On the Existence of the Conditionally Linear Integral in Conservative Holonomic Systems with Two Degrees of Freedom

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Abstract: - For the Lagrange equations of the 2nd kind the problem of the existence of the conditionally linear in the velocities integral, which possesses the property that its total time derivative is identically proportional to the integral itself, is considered. The Lagrange function is assumed to be given in arbitrary generalized coordinates. The conditions for the existence of such integral are reduced to the study of the compatibility of two equations in partial derivatives of the 2nd order for one unknown function of two independent arguments. These equations are written in the invariant form in an arbitrary system of generalized coordinates and the problem is transformed into the investigation of the set of Pfaffian equations.

Key-Words: - linear integral, point transformation, differential invariants and parameters, overdetermined PDEs system, Pfaffian equations

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1 Problem Statement

Let

$$L = \tilde{T} + U = \frac{1}{2} a_{ij}(q^1,\, q^2)\dot{q}^i\dot{q}^j + U(q^1,\, q^2) \quad (1)$$

be the Lagrangian of a holonomic system with two degrees of freedom ($i, j = 1, 2$; the Ricci summation convention is applied throughout the paper). The system is referred to local coordinates $q^1,\, q^2$ chosen on its configuration 2-D manifold M. All occurring functions of coordinates are supposed to be smooth locally up to desired order. The dot denotes the derivative with respect to time.

The Lagrange equations of the 2nd kind, resolved with respect to the generalized accelerations, have the form

$$\ddot{q}^i = -\,\Gamma^i_{jk}\,\dot{q}^j\,\dot{q}^k + a^{ij}\partial_j U\,, \quad (2)$$

where $\Gamma^i_{jk} = \frac{1}{2} a^{li}\left(\partial_k a_{jl} + \partial_j a_{kl} - \partial_l a_{jk}\right)$, $\partial_j = \partial/\partial q^j$, $\|a^{ij}\| = \|a_{ij}\|^{-1}$; $i, j, k, l = 1, 2$.

Consider the expression

$$f(q,\, \dot{q}) = a_i(q)\dot{q}^i - c = 0 \quad (c = const) \quad (3)$$

If its total time derivative due to the system (2) is identically

$$\frac{Df}{Dt} \equiv \lambda(q, \dot{q}) f \quad (4)$$

where λ is a certain function, then the integral curves of the equations (2) with initial conditions such that $f(q_0,\, \dot{q}_0) = 0$ for some value of c belong to the set given by the equation (3).

In the considered case of two degrees of freedom, when the constant c is *arbitrary*, the necessary and sufficient conditions for the existence of the first linear integral (3) were found in [1] and [2]. These conditions have to be satisfied with the functions a_{ij}, U and their derivatives. The functions may be given in any generalized coordinates.

But the identity (4) may be satisfied only for some *exceptional* value of c. When $c = 0$ the expression (3) is called the linear invariant relation of the equations (2). For the case of two degrees of freedom, the criterion for its existence was given by [3].

And when $c \neq 0$ the expression (3) will be called the conditionally linear integral in this paper. Without loss of generality, we assume $c = 1$.

Conditions for the existence of such kinds of integrals were the subject of the papers, [4], [5], [6]. The most general result was obtained in [6], for the case of arbitrary number n of degrees of freedom. It is proved that when a conditionally linear integral exists, there are generalized coordinates of the system in which the Lagrange function is written in the form

$$L = \frac{1}{2} a_{11}{}^{-1} (a_{p1}\, \dot{q}^p - 1)^2 + \frac{1}{2} A_{ij} \dot{q}^i \dot{q}^j + A_i\, \dot{q}^i + A$$
$$(i, j, p = 1, \dots, n) \qquad (5)$$

where A_{ij}, A_i, A do not depend on q^1. The converse conclusion is also true. The conditionally linear integral is $a_{p1}\, \dot{q}^p - 1 = 0$.

However, this result does not clarify whether having the Lagrange function (1) referred to certain given generalized coordinates, it is possible to find (in the local sense) the point transformation of the coordinates so that the transformed Lagrange function takes the form (5) (then the system has the conditionally linear integral), or such transformation does not exist (then there is no conditionally linear integral).

Below we research this problem in the case of a naturally conservative system with two degrees of freedom.

2 Reducing to the Pfaffian System

2.1 Kilmister's Theorem and Its Invariant Reformulation

The Lagrange functions found in [4], [5], can be obtained from (5) under the additional assumption that the products $a_{11}{}^{-1} a_{p1}$ do not depend on q^1. Let us exhibit that in the case of $n = 2$ and of a natural conservative system (i.e. its Lagrangian does not contain linear in the generalized velocities terms) the following theorem is valid. This theorem has been proved first by [4], in a different way.

Theorem. For the existence of a conditionally linear integral in a natural conservative holonomic system with two degrees of freedom, it is necessary and sufficient that there exists the nondegenerate point transformation

$$x = x(q^1, q^2),\ y = y(q^1, q^2) \qquad (6)$$

such that the transformed Lagrange function becomes

$$L = \frac{1}{2} [\, G(x, y) \dot{x}^2 + \dot{y}^2] + \frac{1}{2G} + A(y) \qquad (7)$$

with the force function $U = {}^1\!/_{2G} + A(y)$. The network (x, y) is semi-geodesic, $G(x, y) > 0$.

Indeed, by opening the brackets in the formula (5), we obtain

$$L = \frac{1}{2} [a_{11}\, (\dot{q}^1)^2 + \frac{(a_{21})^2}{a_{11}} (\dot{q}^2)^2 + 2a_{21} \dot{q}^1 \dot{q}^2 - 2\dot{q}^1 - 2 \frac{a_{21}}{a_{11}} \dot{q}^2] + \frac{1}{2a_{11}} + \frac{1}{2} A_{22}(q^2)(\dot{q}^2)^2 + A(q^2)$$

To exclude the terms linear in the velocities, one should require that $a_{11}{}^{-1} a_{21}$ be independent of q^1. Then, omitting the exact time derivatives, whose presence in the Lagrange function does not affect the form of the Lagrange equations, we obtain (7) where $x = q^1 + F,\ \dot{F} = a_{11}{}^{-1} a_{21} \dot{q}^2,\ \dot{y} = \dot{q}^2 \sqrt{A_{22}},\ G = a_{11}$.

If the desired functions (6) exist then the following differential equations

$$\begin{aligned} &\Delta_1 y = a^{ij} \partial_i y\, \partial_j y = 1, \\ &d(2U - \mu^2) \wedge dy = 0, \\ &d(\mu^2) \wedge \omega + 2\mu^2\, d\omega = 0 \end{aligned} \qquad (8)$$

must be compatible. Here $\mu(q^1, q^2)$ is an integrating factor for

$$\nabla x = \mu(q^1, q^2)\, \delta\ (-a^{12} \partial_1 y - a^{22} \partial_2 y\,, a^{11} \partial_1 y + a^{12} \partial_2 y), \qquad \delta = \sqrt{a_{11} a_{22} - (a_{12})^2},$$
$$\omega = \delta\, [-(a^{12} \partial_1 y + a^{22} \partial_2 y) dq^1 + (a^{11} \partial_1 y + a^{12} \partial_2 y) dq^2],$$

($\wedge$ denotes the exterior multiplication).

The system (8) has the invariant form because, when by any reversible point transformation $q \to Q$ the kinematic line element (KLE), [2],

$$ds^2 = 2\tilde{T}\, dt^2 = a_{ij}(q^1, q^2) dq^i dq^j$$

becomes

$$ds^2 = A_{ij}(Q^1, Q^2) dQ^i dQ^j,$$

then for each equation I in (8) we have $I = I'$, where I' is written exactly as I but in Q-variables.

The first and second equations in (8) follow from the fact that the first differential parameters, [7], of the functions (6) are equal $\Delta_1 y = 1$ and $\Delta_1 x = G^{-1} = \mu^2 \Delta_1 y = \mu^2$ in x, y coordinates. The third equation (8) gives the condition that the differential form $\mu\omega$ is exact. This equation can be rewritten in the equivalent form

$$\Delta_2 y + \Delta(\ln|\mu|\,, y) = 0 \qquad (9)$$

where, [7],

$$\Delta_2 \varphi = \frac{1}{\delta} \partial_i (\delta\, a^{ij} \partial_j \varphi), \qquad \Delta(\varphi, \psi) = a^{ij} \partial_i \varphi\, \partial_j \psi$$

are correspondingly the second and mixed differential parameters of functions $\varphi(q^1, q^2)$ and $\psi(q^1, q^2)$.

Thus, we have the system of 3 PDEs with two unknown functions $y(q^1, q^2)$ and $\mu^2 = z(q^1, q^2) > 0$. The problem is to find the compatibility conditions of these equations. When the equations are compatible the conditionally linear integral $G\dot{x} \pm 1 = 0$ takes a place.

2.2 Elimination of One Unknown Function

For a simplification of the formulae following next let us set

$$q^1 = u, \quad q^2 = v, \quad x = x(u,v),$$
$$y = y(u,v), \quad w(u,v) = 2U - z,$$
$$a^{11} = A, \quad a^{22} = B, \quad a^{12} = a^{21} = C,$$
$$\Delta = \sqrt{AB - C^2} = \delta^{-1}, \quad \partial_u y = p, \quad \partial_v y = q,$$
$$\partial^2_{uu} y = r, \quad \partial^2_{vv} y = t, \quad \partial^2_{uv} y = s,$$
$$\partial_u z = P, \quad \partial_v z = Q,$$
$$\partial^2_{uu} z = R, \quad \partial^2_{vv} z = T, \quad \partial^2_{uv} z = S$$

The Pfaffian system

$$dz = P\,du + Q\,dv, \quad dP = R\,du + S\,dv, \quad dQ = S\,du + T\,dv \tag{10}$$

will be of our interest in what follows.

The system of equations (8) and (9) takes the form

$$Ap^2 + 2Cpq + Bq^2 = 1, \; p\ \partial_v w - q\ \partial_u w = 0,$$
$$P\left(\tfrac{A}{\Delta}p + \tfrac{C}{\Delta}q\right) + Q\left(\tfrac{C}{\Delta}p + \tfrac{B}{\Delta}q\right) + 2z\left\{p\left[\partial_v\left(\tfrac{C}{\Delta}\right) + \partial_u\left(\tfrac{A}{\Delta}\right)\right] + q\left[\partial_v\left(\tfrac{B}{\Delta}\right) + \partial_u\left(\tfrac{C}{\Delta}\right)\right] + r\tfrac{A}{\Delta} + t\tfrac{B}{\Delta} + 2s\tfrac{C}{\Delta}\right\} = 0 \tag{11}$$

From the first two equations we find

$$p = \varepsilon\rho\ \partial_u w, \quad q = \varepsilon\rho\ \partial_v w, \tag{12}$$
$$\rho = \frac{1}{\sqrt{\Delta_1 w}}, \;\; \varepsilon = \pm 1,$$

and now one can replace the first two equations (11) by

$$\partial_v(2U - z)\ \partial_u[\Delta_1(2U - z)] - \partial_u(2U - z)\ \partial_v[\Delta_1(2U - z)] = 0 \tag{13}$$

Remark. If $z = const$ then the differential form ω is exact and the fact, that $\Delta_1 U$ depends on U only, follows from (13). Since $z = const$ we have $\mu = const$ and, hence, $\Delta_2 y = 0$ according to (9).
But

$$\varepsilon^{-1}\Delta_2 y = [\partial_u\left(\frac{A}{\Delta}\cdot\frac{\partial_u w}{\sqrt{\Delta_1 w}} + \frac{C}{\Delta}\cdot\frac{\partial_v w}{\sqrt{\Delta_1 w}}\right) + \partial_v\left(\frac{C}{\Delta}\cdot\frac{\partial_u w}{\sqrt{\Delta_1 w}} + \frac{B}{\Delta}\cdot\frac{\partial_v w}{\sqrt{\Delta_1 w}}\right)]\ \Delta = \ \Delta_2 w \cdot \frac{1}{\sqrt{\Delta_1 w}} + A\,\partial_u w\left(-\frac{\partial_u(\Delta_1 w)}{2(\Delta_1 w)^{3/2}}\right) + C\,\partial_v w\left(-\frac{\partial_u(\Delta_1 w)}{2(\Delta_1 w)^{\frac{3}{2}}}\right) + B\ \partial_v w\left(-\frac{\partial_v(\Delta_1 w)}{2(\Delta_1 w)^{3/2}}\right) = 0,$$

whence

$$\Delta_2 w = \Delta_1\left(\frac{w}{\sqrt{\Delta_1 w}}\right)\sqrt{\Delta_1 w}$$

follows.

Thus, $\Delta_1 U$ and $\Delta_2 U$ depend on U only, due to which the Lagrangian (1) has the hidden or explicit cyclic coordinate, [8]. There exists the first integral linear in the velocities. The case $z = const$ is exhausted. (12)

By the substitution of the derivatives (12) in the third equation (11) and carrying out simplifications we obtain

$$\Delta(z, 2U - z) + 2z\{\Delta_2(2U - z) - \frac{1}{2}\Delta(\ln[\Delta_1(2U - z)], 2U - z)\} = 0 \tag{14}$$

(the factor ε was canceled and did not enter the formula).

Thus, the considered problem has been reduced to researching the consistency of the overdetermined (15) system of PDEs (13) and (14) with one unknown function $z(q^1, q^2)$. This system is written in the invariant form.

See that PDEs (13) and (14) are dependent linearly on the second partial derivatives R, S, and T.

2.3 First Prolongation of the Differential System

To simplify a little the next formulae let us consider that, from the outset, the KLE is given in isometric coordinates

$$ds^2 = 2\tilde{T}dt^2 = \Lambda[(dq^1)^2 + (dq^2)^2]$$

where $\Lambda(q^1, q^2) > 0$. As known, such a choice of coordinates is always possible locally. Of course, the following generic conclusions do not depend on a coordinate choice.

Set

$$\partial_u(2U) = \alpha, \quad \partial_u(2U) = \beta, \qquad d\alpha = k du + n dv,$$
$$d\beta = n du + m dv, \qquad d\Lambda = \xi du + \eta dv,$$
$$\partial_u(2U - z) = Y, \quad \partial_v(2U - z) = V$$

In the explicit form, equations (13) and (14) are correspondingly

$$f_1 = -2VY\,\boldsymbol{R} + 2(Y^2 - V^2)\,\boldsymbol{S} + 2VY\,\boldsymbol{T} + \eta Y^3\Lambda^{-1} - (2n + \xi V\Lambda^{-1})Y^2 + (\eta V^2\Lambda^{-1} + 2kV - 2mV)Y + 2nV^2 - \xi V^3\Lambda^{-1} = 0 \tag{15}$$

and

$$f_2 = -2zV^2\boldsymbol{R} + 4zVY\,\boldsymbol{S} - 2zY^2\,\boldsymbol{T} - Y^4 + (\alpha + z\xi\Lambda^{-1})Y^3 + (-2V^2 + \beta V + z\eta V\Lambda^{-1} + 2zm)Y^2 + (\alpha V^2 + z\xi V^2\Lambda^{-1} - 4znV)Y - V^4 + z\eta V^3\Lambda^{-1} + \beta V^3 + 2zkV^2 = 0 \tag{16}$$

To obtain the first prolongation of the differential system (15) and (16) differentiate each equation relative to the independent variables u and v

$$F_1 = \partial_u(f_1) = 0, \quad F_2 = \partial_v(f_1) = 0,$$
$$F_3 = \partial_u(f_2) = 0, \quad F_4 = \partial_v(f_2) = 0$$

These equations contain 4 independent leading derivatives

$$R_1 = \partial_u R,\ R_2 = \partial_v R,\ T_1 = \partial_u T,\ T_2 = \partial_v T \quad (17)$$

provided that $\partial_u S = \partial_v R$ and $\partial_v S = \partial_u T$.

The 4×4 matrix of the coefficients at the quantities (17) is

$$2\Lambda\begin{pmatrix} -YV & Y^2 - V^2 & YV & 0 \\ 0 & -YV & Y^2 - V^2 & YV \\ -zV^2\Lambda^2 & 2zYV\Lambda^2 & -zY^2\Lambda^2 & 0 \\ 0 & -zV^2\Lambda^2 & 2zYV\Lambda^2 & -zY^2\Lambda^2 \end{pmatrix}$$

The rank of this matrix equals 3 if

$$Y^2 + V^2 \neq 0 \quad (18)$$

We suppose that this condition rejects the trivial case and is always fulfilled.

The linear combination

$$-zV\,F_1 + zY\,F_2 + Y\,F_3 + V\,F_4$$

leads to the equation

$$F_5 = h_{11}\boldsymbol{R^2} + h_{22}\boldsymbol{S^2} + h_{33}\boldsymbol{T^2} + h_{12}\boldsymbol{RS} + h_{13}\boldsymbol{RT} + h_{23}\boldsymbol{ST} + h_1\boldsymbol{R} + h_2\boldsymbol{S} + h_3\boldsymbol{T} + h_0 = 0, \quad (19)$$

where

$\boldsymbol{h_{11}} = -2zV^2,\ \boldsymbol{h_{22}} = -8z(Y^2 + V^2),$
$\boldsymbol{h_{33}} = -2zY^2,\quad \boldsymbol{h_{12}} = 4zYV,$
$\boldsymbol{h_{13}} = 6z(Y^2 + V^2),\ \boldsymbol{h_{23}} = 4zYV,$
$\boldsymbol{h_1} = 4Y^4 - 3(\alpha + z\xi\Lambda^{-1})Y^3 + (6V^2 - 2\beta V - z\eta\Lambda^{-1} - 6zm)Y^2 - (3\alpha V^2 + 3z\xi\Lambda^{-1}V^2 + 4znV)Y + 2V^4 - (2\beta + z\eta\Lambda^{-1})V^3 + 2z(2k - 3m)V^2,$
$\boldsymbol{h_2} = (4V - \beta - 2z\eta\Lambda^{-1})Y^3 - (\alpha V + 2z\xi\Lambda^{-1}V - 16zn)Y^2 + [4V^3 - \beta V^2 - 2z\eta\Lambda^{-1}V^2 - 4z(k + m)V]Y - \alpha V^3 - 2z\xi\Lambda^{-1}V^3 + 16znV^2,$
$\boldsymbol{h_3} = 2Y^4 - (2\alpha + z\xi\Lambda^{-1})Y^3 + (6V^2 - 3\beta V - 3z\eta\Lambda^{-1}V - 6zk + 4zm)Y^2 - (2\alpha V^2 + z\xi\Lambda^{-1}V^2 + 4znV)Y + 4V^4 - 3(\beta + z\eta\Lambda^{-1})V^3 - 6zkV^2,$
$\boldsymbol{h_0} = -2\xi\Lambda^{-1}Y^5 + (2\alpha\xi\Lambda^{-1} - 2\eta\Lambda^{-1}V - 3k - 2m)Y^4 - [4\xi\Lambda^{-1}V^2 + 2(n - \alpha\eta\Lambda^{-1} - \beta\xi\Lambda^{-1})V - 2zn\eta\Lambda^{-1} - n\beta - \alpha(3k + 2m) - z\xi(3k + m)\Lambda^{-1}]Y^3 - \{4\eta\Lambda^{-1}V^3 + [5(k + m) - 2(\alpha\xi + \beta\eta)\Lambda^{-1}]V^2 - [z(3m + k)\eta + 2zn\xi)\Lambda^{-1} + \alpha n + (2k + 3m)\beta)]V - 2z(3km - m^2 - 4n^2)\}Y^2 + \{-2\xi\Lambda^{-1}V^4 + 2[(\alpha\eta + \beta\xi)\Lambda^{-1} - n]V^3 + [2zn\eta\Lambda^{-1} + \alpha(3k + 2m) + n\beta + z(3k + m)\xi\Lambda^{-1}]V^2 + 4(k + m)znV\}Y - 2\eta\Lambda^{-1}V^5 + \left(2\beta\eta\Lambda^{-1} - 2k - 3m\right)V^4 + [z(k\eta + 3m\eta + 2n\xi)\Lambda^{-1} + \alpha n + (2k + 3m)\beta]V^3 - 2z(k^2 - 3km + 4n^2)V^2$

The equations (15), (16), and (19) can be resolved with respect to R, S, and T. In virtue of (18), the rank of the submatrix 2×4 which is formed by the first two rows of the matrix written above, equals 2. Hence, according to the linear algebraic system (15) and (16), the solution for R, S, and T can be searched in the form

$$R = Y^2\rho + c_1, \qquad S = YV\rho + c_2,$$
$$T = V^2\rho + c_3$$

Here ρ iz unknown parameter and (c_1, c_2, c_3) is any particular solution of the linear inhomogeneous algebraic system (15-16). Let us pick

$$c_1 = k, \quad c_2 = \frac{1}{2Yz\Lambda}[V^3\Lambda + V(Y^2 - Y\alpha)\Lambda + Yz(-Y\eta + 2n\Lambda) - V^2(z\eta + \beta\Lambda)],$$

$$c_3 = \frac{1}{2Y^2z\Lambda}\{(\alpha\Lambda + z\xi)Y^3 - \Lambda Y^4 + [2zm\Lambda + (\beta\Lambda - z\eta)V]Y^2 + (z\xi - \alpha\Lambda)V^2Y + \Lambda V^4 - (z\eta + \beta\Lambda)V^3\}$$

The substitution of these formulae in (19) leads to the linear equation with respect to ρ because all the terms having the second degree of ρ disappear.

Thus, the algebraic system (15), (16), and (19) specifies the single solution

$$\boldsymbol{R} = k + \frac{1}{2z(V^2 + Y^2)^2\Lambda^2}\{V^3Y^2\Lambda(2z\eta - 5\beta\Lambda) - V^6\Lambda^2 + V^5\Lambda(z\eta + \beta\Lambda) + V^4Y\Lambda(Y\Lambda + \alpha\Lambda - z\xi) + VY^3\Lambda(Yz\eta + 2z\alpha\eta - 4nz\Lambda - 6Y\beta\Lambda + 6\alpha\beta\Lambda + 2z\beta\xi) + Y^4[3(Y - \alpha)^2\Lambda^2 - z\Lambda(\beta\eta + 2k\Lambda + Y\xi - \alpha\xi) + 2z^2(\eta^2 + \xi^2)] + V^2Y^2[(5Y^2 - 5Y\alpha + 3\beta^2)\Lambda^2 - z\Lambda(2m\Lambda - \beta\eta + 2Y\xi + \alpha\xi) + 2z^2(\eta^2 + \xi^2)]\},$$

$$\boldsymbol{S} = \frac{1}{2(V^2 + Y^2)^2z\Lambda^2}\{Y^4z\Lambda(-Y\eta + 2n\Lambda) - V^4\Lambda(Yz\eta - 2nz\Lambda + 7Y\beta\Lambda) + V^5\Lambda(4Y\Lambda - z\xi) + V^2Y^2\Lambda(-2Yz\eta + 2z\alpha\eta - 7Y\beta\Lambda + 6\alpha\beta\Lambda + 2z\beta\xi) + VY^3[(4Y^2 - 7Y\alpha + 3\alpha^2)\Lambda^2 - z\Lambda(\beta\eta + 2k\Lambda + Y\xi - \alpha\xi) + 2z^2(\eta^2 + \xi^2)] + V^3Y[(8Y^2 - 7Y\alpha + 3\beta^2)\Lambda^2 - z\Lambda(2m\Lambda - \beta\eta + 2Y\xi + \alpha\xi) + 2z^2(\eta^2 + \xi^2)]\},$$

$$\boldsymbol{T} = \frac{1}{2(V^2+Y^2)^2 z\Lambda^2}\{3V^6\Lambda^2 + VY^4\Lambda(\beta\Lambda - z\eta) - V^5\Lambda(z\eta + 6\beta\Lambda) + Y^4\Lambda(2mz\Lambda - Y^2\Lambda + Y\alpha\Lambda + Yz\xi) + V^3Y\Lambda(2z\alpha\eta - 2Yz\eta - 4nz\Lambda - 5Y\beta\Lambda + 6\alpha\beta\Lambda + 2z\beta\xi) + V^4[(5Y^2 - 6Y\alpha + 3\beta^2)\Lambda^2 + z\Lambda(\beta\eta + Y\xi - \alpha\xi) + 2z^2(\eta^2+\xi^2)] + V^2Y^2[(Y^2 - 5Y\alpha + 3\alpha^2)\Lambda^2 + z\Lambda(4m\Lambda - \beta\eta - 2k\Lambda + 2Y\xi + \alpha\xi) + 2z^2(\eta^2+\xi^2)]\}$$

After the substitution of the obtained R, S, and T in the right-hand sides of (10) and carrying out the replacements

$$Y = \alpha - P,\ V = \beta - Q,$$

we derive the set of Pfaffian equations (10) closed relative to unknowns $z,\ P$ and Q.

3 Main Result

Thus, the problem of the existence of the conditionally linear integral of the Lagrange equations in the case of two degrees of freedom has been transformed into a study of the closed set of Pfaffian equations. When a nontrivial solution $z = z(u,v)$ is known one can find $y(u,v)$ from (12) $(p = \partial_u y,\ \ q = \partial_v y,\ \ w = 2U - z)$ and then obtain $x(u,v)$ $\left(dx = \sqrt{z}\ \omega\right)$ by quadratures. In xy-coordinates the Lagrangian (1) takes the form (7).

If such a solution does not exist there is no conditionally linear integral of the Lagrange equations.

4 Conclusion

In the considered problem the analysis of the overdetermined nonlinear PDEs system of the *second* order can be changed by the study of the nonlinear PDEs system of the *first* order. Since all the equations of the latter system are polynomials of high degrees with respect to z, $\partial_u z$ and $\partial_v z$ the problem of finding its integrability conditions is hard enough, but there are powerful modern relevant algorithms and computer systems of symbolic computations which would be useful in concrete cases. There is a vast set of corresponding publications. The list of some of them one can find, e.g., in the bibliography of the book, [9], and in later sources.

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