_c^d I(t)\ln\left|1-\frac{t^2}{x^2}\right| dt\right\} - (L_1-L_3)\left\{\int_a^b J(t)\ln\left|1-\frac{t^2}{x^2}\right| dt - \int_c^d J(t)\ln\left|1-\frac{t^2}{x^2}\right| dt\right\}\right], \text{ for } 0<x<a, b<x<c, x>d \quad (13a\text{-}c)$$

It is to be mentioned that the stress component depends on the velocity of the moving punch. However, in the plane of the punches, the normal displacement component is independent of that. Further, we note from equation (13) that the normal displacement component decreases gradually as x tends to infinity.

The SIF at the ends of the punches is defined by:

$$N_1 = \lim_{x\to a+} \sqrt{2(x-a)}\,[\sigma_{yy}(x,0)]_{a<x<b},$$

$$N_2 = \lim_{x\to b-} \sqrt{2(b-x)}\,[\sigma_{yy}(x,0)]_{a<x<b},$$

$$N_3 = \lim_{x\to c+} \sqrt{2(x-c)}\,[\sigma_{yy}(x,0)]_{c<x<d},$$

$$N_4 = \lim_{x\to d-} \sqrt{2(d-x)}\,[\sigma_{yy}(x,0)]_{c<x<d}$$

and using the expressions (12a,b) those are found as:

$$N_1 = \frac{\mu R(M)}{(2-M^2)\sqrt{1-M^2k^2}} \frac{C_1}{a^{\frac{3}{2}}\sqrt{b^2-a^2}},$$

$$N_2 = \frac{\mu R(M)}{(2-M^2)\sqrt{1-M^2k^2}} \frac{1}{b^{\frac{3}{2}}\sqrt{d^2-b^2}}\left[C_1\sqrt{\frac{(d^2-a^2)(c^2-b^2)}{(c^2-a^2)(b^2-a^2)}} - C_2\sqrt{\frac{b^2-a^2}{c^2-b^2}}\right],$$

$$N_3 = \frac{\mu R(M)}{(2-M^2)\sqrt{1-M^2k^2}} \frac{C_2\sqrt{c^2-a^2}}{c^{\frac{3}{2}}\sqrt{(c^2-b^2)(d^2-c^2)}},$$

and

$$N_4 = \frac{\mu R(M)}{(2-M^2)\sqrt{1-M^2k^2}} \frac{1}{d^{\frac{3}{2}}\sqrt{d^2-b^2}}\left[C_1\sqrt{\frac{d^2-c^2}{c^2-a^2}} + C_2\sqrt{\frac{d^2-a^2}{d^2-c^2}}\right]. \quad \text{(14a-d)}$$

The torque applied over the contact regions are given by:
$T_1 = -\int_a^b \sigma_{yy}(x,0)dx$, and $T_2 = -\int_c^d \sigma_{yy}(x,0)dx$, and using (8a,b) and (12a,b) in the above expressions, we obtain:

$$T_1 = \frac{\mu R(M)}{(2-M^2)\sqrt{1-M^2k^2}} \frac{1}{\sqrt{(d^2-b^2)(c^2-a^2)}}\left[\left(C_1\sqrt{\frac{d^2-a^2}{c^2-a^2}} + C_2\frac{c^2}{a^2}\right)\left(1-\frac{a^2}{b^2}\right)\Pi\left(\frac{c^2(b^2-a^2)}{b^2(c^2-a^2)}, r\right) + C_2\left(1-\frac{a^2}{c^2}\right)F(r)\right],$$

$$T_2 = \frac{-\mu R(M)c^2}{(2-M^2)\sqrt{1-M^2k^2}} \frac{1}{a^2\sqrt{(d^2-b^2)(c^2-a^2)}}\left[\left(C_1\sqrt{\frac{d^2-a^2}{c^2-a^2}} + C_2\frac{c^2}{a^2}\right)\left(1-\frac{a^2}{d^2}\right)\Pi\left(\frac{a^2(d^2-c^2)}{d^2(c^2-a^2)}, r\right) - C_1\left(1-\frac{a^2}{c^2}\right)\sqrt{\frac{d^2-a^2}{c^2-a^2}}F(r)\right], \quad \text{(15a,b)}$$

where $F(r) = F\left(\frac{\pi}{2}, r\right)$ and $\Pi(\phi, r) = \Pi\left(\frac{\pi}{2}, \phi, r\right)$ are elliptic integrals of first and third kind respectively, and $r = \sqrt{\frac{(d^2-c^2)(b^2-a^2)}{(d^2-b^2)(c^2-a^2)}}$.

2.1 Problem II

In this section, we consider a semi-infinite homogeneous, isotropic material with punches located at $Y = 0,\ a \le |X| \le b, c \le |X| \le d$ .The equations of equilibrium (neglecting body force), in terms of displacements are:

$$(\lambda + 2\mu)[u_{,XX} + v_{,XY}] + \mu[u_{,YY} - v_{,XY}] = 0,$$
$$(\lambda + 2\mu)[u_{,XY} + v_{,YY}] + \mu[v_{,XX} - u_{,XY}] = 0, \quad \text{(16a,b)}$$

where u, v, λ, μ have already been defined earlier.

Using the same technique as adopted in the problem 1, and using the boundary conditions (on account of symmetry about $X = 0$)

$$v(x,0) = -q, \text{ for } a \le x \le b, c \le x \le d$$
$$\sigma_{yy}(x,0) = 0, \text{ for } 0 < x < a,\ b < x < c, x > d$$
$$\sigma_{xy}(x,0) = 0, \text{ for } x > 0, \quad \text{(17a-e)}$$

we obtain the following integral equations in $D(\xi)$

$$\int_0^\infty D(\xi)\cos(x\xi)d\xi = -\frac{\pi}{2}q, \quad \text{for } a \le x \le b, c \le x \le d \quad \text{(18a,b)}$$

$$\int_0^\infty \xi D(\xi)\cos(x\xi)d\xi = 0, \quad \text{for } 0 < x < a,\ b < x < c, x > d. \quad \text{(19a-c)}$$

where q is a constant.

It is to be mentioned that the above integral equations cannot be obtained using the corresponding expressions of the dynamic problem given by the equations (3a,b) by setting $M = 0$. Now, employing the same method as adopted in the problem I, one can easily obtain

$$[\sigma_{yy}(X,0)]_{a<X<b} = -\frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)}\frac{h(X^2)}{X},$$
$$[\sigma_{yy}(X,0)]_{c<X<d} = -\frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)}\frac{g(X^2)}{X}, \quad \text{(20a,b)}$$

where $h(X^2), g(X^2)$ are same as given by (8a,b) with the exception that C_1, C_2 are to be replaced by D_1, D_2 ,which satisfy (9a,b), when $\frac{(2-M^2)p}{M^2}$ is replaced by $-q$.

The SIF at the ends of the punches are found as:

$$N_1 = -\frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)}\frac{D_1}{a^{\frac{3}{2}}\sqrt{b^2-a^2}},$$

$$N_2 = -\frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)}\frac{1}{b^{\frac{3}{2}}\sqrt{d^2-b^2}}\left[D_1\sqrt{\frac{(d^2-a^2)(c^2-b^2)}{(c^2-a^2)(b^2-a^2)}} - D_2\sqrt{\frac{b^2-a^2}{c^2-b^2}}\right],$$

$$N_3 = -\frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)}\frac{D_2\sqrt{c^2-a^2}}{c^{\frac{3}{2}}\sqrt{(c^2-b^2)(d^2-c^2)}},$$

and

$$N_4 = -\frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)}\frac{1}{d^{\frac{3}{2}}\sqrt{d^2-b^2}}\left[D_1\sqrt{\frac{d^2-c^2}{c^2-a^2}} + D_2\sqrt{\frac{d^2-a^2}{d^2-c^2}}\right]. \quad \text{(21a-d)}$$

Using the results:
$T_1 = -\int_a^b \sigma_{yy}(x,0)dx$, and $T_2 = -\int_c^d \sigma_{yy}(x,0)dx$, the torque applied over the contact regions are found as:

$$T_1 = \frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)} \frac{1}{\sqrt{(d^2-b^2)(c^2-a^2)}} \left[\left(D_1 \sqrt{\frac{d^2-a^2}{c^2-a^2}} + D_2 \frac{a^2}{c^2} \right) \left(\frac{c^2}{b^2} - 1 \right) \Pi \left(\frac{c^2(b^2-a^2)}{b^2(c^2-a^2)}, r \right) - D_2 \left(1 - \frac{a^2}{c^2} \right) F(r) \right],$$

$$T_2 = \frac{-2\mu(\lambda+\mu)}{(\lambda+2\mu)} \frac{c^2}{a^2\sqrt{(d^2-b^2)(c^2-a^2)}} \left[\left(D_1 \sqrt{\frac{d^2-a^2}{c^2-a^2}} + D_2 \frac{a^2}{c^2} \right) \left(\frac{d^2}{a^2} - 1 \right) \Pi \left(\frac{a^2(d^2-c^2)}{d^2(c^2-a^2)}, r \right) + D_1 \left(1 - \frac{a^2}{c^2} \right) \sqrt{\frac{d^2-a^2}{c^2-a^2}} F(r) \right]. \quad (22a,b)$$

3 Numerical Discussions

In this section, numerical results for problem I for the values of the parameters associated with the problem have been presented graphically. Computations of SIF and torque applied over the contact regions have been done taking $\lambda = \mu$ and $\frac{b}{p} = 10, \frac{c}{p} = 20, \frac{d}{p} = 30$, i.e., taking the variations in the position of inner edge of the first pair of punches only. As the velocity of the punch is less than Rayleigh wave velocity, we take the value of $M \leq 0.9194$. From Figures 2, it is clear that the value of the SIF at the inner edge of the first pair of punches decreases, while the same at all other edges increases with the increase in the values of $\frac{a}{p}$. In other words, as the length of the contact region of the inner pair of punches reduces keeping length of the other punches fixed, the value of the SIF decreases. From these graphs, we note that the values of the SIF gradually decrease as $\frac{V}{c_2}$ increases and tend to zero as $\frac{V}{c_2}$ tends to 0.9194, as expected. Variations in absolute values of the torque applied over the contact regions of both pairs of punches have been presented in Figure 3. It is seen from the figures that variations in absolute value of the torque over both the contact regions are of similar character, i.e., the value of absolute value of the torque over both the contact regions decreases with the increase in the values of $\frac{V}{c_2}$, and tends to 0 as $\frac{V}{c_2}$ tends to 0.9194. Nevertheless, the magnitude of the absolute value of the torque over the outer pair of punches is significantly higher than that over the inner pair of punches.

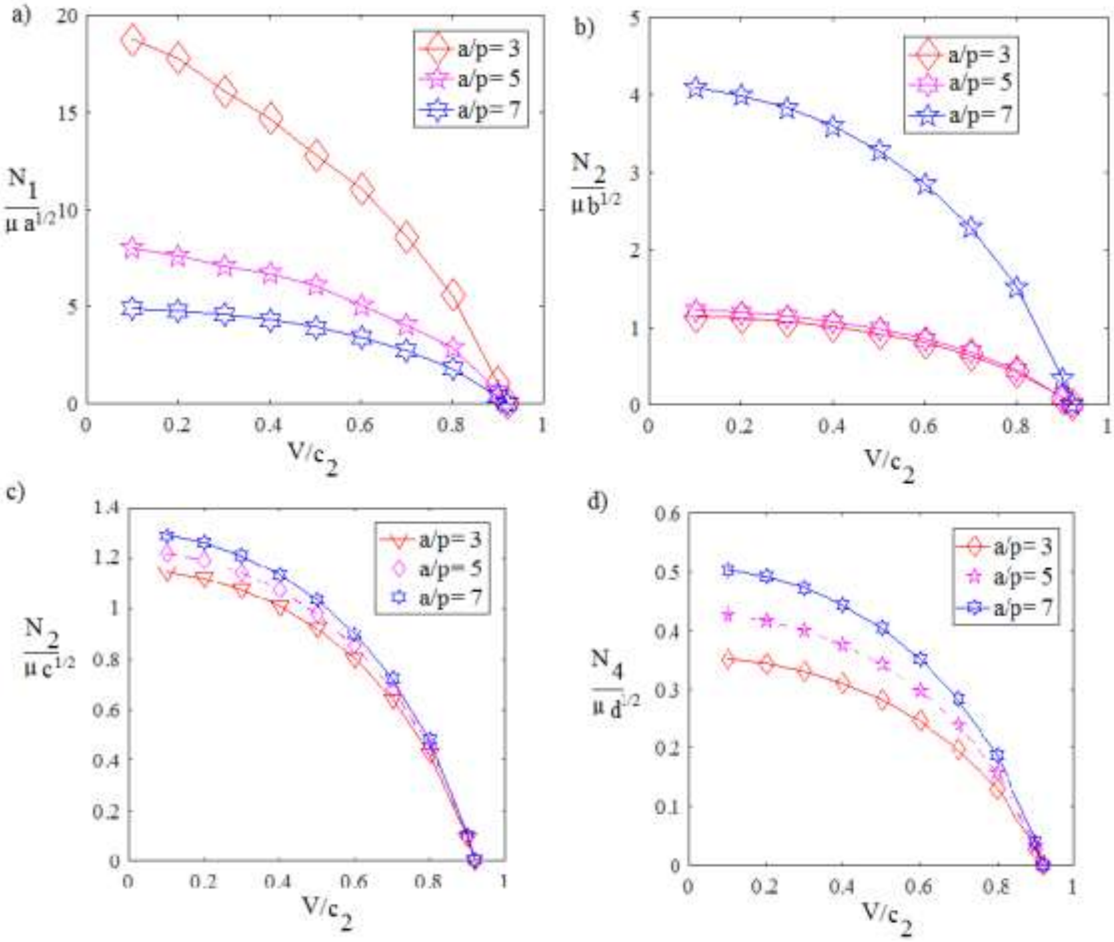

Fig. 2: Variations of S.I.F. with velocity of the punches

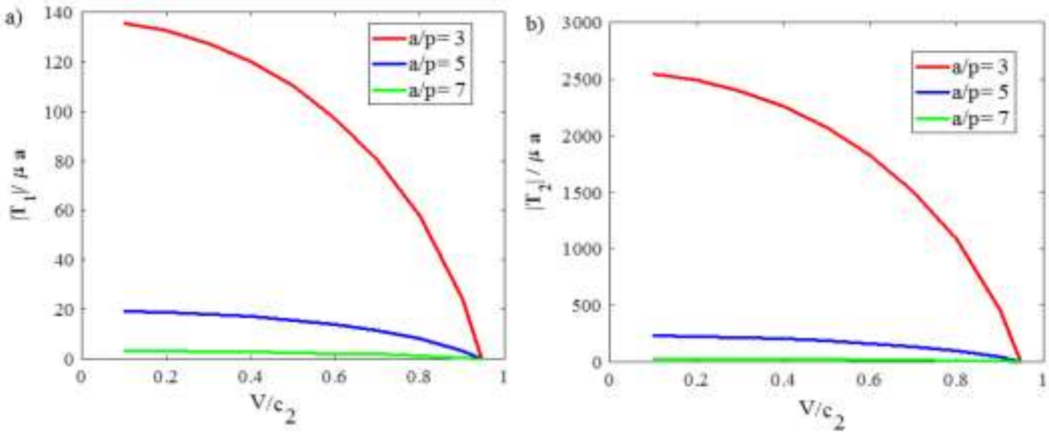

Fig. 3: Variations of torque with velocity of the punches

4 Conclusion

In this work, both the dynamic and static problems of finding stress components under four punches, located closely to each other over an elastic half space, and moving steadily in a fixed direction are solved. The integral transform technique has been employed to study the behavior of SIF at the punch ends, and the absolute value of the torque over the contact regions with the variation in the parameters involved in the problem. As the shape of an indenter may vary, and the rigidity of the semi-infinite medium over which the indenter acts is not always uniform, it is reasonable to assume the normal component of the displacement along the contact regions as a function. However, to avoid complexity in mathematical calculations, we have assumed that the frictions less indenters are flat. The effect of the variation in the velocity of the punches on the SIF at the punch ends, and on the absolute value of the torque over the contact regions have been studied. How the reduction in length of the inner pair of punches, keeping their outer edge fixed, affects the

SIF at the punch ends and the torque over the contact regions, have also been presented graphically. Outcomes of this work are obtained considering the indentation of a half-plane by four rigid flat indenters, but this form of indentation problem with an even number of punches (6,8,…) can be solved. In those cases, the function $A(\xi)$ (see equation 5) is to be changed by adding more similar integrals, and the method of computation will be more complex.

Future scope of the study: After going through this study, one can study the impact of the variations in length of the contact regions of the outer pair of punches, by considering the variations in the values of $\frac{c}{p}$ or $\frac{d}{p}$. The effect of alteration of the distance between the pair of punches, by considering the variations in the values of $\frac{b}{p}$ or $\frac{c}{p}$, on the SIF and absolute value of the torque can also be studied. This work also suggests that a similar problem of indentations can be extended with any even number of punches, and can also be solved with different shapes of indenter instead of the flat one.

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