Derivation of Key Characteristics of Columns in Frames with Sway

JOSTEIN HELLESLAND
Mechanics Section, Department of Mathematics
University of Oslo, Oslo
NORWAY

Abstract: - The paper paper examines columns of elastic frames that are able to displace laterally under axial load. Columns on the same level (story) of such frames may contribute to lateral support (stability) or require lateral support (supported columns). In the former case, maximum moments will occur at column ends, but may occur between column ends in the latter case. The column response, including the formation of moments and shears, is often complicated, often calling for approximate analysis methods and information that presently may not be readily available. Main attention of the paper is paid to providing results that will be helpful in providing, in a rather simple manner, improved understanding of column mechanics, and that may be helpful also as a supplement to full second-order analyses. Towards this goal, the major objective of the paper is threefold: (i) to identify characteristic points, or behavioral"landmarks" in the axial load-moment solution space, through the study of rotationally restrained elastic columns and two-column panels with sidesway; (ii) to derive simple, novel, closed-form expressions for these, thereby providing a tool simplifying the establishment of the variation of end and maximum moments versus axial load; (iii) to identify to what extent isolated (single) column analyses may adequately represent horizontally interacting columns, and potential unwinding, in frames.

Key-Words: - Structural mechanics, elasticity, second-order analysis, stability, framed column behavior, sidesway.

Received: March 25, 2023. Revised: August 27, 2023. Accepted: September 28, 2023. Published: November 14, 2023.

1 Introduction

1.1 General

This paper examines columns of elastic frames that are able to displace laterally under load. Columns in such a frame, or story of a frame, may contribute to the lateral stiffness (support), or they are may require lateral support by the others to maintain lateral story stability. Whether a column falls into one or the other category depends on its relative axial load level. In cases with no transverse load between column ends, the maximum moment will occur at the end of a supporting column, but may occur between the ends of a supported column. Computer programs that account for geometric nonlinear (second-order) effects are available for examining such problems. The complexity of such analysis methods often obscures the understanding of the mechanics of individual members in a frame. In this regard, simplified methods are often helpful as a complement.

There is an extensive literature on approximate maximum design moment calculations for columns of reinforced concrete or steel frames. Much of such work has been incorporated into relevant design codes such as, [1], [2], [3], [4]. In such provisions, columns are either considered braced (laterally supported) or unbraced (laterally supporting).

In contrast, easy-to-apply methods allowing prediction of not only more correct maximum moments, but also column end moments (of importance for connected foundations and beams, etc.) and shears have, apart from some isolated cases, such as, [5], seemingly been almost non-existent until more recently, [6].

Approximate methods are generally based on analysis of columns isolated from the frames of which they are a part by incorporating appropriate end restraints at column ends. These are supposed to reflect the interaction with the surrounding structure. Well known, and rationally justified restraint assumptions for regular frames, are generally adopted also for more irregular frames. Although important, the author has not come across readily available literature on this subject.

Based on engineering practice and research over several years, the author has felt a need for investigating this end restraint issue, and has has long recognized the potential usefulness of approaches that would allow rather easily the establishment of the often complex moment formation in columns for any load level. This has been the driving force behind the present paper, which is based on a more detailed report, [7].

1.2 Object and Scope

The major objective of the work is threefold. Based on a study over the full range of axial loads of the mechanics of the response of rotationally restrained elastic columns and two-column panels with sidesway, it has been to:

(i) identify characteristic points, or behavioral "landmarks", in the axial load-moment solution space;

(ii) establish simple, novel, closed-form expressions defining such characteristic points, and thereby allowing easy establishment of the overall variation of end and maximum moments versus axial load;

(iii) identify to what extent isolated (single) column analyses may adequately represent horizontally interacting columns, and potential unwinding, in frames.

Apart from being of use in providing improved understanding of column mechanics, and the teaching of the subject, such results may also be a helpful supplement to full, second-order structural analyses, in the assessment of the often complicated column response.

Towards the goals of the paper, second-order (small rotations) theory is developed in sufficient detail to allow the establishment of closed-form expressions of key characteristics, and to obtain exact moment-load relationships for isolated, restrained columns. The scope is limited to linear elastic columns with uniform axial load and uniform sectional stiffness.

2 Supporting an Supported Sway Columns

A brief review of the overall frame sidesway mechanics is in order to provide the proper context for subsequent sections. Fig. 1(a) shows a laterally loaded frame that is not braced against sidesway, i.e, it is free to sway. In the absence of axial forces in the columns, the lateral load ($H = 4$) gives rise to a first-order sway displacement Δ_0, equal at all column tops when axial beam deformations are neglected. The lateral load is resisted by column shears (V_0) that are proportional to the relative lateral stiffness of the columns. If this stiffness is the same in all columns, first-order shears of $V_0 = H/4 = 1.0$ result in each of the columns

Under the additional action of axial forces, caused by vertical loads on the frame, Fig. 1(b), the displacement increases to $\Delta = B_s\Delta_0$, where B_s is a sway displacement magnification factor that reflects the global (story, or system) second-order effects of axial loads. Had the individual columns, having different axial load levels in the example, been free to sway independently of each other, some would sway more (the more flexible ones) and some less (the stiffer ones) than the the overall frame. Since the columns are interconnected and forced to act together, a redistribution of shears will take place from the more flexible

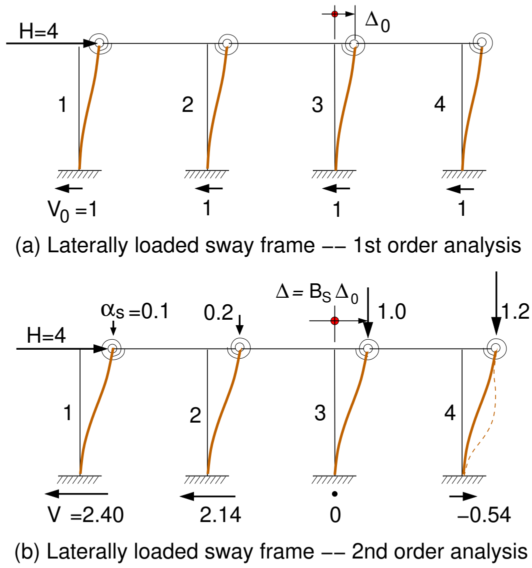

Figure 1: Effects of axial loads on column shears in multibay sway frame: (a) First and (b) second-order analysis ($B_s = 2.67$).

ones (column 3 and 4 in the example) to the stiffer ones (column 1 and 2).

For the vertical loads in the figure, given nondimensionally in terms of the free-sway load index $\alpha_s (= N/N_{cs})$ of each column, the redistributed shears are those shown in Fig. 1(b) (calculated using Eq. (31), to be presented later).

The distinction between (a) laterally "supporting sway columns" ("bracing columns", "leaned-to columns"), as illustrated by column 1 and 2 in the example, and (b) "supported columns" ("braced", "leaning columns"), illustrated by column 4, is found to be useful. A supported column may be visualized as one being allowed to undergo a given sidesway and then braced against further sway by the rest of the frame.

3 Second-order Analysis

3.1 General

Theoretical results relevant for multibay frames with sidesway will be obtained both for panel frames and for individual frame columns considered in isolation from the frame. Two-column panel frames (Fig. 2) allow horizontal interaction between columns to be considered, whereas a simpler model consisting of a single column (Fig. 3(a)) allow explicit expressions to be derived for a number of column characteristics. The latter is of considerable interest, and is pursued below using an analysis tailor-made for this purpose. Theoretical results for panel frames, and limitations of single column models to represent such frames, will be presented and discussed later.

The single column considered is shown in Fig. 3(a). It is laterally loaded (shear V), is initially

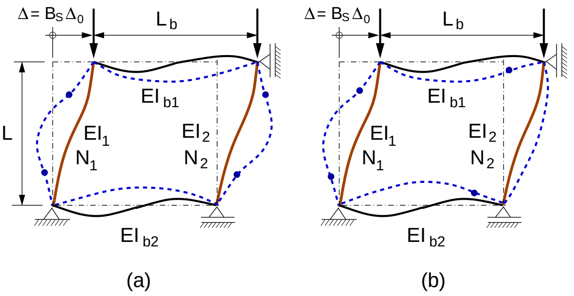

Figure 2: Initial and final deflection shapes of panel frames: (a) Col. 2 is slightly stiffer than Col. 1; (b) Col. 2 is significantly stiffer than Col. 1

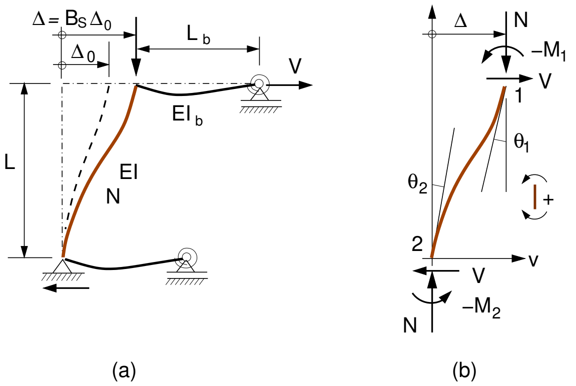

Figure 3: (a) Single column model and (b) sign convention.

straight, and has length L, uniform axial force N and section stiffness EI. Only in-plane bending is considered.

The rotational end restraints, reflecting the rotational interaction with the rest of the frame of which the column may be a part, can conveniently be represented by stiffness coefficients k_1 and k_2 (equal to the moment required to give a unit rotation), respectively, at ends 1 (top) and 2 (bottom). Nondimensionally, they can be defined by

$$\kappa_j = \frac{k_j}{(EI/L)} \qquad j = 1, 2 \qquad (1)$$

at an end j, or by other similar factors, such as the well known G factors, [1], [2]. Unlike the κ factors above, that are relative stiffness factors, the G factors are relative, *scaled* flexibility factors. In a generalized form they can be defined, [8], by

$$G_j = b_o \frac{(EI/L)}{k_j} \quad (= \frac{b_o}{\kappa_j}) \qquad j = 1, 2 \qquad (2)$$

where b_o is simply a reference (datum) factor by which the relative restraint flexibilities are scaled. Common datum values are $b_o = 6$ and $b_o = 2$ for regular unbraced and braced frames, respectively. These values are tacitly implied in ACI, [1], and AISC, [2], and similarly in Eurocodes, [3], [4]. For additional discussion, see for instance, [9], [10], [11].

It should be noted, if not otherwise stated, that the reference factor $b_o = 6$ is used throughout this paper in presentations in terms of G factors!

When the rotational restraints at a column end j is provided by beams, the restraint stiffness at that end can be written

$$k_j = k_b = \sum bEI_b/L_b \qquad j = 1, 2 \qquad (3)$$

Here, the summation is over all beams that are restraining the considered column end (joint), and b is the bending stiffness coefficient. For beams with negligible axial forces, it is typically $b = 3$ for beams pinned at the far end, $b = 4$ for beams fixed at the far end and $b = 2$ for beams in symmetrical single curvature bending. For beams in antisymmetrical double curvature bending, $b = 6$.

Computations in the present paper are carried out for the one-level cases defined in the figures above. However, derived results are also applicable to columns in multistory frames provided appropriate rotational restraints are assigned to the column ends. An approximate approach for multistory frames is to consider the "vertical interaction" between columns meeting at a joint by assigning a fraction of the total restraining beam stiffness sum k_b at a joint to the columns at the joint in proportion to their EI/L values. This assumption leads to

$$k_j = k_b \cdot (EI/L)/(\sum EI/L) \qquad j = 1, 2 \qquad (4)$$

where EI/L in the numerator is for the considered column, and the summation is over all columns meeting at the considered joint. This approach is generally adopted by codes (e.g., [1], [2], [3], [4]) and is widely covered in the literature (see for instance, [8], [10]). It is in particular found to be acceptable for multilevel frames with stiff beams, [5], [12]. By substituting Eq. (4) into Eq. (2), the conventional G factor expression, such as in [1] and [2], is obtained for $b/b_o = 1$.

3.2 Basic Slope-Deflection Equations

For completeness, the major elements of the second-order (small rotations) analysis for the single column in Fig. 3 are developed in sufficient detail to allow the establishment of closed form solutions in suitable forms for use and discussion later in the paper.

The adopted sign convention is shown in Fig. 3(b). Clockwise acting moments and rotations are defined to be positive. Consequently, the moments shown in Fig. 3, acting counter-clockwise, are negative ($-M_1$ and $-M_2$).

The member stiffness relationship, expressed by the slope-deflection equations for a compression member, can be derived from the differential equation ($M = -EIv''$), and given by the matrix equation below:

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} C & S & -(C+S)/L \\ S & C & -(C+S)/L \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \Delta \end{bmatrix} \quad (5)$$

Here, the upper case C and S are the socalled stability functions defined by

$$C = \frac{c^2}{c^2 - s^2} \quad ; \quad S = \frac{s^2}{c^2 - s^2} \quad (6)$$

where

$$c = \frac{1}{pL^2}\left(1 - \frac{pL}{\tan pL}\right) ; \; s = \frac{1}{pL^2}\left(\frac{pL}{\sin pL} - 1\right) \quad (7)$$

and

$$pL = L\sqrt{N/EI} \quad (= \pi\sqrt{\alpha_E}) \quad (8)$$

In first-order theory, when the axial force is zero, C and S take on the familiar values of 4 and 2, respectively. The formulation above was used in [10], [11]. The lower case c and s functions are flexibility factors; they are equal to 1/3rd of the corresponding functions (ϕ and ψ) in [13]. Alternative formulations are also available ([9], [14], [15], [16]).

3.3 End Moments and Shears

From moment equilibrium at the joints ($M_1 + k_1\theta_1 = 0$, $M_2 + k_2\theta_2 = 0$), the end rotations can be expressed by the end moments (e.g., $\theta_1 = -M_1/k_1$). Substitution of these into Eq. (5) gives

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} -(C+S)\,EI\Delta/L^2 \\ -(C+S)\,EI\Delta/L^2 \end{bmatrix} \quad (9)$$

from which the total end moments according to second-order theory can be solved for and expressed by

$$M_1 = -\frac{(C+S)\,(b_{22} - b_{12})}{b_{11}b_{22} - b_{12}b_{21}} \cdot \frac{EI\Delta}{L^2} \quad (10)$$

$$M_2 = -\frac{(C+S)\,(b_{11} - b_{21})}{b_{11}b_{22} - b_{12}b_{21}} \cdot \frac{EI\Delta}{L^2} \quad (11)$$

Here, with the rotational stiffness of the end restraints taken as $k = 6(EI/L)/G$ (Eq. (2) with $b_o = 6$),

$$b_{11} = 1 + \frac{CG_1}{6}; b_{12} = \frac{SG_2}{6}; b_{21} = \frac{SG_1}{6}; b_{22} = 1 + \frac{CG_2}{6}$$

When the determinant $D = b_{11}b_{22} - b_{12}b_{21}$ approaches zero, the end moments M_1 and M_2 approach infinity. This happens when the axial load approaches the critical (instability) load. This limit is independent of the lateral displacement Δ. The shear, found from statics of the displaced column ($VL = -(M_1 + M_2 + N\,\Delta)$), can in non-dimensional form be written

$$\frac{V}{EI\,\Delta/L^3} = -\frac{M_1 + M_2}{EI\,\Delta/L^2} - (pL)^2 \quad (12)$$

First-order results. It is convenient to present results in terms of first-order results, found by substituting $C = 4$ and $S = 2$ into Eqs. (10) and (11). The resulting sway magnified first-order moments become

$$B_s M_{02} = -\frac{6(G_1 + 3)}{2(G_1 + G_2) + G_1\,G_2 + 3} \cdot \frac{EI\Delta}{L^2} \quad (13)$$

and

$$B_s M_{01} = B_s M_{02}\,\frac{G_2 + 3}{G_1 + 3} \quad (14)$$

and the corresponding sway magnified first-order shear (from statics),

$$B_s V_0 = -B_s\,(M_{01} + M_{02})/L \quad (15)$$

3.4 Location and Magnitude of Maximum Column Moment

The moment expression of the axially loaded column subjected to the total end moments M_1 and M_2, can be found from the differential equation ($M = -EIv''$) and expressed in the familiar form given by

$$M(x) = M_2\Big[\frac{\mu_t - \cos pL}{\sin pL}\sin px + \cos px\Big] \quad (16)$$

where

$$\mu_t = -M_1/M_2 \quad (17)$$

This end moment ratio, between total end moments (from the second-order analysis), becomes positive when the end moments at the two ends act in opposite directions ("single curvature"). From the maximum moment condition, $dM(x)/dx = 0$, the location x_m of the maximum moment between ends can be obtained as

$$\tan px_m = \frac{\mu_t - \cos pL}{\sin pL} \quad (18)$$

Noting that the period of Eq. (18) is π, px_m can be expressed by

$$px_m = \arctan\frac{\mu_t - \cos pL}{\sin pL} + n\pi \qquad n = 0, 1, .. \quad (19)$$

If, for any value of n (in practice n=0 or 1 for a restrained column, and n=0 for a pin-ended column),

$$0 < px_m < pL \quad (20)$$

then maximum moment is located between ends. If no px_m value can be found that satisfies Eq. (20), the

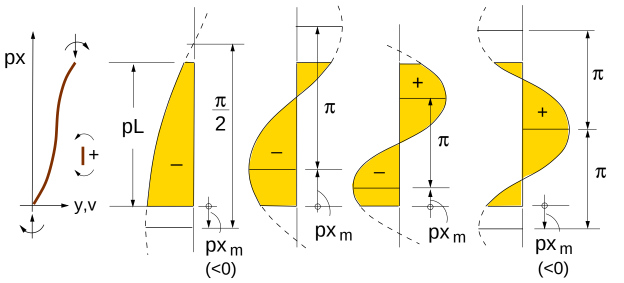

Figure 4: Moment diagrams and location of maximum moments at different axial load levels.

maximum moment is located at one column end and is then equal to the largest end moment.

Fig. 4 shows different moment diagrams that may result. The column length is represented by $pL = \pi\sqrt{\alpha_E}$. In the first case, with $px_m < 0$, the maximum moment is at the lower end. In the other three cases, maximum moments occur between ends. In the third case, two maxima occur, one on either side of the column. This may occur when end moments are equal or nearly equal and acting in opposite directions.

By substituting Eq. (18) into Eq. (16), the maximum moment can be solved for and expressed as

$$M_{max} = B_{tmax} M_2 \tag{21}$$

where

$$B_{tmax} = \frac{1}{\cos px_m} \qquad for \quad \mu_t > \cos pL \tag{22}$$

$$B_{tmax} = 1.0 \qquad for \quad \mu_t \leq \cos pL \tag{23}$$

The maximum moment may become positive or negative, as seen in Fig. 4, but in practice it is the absolute value that is of interest. For a restrained column with $N > N_E$ (i.e., $\alpha_E > 1$), maximum moment will always be between ends (see also Eq. (35)).

Apart from the more general criteria set up above for maximum moment to form between ends of the column, the above derivation follows that given in [10] and [11] for pin-ended members, for which $\alpha_E \leq 1$, and consequently $N \leq N_E$.

3.5 Presentation of Moments and Shears

The analysis presented above is used to compute moments and shears in a column for different axial loads and restraints as a function of the total relative lateral displacement

$$\Delta = B_s \Delta_0 \tag{24}$$

between column ends, where B_s is a sway displacement magnifier that represents the global (system, story) second-order effects ("$N\Delta$" effects). This magnifier is a function of the properties and axial loads of all members in the frame system. Other notations for B_s are, for instance, δ in ACI, [1], and B_2 in AISC, [2]. A number of available B_s expressions in the literature, such as, [5], [17], [18], [19], are reviewed in [5], which also includes a general B_s proposal that cover both lateral supporting and lateral supported columns.

The local second-order ("$N\delta$") effect in an individual column with a specified sidesway $\Delta = B_s\Delta_0$, can be quantified by the ratios of second-order analysis results for a given axial load N and results for $N = 0$ (first-order). These ratios, or local magnifiers, are denoted B_{max}, B_1, B_2 and B_v for maximum moment, end moments at column end 1 and 2, and shear force, respectively.

Thus, for frames with sway due to lateral and axial loading only, total load effects can be expressed by

$$M_{max} = B_{max} B_s M_{02} \tag{25}$$

$$M_1 = B_1 B_s M_{01} \tag{26}$$

$$M_2 = B_2 B_s M_{02} \tag{27}$$

$$V = B_v B_s V_0 \tag{28}$$

Since the load effects above are all explicit functions of the lateral displacement Δ (Eqs. (10), (11), (12)), Δ will cancel out in the ratios. As a result, the local B coefficients are independent of the magnitude of Δ. The B coefficients depend only on the end restraints, which is uniquely defined by the first-order moment gradient (Eq. (14)), and on the axial load level defined for instance by the nondimensional load parameters α_E or α_s (Eq. (29)).

M_{max} is expressed as a function of the first-order moment at end 2, which per definition is taken as the end with the larger first-order end moment (absolute value). Note that B_{max} above is different from the B_{tmax} in Eq. (21) (and there applied to M_2 obtained from the second-order analysis).

3.6 Pseudo-critical Loads and Load Indices

In describing the response of "framed" columns, that may be part of a larger frame, specific, individual member axial load indices are often helpful when considering the columns in isolation, and will be used in this paper. For an elastic, framed member of length L, uniform axial load N and uniform sectional bending stiffness EI along the length, the relevant load indices are primarily those defined below by the expressions in Eq. (29), which are given in terms of the critical loads defined by Eq. (30).

$$\alpha_{cr} = \frac{N}{N_{cr}} \; ; \; \alpha_s = \frac{N}{N_{cs}} \; ; \; \alpha_b = \frac{N}{N_{cb}} \; ; \; \alpha_E = \frac{N}{N_E} \tag{29}$$

where

$$N_{cr} = \frac{N_E}{\beta^2} \; ; \; N_{cs} = \frac{N_E}{\beta_s^2} \; ; \; N_{cb} = \frac{N_E}{\beta_b^2} \; ; \; N_E = \frac{\pi^2 EI}{L^2} \quad (30)$$

Here,
– N_E is the socalled Euler load (critical load of a pin-ended column), which is a convenient reference load parameter in several contexts; β is the effective length factor, taken equal to β_s and β_b for the free-to-sway and the braced case, respectively.
– N_{cr} is the critical load of the column at system (frame, story) instability; N_{cs} and N_{cb} are critical loads of the column when considered in isolation from the rest of the frame, but with rotational restraints (to be assumed) corresponding to the respective bending mode (sway or braced) of the real frame; N_{cs} is determined for the column considered completely free to sway, and N_{cb} for the column considered fully braced. Except when the frame consists of a single column, these are strictly pseudo-critical loads. They can be very useful in column characterization and discussion.

As defined, the load indices are interrelated. For instance, $\alpha_E = \alpha_s/\beta_s^2$ or $\alpha_E = \alpha_b/\beta_b^2$. The free-sway critical load is defined by $\alpha_s = 1.0$ or, for instance, $\alpha_E = 1/\beta_s^2$, and the braced critical load by $\alpha_b = 1.0$ or $\alpha_E = 1/\beta_b^2$.

4 Single Column Response – Stationary Restraints

General. Typical moment and shear response versus increasing axial load of columns with stationary end restraints, computed with the presented second-order analysis, are shown in Fig.5, Fig.6 and Fig.7. The columns are illustrated by the inserts in the up--per left hand parts of the figures. The column top is-initially displaced laterally by an amount $B_s\Delta_0$ (sway-magnified first-order displacement) and then kept constant at this value (in a real case, by the action of the overall frame of which the column may be considered isolated from).

The results in Fig. 5 are typical for columns with moderately flexible end restraints, whereas Fig. 6 are more representative for a column with stiff restraints at one end, and Fig. 7 for columns with very stiff, and nearly equal, rotational restraints at both ends.

The moments and shear are shown nondimensionally in terms of the respective B factors in Eqs. (25) to (28), and the axial forces are given nondimensionally in terms of load indices α_s and α_E (Eq. (29)). All results in the figure are given in terms of B_sM_{02}. Therefore, the B_1 response is represented by $B_1 \cdot M_{01}/M_{02}$

The curves labeled $B_{2,secant}$ are secant approximations to the end moment curves discussed later

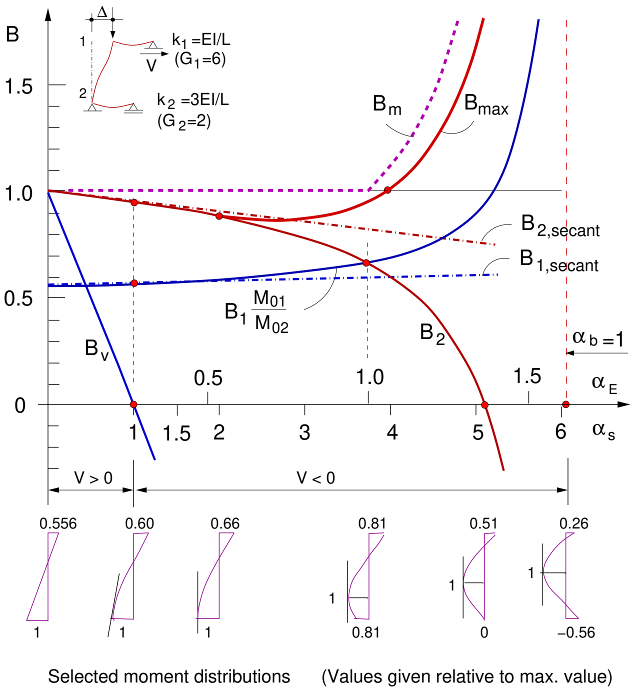

Figure 5: Moments and shear versus axial load level in column with relative flexible end restraints ($\beta_s = 1.932$, $\beta_b = 0.785$, $\alpha_E = 0.268\alpha_s$)

(Section 5.9). The curves labeled B_m, are maximum moment predictions in accordance with present procedures in major design codes, and will also be discussed later (Section 7).

It is seen in the figures that both moments and shears approach infinity (in either the positive or negative direction) as the axial load approaches the elastic critical load N_{cb} of the fully braced column. The corresponding load index at that instance is $\alpha_b = 1$, or $\alpha_E = 1/{\beta_b}^2$. For an elastic column, the braced critical load is independent of whether the column is fully braced at zero or at a non-zero end displacement.

If the considered column was a part of a larger, unbraced frame, it should be emphasized that system instability of the frame may be reached for an axial load (N_{cr}) in the column that is well below its fully braced value N_{cb}. Also, lateral sway magnification (B_s) will most often, but not always, reach large, unacceptable values well before this load level.

Shears. In the absence of axial load, a shear of $V = B_sV_0$ is required to give a sway column a displacement $\Delta = B_s\Delta_0$. When an axial load is applied, the overturning moment is increased. This tends to further increase the sway (if it had been free to sway). To maintain the displacement at the original specified value, V (or B_v) must decrease, as seen in the figures. The variation of B_v is almost linear up to and some-

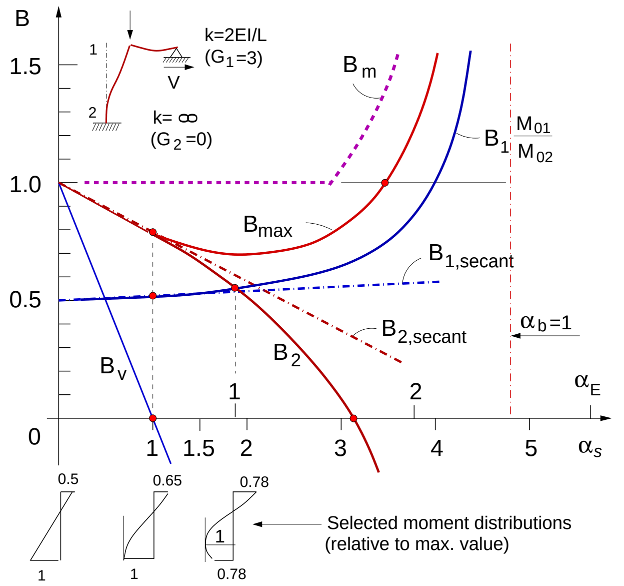

Figure 6: Moments and shear versus axial load level in column with stiff end restraints, $G_1 = 3$, $G_2 = 0$ ($\beta_s = 1.373$, $\beta_b = 0.626$)

what beyond $\alpha_s = 1$, and can in this range be given by $V = B_v B_s V_0$ with

$$B_v = 1 - \alpha_s \tag{31}$$

B_v is zero at $\alpha_s = 1$ (or $\alpha_E = 1/\beta_s^2$) and becomes negative for $\alpha_s > 1$. This linear approximation becomes increasingly inaccurate (giving too small negative values) at larger α_s values, [5]).

At $\alpha_s = 1$, the function of the column changes. For $\alpha_s < 1$ ($B_v > 0$), it is capable of resisting external lateral loads, and is capable of supporting, or bracing, columns with higher axial loads. For $\alpha_s > 1$ ($B_v < 0$), it will require lateral support, or bracing, by the rest of the story (structure). Zero shear is the instability criterion for a column that is free to sway.

Moment at end 2 (with the stiffer end restraint). The largest (absolute value) of the first-order end moment will always develop at the end with the largest rotational restraint stiffness. This can be seen in the figures, and also in the first-order moment diagram (for $N = 0$ ($\alpha_E = 0$)) at the bottom of Fig. 5. The last moment diagram at the bottom is obtained for an axial load of 0.998 α_b

This end, with the largest moment, is conventionally denoted end 2 and the moment M_2 (or nondimensionally, B_2). So also here. It decreases continuously with increasing load level. At some point, it becomes zero and changes direction (at about $\alpha_s = 5.1$ ($\alpha_E = 1.37$) in Fig. 5), and eventually approaches minus infinity (at $\alpha_b = 1$). The responses of the stiffer restraint cases, Fig.6 and Fig.7, are similar.

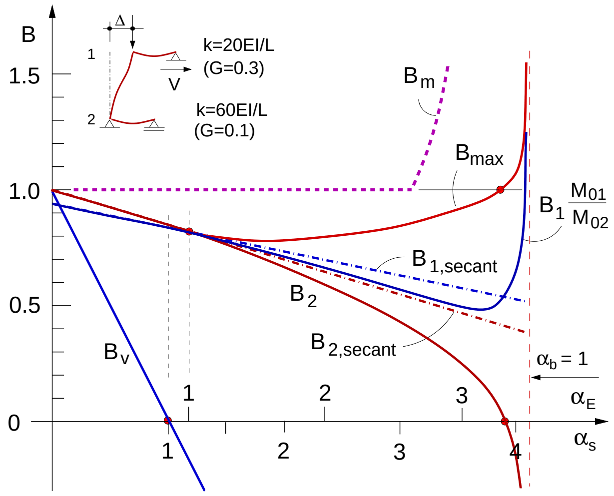

Figure 7: Moments and shear versus axial load level in column with very stiff and almost equal end restraints ($\beta_s = 1.065$, $\beta_b = 0.532$, $\alpha_E = 0.882\alpha_s$)

Moment at end 1 (with the smaller end restraint). The M_1 response, represented by $B_1 \cdot M_{01}/M_{02}$ in Fig. 5 and Fig.6, is typical for cases with relatively low restraint stiffness at end 1. The moment at end 1 stays fairly constant, or increases slowly, until the end moments at the two ends become equal at $\alpha_E = 1$. From there, it increases more sharply towards infinity at $\alpha_b = 1$. ($\alpha_E = 1/\beta_b^2$). In the case with stiff restraints also at end 1, such as in Fig. 7, both end moments decrease markedly at first with increasing axial load.

Equal or nearly equal end moments. If end restraints at the two ends had been exactly equal, the end moments would also be equal. With increasing axial load, the end moments decrease continuously to zero at $\alpha_b = 1$. Up to this point, the column deflection mode would be one in antisymmetrical double curvature bending. At $\alpha_b = 1$, the smallest disturbance would cause the deflected shape to change suddenly from double curvature bending into the braced buckling mode. This phenomenon is often referred to as *unwinding* ([20], [21]). Differences in end restraints will always be present in practical cases.

Maximum moment. The maximum moment (M_{max}, B_{max}) is initially, for zero axial load, equal to the moment at end 2, and stays at end 2 up to an axial load point at which it starts forming away from end 2. Then, following an initial decrease, B_{max} starts increasing with increasing load level, and approach plus infinity for axial loads approaching the braced critical load.

5 "Landmarks" in Single Column Response

5.1 Key Characteristics

Definition of a number of key characteristics, or "landmarks" in the moment versus axial load "map", are pursued here in order to facilitate a good understanding of the mechanics of laterally displaced columns, and for enabling a quick establishment of moment-axial load relationships. The results may be of use both for researchers and teachers of column mechanics, and for practicing engineers. The characteristic points considered are identified by bullet points in Fig. 5, Fig. 6 and Fig.7.

5.2 Zero Shear Force

The shear resistance of a sway columns becomes zero at $\alpha_s = \alpha_E\,\beta_s^2 = 1$. For $\alpha_s > 1$, the column requires lateral support. The effective length (buckling length) factor β_s, defining N_{cs} (Eq. (30)) at this stage, can be determined exactly, or graphically from diagrams such as the well known alignment charts (in terms of G factors), or also from one of the many approximate methods available. A summary and evaluation of such methods are given in [8]. One of several expressions, derived directly from column mechanics, is defined by

$$\beta_s = \frac{2\sqrt{R_1 + R_2 - R_1 R_2}}{R_1 + R_2} \tag{32}$$

where

$$R_j = \frac{k_j}{k_j + c\,EI/L} \qquad \left(= \frac{1}{1 + (c/b_0)G_j} \right) \tag{33}$$

The c factor is a constant, to be taken as $c = 2.4$ in conjunction with Eq. (32). R_1 and R_2 are degree-of-rotational-fixity factors at ends $j = 1$ and $j = 2$, respectively. $R = 0$ at a pinned end and $R = 1$ at a fully fixed end.

Application to the case in Fig. 5 gives $\beta_s = 1.950$ (exact 1.932), and to Fig. 7, $\beta_s = 1.076$ (exact 1.065). In general, predictions are within 0% and +1.7% of exact results for positive R values.

5.3 Moments approach Infinity

The outer axial load limit of the $M - \alpha$ relationship is defined by $\alpha_b = \alpha_E\,\beta_b^2 = 1$, representing braced instability. Similarly to free-sway instability, there is a number of tools available for determining the corresponding effective length factors β_b (Eq. (30)). One of several alternative formulations, [8], for this case is

$$\beta_b = 0.5\sqrt{(2 - R_1)(2 - R_2)} \tag{34}$$

where the R factors are identical to those defined by Eq. (33) with $c = 2.4$.

This expression is accurate to within -1% and +1.5% for positive R values. Applied to the case in Fig. 5, it gives $\beta_b = 0.785$ (= exact), and applied to Fig. 7, $\beta_b = 0.536$ (exact 0.532).

For approximate system instability calculations of multi-column frames, the method of means, [8], or similar methods may be used.

5.4 Maximum Moment leaves Column End

The load index at which the maximum moment starts to form away from end 2 of the column can be expressed either in terms of the column's α_E value or its free-sway stability index α_s, as

$$0.25 \le \alpha_E \le 1.0 \quad \text{and} \quad 1.0 \le \alpha_s \le \beta_s^2 \tag{35}$$

These limiting values can be found from Eq. (18) as the load at which $\cos pL = \mu_t$ becomes zero (x_m, measured from end 2). The minimum value is obtained for $\mu_t = 0$ (for a column pinned at one end) as $pL = \pi/2$ and α_E = 0.25. Similarly, the maximum is obtained for a column with equal end restraints, i.e., with $\mu_t = -1$ (antisymmetrical curvature), as $pL = \pi$ and $\alpha_E = 1.0$ (or $\alpha_s = \beta_s^2$). The moment gradient expression, $dM/dx = N\,\theta + V$, also gives useful information as demonstrated in [7]. For instance, for a case with full fixity at an end (e.g., Fig. 6), θ is zero. Thus, the gradient becomes negative, and the maximum moment leaves the end as V becomes negative when α_s exceeds 1.0.

5.5 Maximum Moment exceeds $B_s M_{02}$

The axial load level $\alpha = \alpha_m$ at which the maximum moment exceeds the larger sway-magnified first-order end moment ($B_s M_{02}$), i.e. when B_{max} in the figures exceeds 1.0, is of considerable interest in approximate analyses, and in design, as the maximum moment may be taken conservatively equal to $B_s M_{02}$ below this load level.

Closed form solutions of load levels at which B_{max} = 1 could not be derived. For this reason, they were calculated for a wide range of restraint combinations using the theory presented above. Results are presented in Fig. 8 in terms of the free-sway load index $\alpha_{s,m}$, and in Fig. 9 in terms of the braced load index $\alpha_{b,m}$. The peaks in the figures are obtained when end restraints are equal, in which cases moments remain less than $B_s M_{02}$ ($B_{max} < 1.0$) until unwinding take place at the braced critical load level ($\alpha_b = 1$). It should be recalled that all G factors in the figures are defined with $b_0 = 6$ (Eq. (2)).

Based on these results for single columns with stationary (invariant) end restraints, it can be concluded from the figures that

$$B_{max} < 1 \quad \text{for} \quad \alpha_s < 3.5 \quad \text{or} \quad \alpha_b < 0.5 \tag{36}$$

for "practical" columns (for panels, see Eq. (45)).

The smaller $\alpha_{b,m}$ values for $G_1 > 20$ ($k > 0.3EI/L$) are of no practical significance since such columns are essentially pin-ended. A perfect pin ($G = \infty$) is difficult to achieve at a connection to the surrounding structure. For instance, the Commentary to AISC, [2], indicates $G = 10$ as an upper, practical flexibility value in effective length predictions.

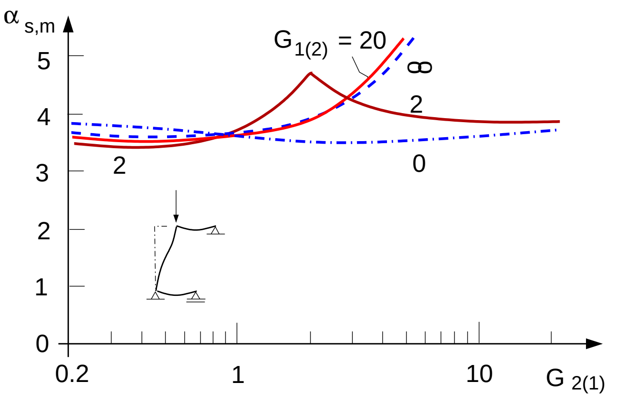

Figure 8: Axial load level in terms of $\alpha_s = N/N_{cs}$ at which the maximum moment between ends is equal to the larger sway-modified end moment.

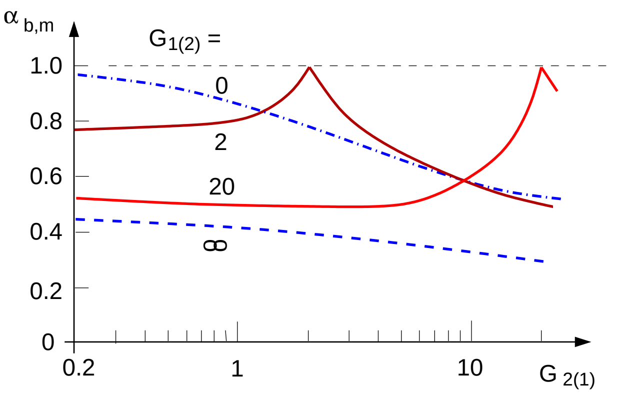

Figure 9: Axial load level in terms of $\alpha_b = N/N_{cb}$ at which the maximum moment between ends is equal to the larger sway-modified end moment.

Many structural codes accept that local second-order effects can be neglected if they do not lead to maximum moments in excess of 1.05 to 1.10 times $B_s M_{02}$. With such criteria, the load index limits above would be somewhat greater. For more details, see the report, [7], on which this paper is based.

5.6 Equal End Moments, $M_1 = M_2$

For M_1 and M_2 to become equal, it can be seen from Eqs. (10) and (11) that this requires the stability functions C and S to become equal, and thus also $c = s$ (Eq. (6)). Rewriting of $c = s$ results in $2\sin pL - pL\cos pL = pL$, the solution of which is $pL = \pi$. Consequently,

$$M_1 = M_2 \quad \text{for} \quad \alpha_E = 1 \quad (\text{or } \alpha_s = \beta_s^2) \tag{37}$$

The moments at $\alpha_E = 1$, found by evaluating Eq. (11) at $pL = \pi$, become

$$M_1 = M_2 = -\frac{12}{G_1 + G_2 + 2 \cdot 1.216} \cdot \frac{EI\Delta}{L^2} \tag{38}$$

or, in terms of the sway magnified first-order moments (Eqs. (13) and (14)),

$$B_2 = \frac{M_2}{B_s M_{02}} = \frac{4(G_1 + G_2) + 2G_1G_2 + 6}{(G_1 + G_2 + 2.432)(G_1 + 3)} \tag{39}$$

$$B_1 = \frac{M_1}{B_s M_{01}} = B_2 \frac{G_1 + 3}{G_2 + 3} \tag{40}$$

5.7 Zero End Moment, $M_2 = 0$

In order to obtain $M_2 = 0$ ($B_2 = 0$), at the end with the larger first-order end moment, it can be seen from Eq. (11) that this requires that $C - S = -6/G_1$. Thus,

$$\frac{1 - \cos pL}{pL \sin pL} = -\frac{G_1}{6} \tag{41}$$

At $M_2 = 0$ there is, as expected, no interaction with G_2 at end 2. If accurate solutions (by solving for $pL (= \pi\sqrt{\alpha_E})$ by iterations) are not required, a reasonably simple approximation that gives results within about ± 2 percent, has been found to be given by

$$\alpha_E = \frac{4 + 1.1\,G_1}{1 + 1.1\,G_1} \tag{42}$$

For the columns in Fig.5, Fig.6 and Fig.7, B_2can be se-en to become zero at about α_E = 1.37, 1.67 and 3.4, respectively. These correspond well with the comparative values by Eq. (42) of 1.40, 1.67 and 3.3.

5.8 End Moments at $\alpha_s = 1$

The value of end moments at $\alpha_s = 1$, which represent the supporting column limit, is of significant interest, in particular for developing approximate, linearized end moment relationships (see below). Values of B_1 and B_2 at $\alpha_s = 1$ are labeled B_{1s} and B_{2s}, respectively. It was not possible to find simple, closed-form expressions for these in the general case with arbitrary end restraints. Therefore, results were computed for a wide range of end restraint combinations. They are plotted in Fig. 10.

B_{2s} coincides with B_{1s} in the case with equal end restraints (dash-dot borderline $G_1 = G_2$ in the figure). Results for B_{1s} (dashed lines) and B_{2s} (solid

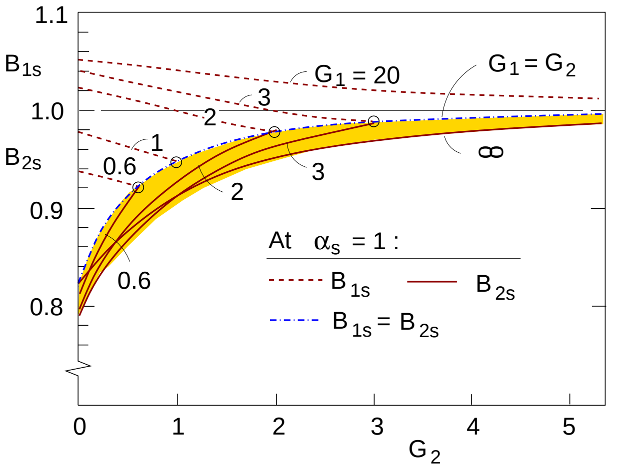

Figure 10: End moment factors at $\alpha_s = 1$ versus restraints (G_2 of the stiffer end restraint and G_1 of the more flexible restraint).

lines) are located above and below the borderline, respectively. Corresponding B_{1s} and B_{2s} curves terminates therefore at the dash-dot curve.

As seen from the figure for $G_2 = 0$ (fixed end), B_{2s} may have values ranging between 0.79 and 0.82, and B_{1s} between about 0.82 and 1.05.

5.9 Secant Approximations of End Moments

The results above are informative and can for instance be used to establish useful linear approximations of the column end moment curves. The secants through the moment points at $\alpha_s = 0$ and $\alpha_s = 1$ are given by

$$B_{1,secant} = \frac{M_1}{B_s M_{01}} = 1 - (1 - B_{1s})\,\alpha_s \qquad (43)$$

and

$$B_{2,secant} = \frac{M_2}{B_s M_{02}} = 1 - (1 - B_{2s})\,\alpha_s \qquad (44)$$

where $B_{1,s}$ and $B_{2,s}$ are the moment factor values at $\alpha_s = 1$ shown in Fig. 10.

Such secants provide close approximations to the end moment curves also well beyond $\alpha_s = 1$, as can be seen for the cases in Fig.5, Fig.6 and Fig.7. These cases were computed with pairs of (B_{1s}, B_{2s})-values read from Fig.10 of approximately (1.02,0.955),(1.04, 0.79) and (0.89, 0.85), respectively. Some efforts to derive approximate closed-form secant expressions are presented in [6].

6 Non-stationary Restraints

6.1 Panel Columns – General

In the preceding sections, end restraint stiffnesses of the single columns were given as constant (stationary, invariant) values. These results are applicable also to columns of a general frame provided the chosen column end restraints adequately represent the interaction with the surrounding structure (other columns on the same level (horizontal interaction) and columns on other levels (vertical interaction) as discussed in Section 3.1).

To what extent stationary restraint values can adequately represent horizontal interaction is considered in this section for panel frames with two columns (Fig. 2). More specifically, the purpose of the panel analyses is, apart from studying the general mechanics of behavior and interaction between the panel columns, to clarify to what extent the column behavior in the panels will affect conclusions reached earlier from the single column studies. The author is not familiar with available literature reporting on such aspects.

The panels are analysed using a stiffness formulation of the second-order theory that incorporates the slope-deflection equations given earlier (Section 3.2). Axial forces in beams are neglected.

Two panels, giving deflection shapes indicated in Fig. 2, are investigated. They are labeled Panel 1 and Panel 2, respectively. The axial column loads N_1 (left column) and N_2 (right column) are the same in the columns of both panels. Thus, $N_1 = N_2 = N$.

Member stiffnesses, expressed in terms of EI/L, of Panel 1 are: $EI_1/L = EI/L$, $EI_2/L = 1.1EI/L$, $EI_{b1}/L_b = 0.333EI/L$ and $EI_{b2}/L_b = 1.667EI/L$. End 2 of each column is taken to be at the base (bottom), which is connected to the stiffer beam (5 times that at the upper beam), and thus has the larger first-order end moment.

Member stiffnesses of Panel 2 are similarly: $EI_2/L = 2EI/L$ (twice that of Column 2 of Panel 1). Otherwise it has the same properties as Panel 1.

6.2 Isolated Column Analysis

For comparison reasons, results are also obtained for each of the two columns of each panel by considering them in isolation using the theory derived in Section 3.2. The stationary end restraints used in these isolated column analyses (Section 3.1) are determined by assuming a hinged support at locations of the first-order inflection points in the panel beams. Resulting G values (Eq. (2)) are given in the figure inserts (Fig. 11, Fig. 12, Fig. 13 and Fig.14).

Alternatively, the restraint stiffness assessments could have been based on the conventional approach of assuming antisymmetrical beam bending stiffness ($k_b = 6EI_b/L_b$), which is the most common approach in codes (ACI, AISC, etc.), and is equivalent to assuming a hinged support at midspan. With the EI/L values given above, $k_1 = k_{b1} = 2EI/L$ and $k_2 = k_{b2} = 10EI/L$ are obtained. Eq. (2) then gives (G_1, G_2) values of (3, 0.6) for Column 1 of Panel 1,

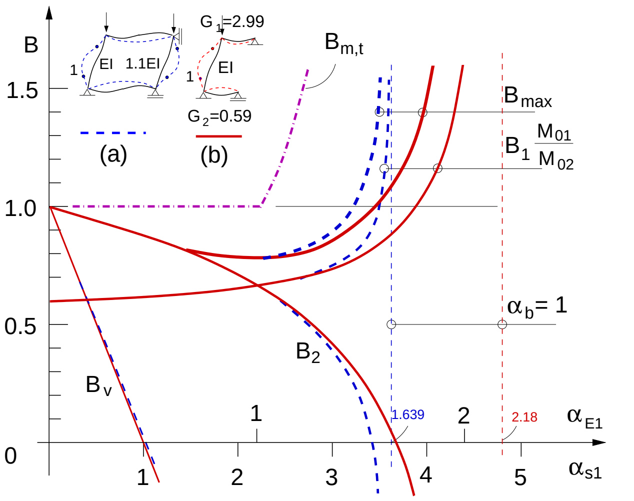

Figure 11: Moments and shear versus axial load level for two cases: (a) Column 1 (left hand) of Panel 1; (b) Column 1 in the panel considered in isolation with approximate restraints.

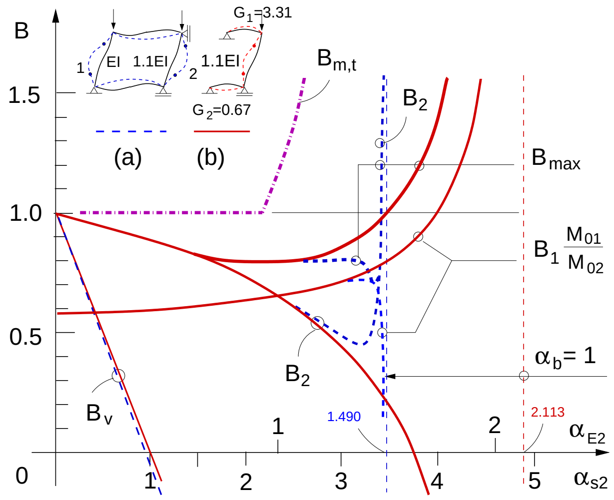

Figure 12: Moments and shear versus axial load level for two cases: (a) Column 2 (right hand) of Panel 1; (b) Column 2 in the panel considered in isolation with approximate restraints.

(3.3, 0.66) for Column 2 of Panel 1, (3, 0.6) for Column 1 of Panel 2 and (6, 1.2) for Column 2 of Panel 2. These are not too different from the values given in the figure inserts (Fig.11, Fig.12, Fig.13, Fig.14) (based on as sumed support locations at first-order beam inflection points).

Still another alternative would be to base beam stiffnesses on first-order theory, [5]. In the considered cases, resulting differences in computed B values according to these three alternatives will be minor, and will not alter the conclusions drawn below on inflection-point based restraints. Additional details are given in [7].

6.3 Panel Results and Characteristics

Results for the two columns of Panel 1 are shown in Fig.11 and Fig.12 and for the two columns of Panel 2 in Fig.13 and Fig.14. Panel column results are shown by dashed (blue) lines and the isolated column results by solid (red) lines. The two cases are shown by inserts in the upper left corner of the figures. The dash-dot curves labeled $B_{m,t}$, representing present design maximum moment magnifiers, will be discussed later.

Results are plotted versus the nominal load indices α_E. In addition, abscissas in terms of the free-sway index α_s of the respective isolated columns ($\alpha_s = \alpha_E\, \beta_s{}^2$) are added for the convenience of reading and interpretation of results. Thus, zero shear of the isolated columns will always be obtained for $\alpha_s = 1$ in the figures.

System instability. System instability of the panels will always be initiated by the most flexible column. Since both columns are connected to the same beams, Column 1, with the highest load index α_E of the two columns, is most flexible. Panel 1, with reasonably close load indices ($\alpha_{E1} = 1.1\alpha_{E2}$), represents a more practical case than Panel 2, with rather big difference in load indices ($\alpha_{E1} = 2\alpha_{E2}$).

Restraint softening and critical loads. The rotational stiffnesses of the panel beams are gradually reduced with increasing axial column load, as the beams "unwind" from double curvature towards more single curvature type bending, as illustrated in Fig. 2 and in the figure inserts. For instance, in the case of Panel 1, Fig. 11, this beam softening implies a restraint stiffness reduction to about one third of the initial (double curvature) stiffness. As a consequence, braced panel instability, initiated by the most flexible panel Column 1, is seen to result at a lower axial load (1.639 α_{E1}) than that of the isolated single Column 1 (2.18 α_{E1}).

Furthermore, a rather sudden reversal ("unwinding") of end moments is seen to take place in the stiffer Column 2 for loads close to the panel critical load (Fig. 12).

In the case of Panel 2, with Column 2 being significantly stiffer than the Column 1 ($\alpha_{E2} = \alpha_{E2}/2$), only a partial unwinding of beams may take place (Fig. 2(b)). Close to panel instability, the stiff Column 2 unwinds from double to single curvature bending (Fig. 14), reflecting a braced effective length factor that is greater than 1.0. This again implies that Column 2 contributes (through the beams) to the restraint of the more flexible Column 1.

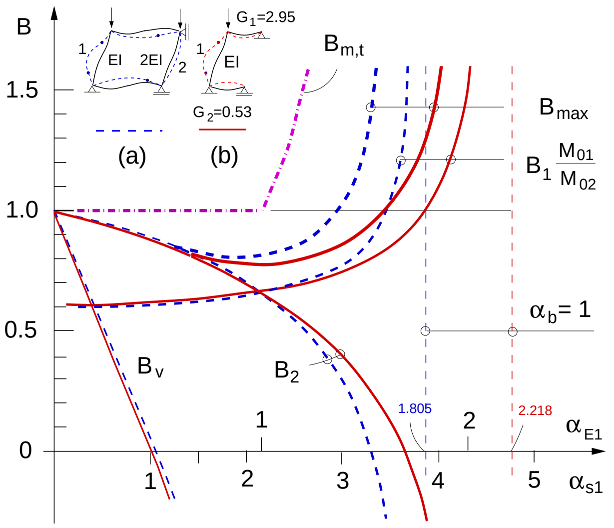

Figure 13: Moments and shear versus axial load level for two cases: (a) Column 1 (left hand) of Panel 2; (b) Column 1 in the panel considered in isolation with approximate restraints.

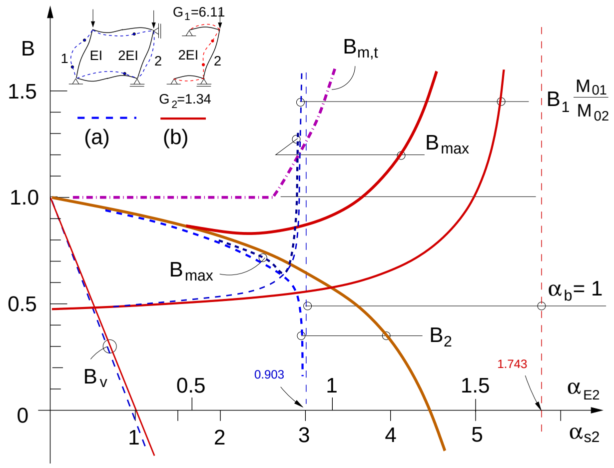

Figure 14: Moments and shear versus axial load level for two cases: (a) Column 2 (right hand) of Panel 2; (b) Column 2 in the panel considered in isolation with approximate restraints.

6.4 Non-stationary vs. Stationary Restraints

By comparing results for isolated columns and panel columns it is possible to draw some conclusions that should be useful in practical design work:

Shears. The axial load at which shears become zero in the panel columns, can be predicted well by the free-to-sway critical load of the corresponding isolated column (Eq. (32)). The difference between the two is found to be about ± 1% for the columns of Panel 1, and about ± 3.8% for those of Panel 2 at $\alpha_s = 1$.

Instability. In lieu of a full system instability analysis, it is found that braced critical loads of panel columns can be computed approximately from Eq. (34) with softened (reduced) end restraints discussed above (and as applied in $B_{m,t}$ predictions later). Alternatively, approximate yet quite accurate system instability analyses can be carried by the "method of means", [8].

Moments. The moments of the panel columns can be seen to initially follow the isolated column moments quite closely. Consequently, the panel columns respond initially with almost stationary end restraints that are (by implication) nearly equal to the restraints employed in the isolated column analyses. Thus, Eq. (35) also provides adequate limits for when maximum moments leave the ends of panel columns.

Maximum moments exceed $B_s M_{02}$. Due to the softening of restraints, the maximum moment branch of the panel columns is pressed upwards at lower load levels than in the case of the isolated columns. This affects the limits given in Eq. (36). For the panel columns, it is found that

$$B_{max} < 1 \quad \text{for} \quad \alpha_s < 3 \quad \text{or} \quad \alpha_b < 0.8 \qquad (45)$$

provide reasonably conservative limits. Although these are based on a rather limited number of panel columns, they are believed to be reasonably representative. It should be noted that the α_s and α_b values in Eq. (45) are computed with the critical loads N_{cs} and N_{cb}, respectively, of the panel columns, rather than those of the isolated single columns with stationary end restraints.

While N_{cs} of the isolated and the panel column are almost identical, N_{cb} of the panel column may be lowered (by the mentioned restraint softening) considerably below that of the single column.

Maximum moment accuracy. For the more flexible Column 1 of Panel 1 (Fig. 11, $EI_2 = 1.1EI_1$), it is found that isolated column predictions of maximum moment are at most 5% below those of the panel for axial loads below about 85% of the panel column's instability load.

In the case of the most flexible Column 1 of Panel 2 (Fig. 13, $EI_2 = 2EI_1$), this is the case for axial loads below about 65% of the panel column's instability load. This is still a rather high load level in design practice.

For the stiffer Column 2 of the two panels, the isolated column predictions are conservative up to load levels very close to the panel columns' instability loads (Fig.12,Fig.14). This is typical for the stiffer columns in frames.

Equal end moments. The equal end moment results for the isolated single columns (Eqs. (37) - (40)), are also found to apply to the panel columns, except for unusual cases of panel columns with critical loads greater then $\alpha_E = 1$ (e.g., Fig. 14).

Zero end moment. The applicability of Eqs. (41) and (42) for prediction of zero end moments ($M_2 = 0$; $B_2 = 0$) are found to be more limited for panel columns. For the most flexible of the panel columns (columns 1), Eqs. (41) and (42) will give some 7-10% too large axial load values. The limits are not applicable at all to the stiffer panel columns (Columns 2), due to the sudden moment reversals or unwinding that takes place in these columns close to frame instability (brought on by the most flexible columns).

End moments and secant approximations. Moments at $\alpha_s = 1$ (B_{1s} and B_{2s}) in Fig. 10, and the secant expressions in Eqs. (43) and (44) are applicable also to the panel columns.

7 Maximum Moment in Present Design Codes

Maximum moment approximations in present design practice can be given by $M_{max} = B_m B_s M_{02}$, where

$$B_m = \frac{C_m}{1-\alpha_b} \geq 1.0; \quad C_m = 0.6 + 0.4\mu_o (\geq 0.4) \tag{46}$$

$$\mu_0 = -\frac{B_s M_{01}}{B_s M_{02}} = -\frac{M_{01}}{M_{02}} \tag{47}$$

For the sake of comparison, predictions by this expression are included in the presented moment-axial load figures. This B_{max} approximation is common in major design codes (e.g., [1], [2], [3], [4]).

Above, α_b is the critical braced load index (Eq. (29)), C_m is a first-order moment gradient factor (that accounts for other than uniform first-order moments), and μ_0 is the ratio of first-order end moments, taken to be positive when the member is bent in single curvature by these moments, and negative otherwise. In some codes, [3], C_m is limited to 0.4.

Single restrained columns. B_m predictions by Eq. (46), with N_{cb} ($\alpha_b = N/N_{cb}$) computed for end restraints given in the respective figures for the laterally displaced column analyses, are shown in Fig.5, Fig.6 & Fig.7. These are based on the first-order end moment ratios given by $\mu_0 = -(G_2 + 3)/(G_1 + 3)$ (from Eq. (14)).

Isolated panel columns. B_m predictions denoted $B_{m,t}$ (subscript t for tangent) for the panel columns considered in isolation are shown in Fig.11, Fig.12, Fig.13&Fig.14.The first-order moment ratios (μ_0)used in the calculations are those obtained from the first-order panel analyses. N_{cb} ($\alpha_b = N/N_{cb}$) are based on assumed rotational beam restraint stiffnesses of $k_b = 2EI_b/L_b$, (implying horizontal tangent at beam midlength). These are 1/3rd of stiffnesses for beams bent in antisymmetrical curvature bending, and reflects, in an approximate sense, the restraint softening discussed earlier, and are in accordance with most codes of practice.

Column 2 of Panel 2 (Fig. 14) represents a case in which the restraint softening assumption is not very good. This is not surprising for this special case (with Column 2 having twice the EI/L of Column 1): Rather than receive restraint, Column 2 has been found earlier to contribute to the restraint of Column 1. If instead, B_m in this case had been computed with the panel critical load, which is an accepted alternative by the codes, the rising branch would have commenced before (moved towards the left), giving very conservative predictions also for this case.

Conclusions. It is clear that B_m ($B_{m,t}$) given by Eq. (46) is generally very conservative for columns in frames with moments caused by sidesway, and in particular for common cases of panel columns due to restraint softening. There is room for considerable improvements in maximum moment predictions. Presentation of some recent efforts, [6], towards this end is outside the scope of this paper.

8 Summary and Conclusions

Theoretically derived, closed form expressions have been presented for a number of member response characteristics that enable quick establishment of several typical points along the moment-axial load curves.

These represent not only a novel contribution, but also a tool that will be useful in providing a general understanding of the often complicated column response, in practical design work and as a complement to full second-order analyses.

The appropriateness of analyzing framed columns by single column models with stationary end restraints are investigated using laterally displaced single columns isolated from laterally displaced two-column panels.

For panel cases with smaller differences in column stiffnesses, it is found that single column models describe the maximum moment response of the most flexible of the panel columns quite closely for axial load levels as high as 85% of the critical panel loading. For the stiffer panel column, this load level is higher.

For panel frames with large stiffness differences between interacting columns, unwinding may occur and reduce the load level at which the most flexible panel column can be described well with a single column model. Even so, it is quite well for axial load levels as high as 65% of the critical panel loading. For the stiffer panel column, this load level is higher, and very close to the critical panel loading.

Results are computed whereby end moments described by secants to the axial load - moment curve can be calculated. They are found to provide good end moment approximations over a wide axial force range, and are useful in particular for the assessment of moments in adjacent restraining members, including foundations.

Present maximum moment approximations in structural design codes are generally found to be very conservative for columns with moments solely due to sidesway. There is room for considerable improvements in such predictions, and particularly so for columns restrained by stiff beams. Axial load limits are established below which maximum column moments in frames with sidesway will be less than the sway-magnified first-order end moment ($B_s M_{02}$). By taking $B_m = 1$, for lack of better values, will allow more economical designs for a wide axial force range.

Acknowledgment:
The author has long been interested in the paper topic, both as a practicing engineer and researcher. During a stay at the Univ. of Alberta (UA), Edmonton, Canada, sponsored by the Research Council of Norway and a research associateship at UA (1981), this interest became more focused. Input and useful discussions by the now deceased Professor J. G. MacGregor (at UA) was greatly appreciated. So was the running of the initial computer analyses of the panels by S.M.A. Lai, then a PhD student at UA.

References:

[1] ACI (American Concrete Institute). *Building code requirements for structural concrete, and Commentary (ACI 318-M14 and ACI 318RM-14)*. Farmington Hills, MI, 2014.

[2] AISC (American Institute of Steel Construction). *Specifications for structural steel buildings, and Commentary (ANSI/AISC 360-16)*. Chicago, IL, 2016.

[3] CEN (European Committee for Standardisation). *Design of concrete structures, part 1-1. Eurocode 2*. Brussels, 2004.

[4] CEN (European Committee for Standardisation). *Design of steel structures, part 1-1. Eurocode 3*, Brussels, 2005.

[5] Hellesland J. Extended second order approximate analysis of frames with sway-braced column interaction. *J. of Construct. Steel Research*, Vol. 65, No. 5, 2009, pp. 1075–1086.

[6] Hellesland J. New and extended design moment formulations for slender columns in frames with sway. *Eng. Struct.*, Vol. 203, No. 1 (109804), 2020.

[7] Hellesland J. Mechanics of columns in sway frames – Derivation of key characteristics. *Research Rep. in Mechanics*, Dept. of Math., Univ. of Oslo, Norway, No.1-Feb, 2019.

[8] Hellesland J. Evaluation of effective length formulas and application in system instability analysis. *Eng. Struct.*, Vol. 45, No. 12, 2012, pp. 405–420.

[9] Chen WF, Lui EM. *Stability design of steel frames*, CRC Press, Boca Raton, Fl, 1991.

[10] Galambos TV. *Structural members and frames*. Prentice Hall, Inc., Englewood Cliffs, NJ, 1968.

[11] Galambos TV, Surovek AE. *Structural stability of steel: concepts and applications for structural engineers*. John Wiley & Sons, Inc., Hoboken, NJ, 2008.

[12] Lai SM A., MacGregor JG, Hellesland J. Geometric nonlinearities in nonsway frames. *J. Struct. Eng.*, ASCE, Vol. 109, No. 12, 1983, pp. 2770–2785.

[13] Timoshenko SP, Gere JM. *Theory of elastic stability*, 2nd ed. McGraw-Hill Book Co, 1961.

[14] Livesley RK. The application of an electronic digital computer to some problems of structural analysis. *The Structural Engineer*, 1956, January.

[15] Livesley RK. *Matrix methods of structural analysis*, 2nd ed., Pergamon Press, New York, NY, 1975.

[16] Bleich F. *Buckling strength of metal structures*. McGraw-Hill Book Co., Inc., New York, NY, 1952.

[17] LeMessurier, WM. A practical method of second order analysis. *Eng. J.*, AISC, Vol. 14, No. 2, 1977, pp. 49–67.

[18] Lui, EM. A novel approach for K factor determination. *Eng. J.*, AISC, Vol. 29, No. 4, 1992, pp. 150–150.

[19] Aristizabal-Ochoa JD. K-factor for columns in any type of construction: Nonparadoxical approach. *J. Struct. Eng.*, ASCE, Vol. 120, No. 4, pp. 1272-1290.

[20] Ketter RL. Further studies of the strength of beam-columns. *J. of the Struct. Div.*, ASCE. Vol. 87, No. ST6, 1961, pp. 135-152.

[21] Ziemian, RD (ed.). *Guide to stability design criteria for metal structures*, 6th ed., John Wiley & Sons, Inc., Hoboken, New Jersey, USA, 2010.

Notation:

B_m = approximate maximum moment magnification factor;
B_{max} = exact maximum moment magnification factor;
B_s = sway magnification factor;
B_{tmax} = maximum moment magnification factor in second-order analysis;
B_1, B_2 = end moment factors, at end 1 and 2;
EI, EI_b = cross-sectional stiffness of columns, and beams;
G_j = relative rotational restraint flexibility at member end j;
H = applied lateral story load (sum of column shears and bracing force);
L, L_b = lengths of considered column and of restraining beam(s);
M_{0j}, M_j = moment in first-order and second-order analysis, at end j;
N = axial (normal) force;
N_{cr} = critical load in general ($= \pi^2 EI/(\beta L)^2$);
N_{cb}, N_{cs} = critical load of columns considered fully braced and free-to-sway, respectively;
N_E = Euler buckling load of a pin-ended column ($= \pi^2 EI/L^2$);
R_j = rotational degree of fixity at member end j;
V_0, V = first-order, and total (first+second-order) shear force;
k_j = rotational restraint stiffness (spring stiffness) at end j;
p = $\sqrt{N/EI}$;
α_{cr} = member (system) stability index ($= N/N_{cr}$);
α_b, α_s = load index of column considered fully braced and free-to-sway, respectively;
α_E = nominal load index of a column ($= N/N_E$);
β_b, β_s = effective length factor corresponding to N_{cb} and N_{cs};
Δ_0, Δ = first-order, and total lateral displacement;
κ_j = relative rotational restraint stiffness at end j ($=k_j/(EI/L)$).

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflicts of Interest

The author has no conflicts of interest to declare that are relevant to the content of this article.