A revision of M. Asch notion of Discrete control

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Abstract : We revisit the notion of discrete control by M. Asch considering the case where the control is discreet. This point of view leads to problems of controllability with or without constraints on the control. This article focuses on the case of hyperbolic equations although similar developments can be done for other PDE's. The main tool used is the HUM method which is "adapted" to the constraints.

Key-Words : Hyperbolic equations, Discrete control, HUM method, Mechanical Systems, Inverted Pendulum, Mathematical Models in Mechanics.R

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Nomenclature

edo	Ordinary differential equation.
edp	Partial differential equation.
sd-o	Semi-discrete observation.
HUM	Hilbert Uniqueness Method.

Greek symbols

Δ	Laplacien operator.
α	The coefficient of heat transfer with a constant rate.

Subscripts

$C_h W$	Semi-discrete observation

1. Control for a differential equation

Similar works with our study that are worth mentioning are [1],[2],[3],[4],[5]. Specifically, let $y(t)$ be the temperature of a small object, controlled by the temperature of its environment, $k(t)$. Suppose that initially the object is at temperature y_0 and the heat transfer takes place at a constant rate of α. This system can be described by an ordinary differential equation

$$y'(t) = \alpha[k(t) - y(t)],$$
$$y(0) = y_0.$$

If we can control the ambient temperature $k(t)$, we could ask that the object reach a given temperature at time $t = T$, say $y(T) = y_1$. Is there a control? Can it be calculated ?.

The equation admits an explicit solution,

$$y(t) = e^{-\alpha t} y_0 + \alpha e^{-\alpha t} \int_0^t e^{\alpha s} k(s) ds.$$

Replacing the solution $y(T) = y_1$ we obtain,

$$\alpha \int_0^T e^{\alpha t} k(t) dt = e^{\alpha T} y_1 - y_0.$$

There is an infinite number of solutions, $k(t)$, to this equation. For example, for a constant control, $k(t) = k_0$, one computes easily,

$$k(t) = k_0 = \frac{e^{\alpha T} y_1 - y_0}{e^{\alpha T} - 1}.$$

1.1 Optimal control for differential equation ordinary

Let's find the control, $k(t)$, which minimizes the norm $L^2(0,T)$,

$$E_k(T) = \int_0^T |k(t)|^2 \, dt.$$

Such control exists and it is unique. The control function takes the form $k(t) = v_0 e^{\alpha(t-T)}$,

Or

$$v_0 = 2\frac{y_1 - e^{-\alpha T} y_0}{1 - e^{-2\alpha T}}.$$

We will see, later, how to compute such a control in a more general framework for partial differential equations.

Remark :

1/ A check exists for all $T > 0$; the initial and final states are arbitrary.

2/ The control found is the one that leads the solution to $y(t) = y_1$ and minimizes $E_k(T)$.

1.2 Control of an EDO system

The case of a system, for xa vector function of dimension n, is written :

$$x'(t) = Ax(t) + Bu(t),$$
$$x(0) = x_0,$$

where$A$ is a square matrix $(n \times n)$, B is a rectangular matrix $(n \times r)$ and u is the control of dimension r. When A is diagonalisable, each eigenmode can be controlled arbitrarily.

Theorem : A system $x' = Ax + Bu$ is said to be **controllable** if the controllability matrix $[B\ AB\ \dots\ A^{n-1}B]$ is of rank equal to n, the order of the system. When this is the case, one can control the system using a linear feedback control $u = -Kx$. This allows to write:

$$x' = (A - BK)x = A_c x,$$

and we can place the eigenvalues of the matrix A_c in the half-space $Re(\lambda) < 0$.

Example : Consider the equation for a simple pendulum:

$$\theta'' + \Omega^2 \theta = 0,$$

The equation is written in the form of a system as follows:

$$\frac{d}{dt}\begin{matrix} 0 \\ \omega \end{matrix} = \begin{matrix} 0 & 1 \\ -\Omega^2 & 0 \end{matrix}\ \begin{matrix} 0 \\ \omega \end{matrix}$$

and the eigenvalues of matrix A are obtained by $\det(\lambda I - A) = 0$ and so $\lambda_1 = i\Omega,\ \lambda_2 = -i\Omega$.

The system is marginally stable - it oscillates, periodically, constantly.

In general, for non-linear oscillations,

$$y'' + V'(y) = 0,$$

the energy E, the amplitude y_{max} and the period T are defined by

$$E = \frac{1}{2}(y')^2 + V(y) = V(y_{max})$$

and

$$T = 4\int_0^{y_{max}} \frac{dy}{\sqrt{2(E - V(y))}}.$$

In order to obtain energy, we multiply the equation by y' and we integrate. Energy is more kinetic potential. When the kinetic energy is zero, the oscillation is at full amplitude: $E = V(y_{max})$.

We are interested here in the feedback control of an unstable system. Some examples of application are: the autopilot of an airplane, the monitoring of a nuclear reactor, the control of chemical processes, and the thermostat of a heating system.

1.3 The inverted pendulum - analytical solution and stability

The physical problem considered is that of the inverted pendulum, which can model for example a rocket on a launch pad or a prosthetic leg.

At continuous time, we have the following equation:

$$\theta'' - \Omega^2\theta = 0,$$

we linearized with $sin\theta \simeq \theta$. This is rewritten as a system of first-order equations:

$$\theta' = \omega,$$
$$\omega' = \Omega^2\theta + u,$$

whereθ represents the angular position, ω the angular velocity and u the feedback control that we want to calculate.

It can be shown that the solution (equation without control) is given by

$$\begin{pmatrix}\theta \\ \omega\end{pmatrix} = \begin{pmatrix} cosh\Omega t & \Omega^{-1} sinh\Omega t \\ \Omega sinh\Omega t & cosh\Omega t\end{pmatrix}\begin{pmatrix}\theta_0 \\ \omega_0\end{pmatrix}$$

and that it grows without limit ...

$$\lim_{t\to\infty} \begin{pmatrix}\theta \\ \omega\end{pmatrix} = \infty.$$

Example : The matrices A and B so that we can form the following system:

$$x' = Ax + Bu.$$

Are

$$\frac{d}{dt}\begin{pmatrix}\theta \\ \omega\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Omega^2 & 0\end{pmatrix}\begin{pmatrix}\theta \\ \omega\end{pmatrix} + \begin{pmatrix}0 \\ 1\end{pmatrix} u.$$

Eigenvalues are the solution of the characteristic equation$\det(\lambda I - A) = 0$and so$\lambda_1 = \Omega, \lambda_2 = -\Omega$.So the system is unstable.

The system is controllable since the matrix of controllability :

$$C = [B \,\vdots\, AB] = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$$

which is clearly of rank 2.

1.4 Stability in the phase plan

In the general case, the stability of a system of ordinary differential equations

$$y' = F(y)$$

is obtained from the examination of the eigenvalues of the **linear stability matrix**,

$$S_L = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2}\end{bmatrix}$$

and we examine S_L at the **critical points**y^* which satisfies $F(y^*) = 0$. Linear stability implies the stability of the nonlinear system in a neighborhood of the critical points.

2. Control of an EDP

Consider a chord on the interval $[0, 1]$. For small oscillations, its motion can be described by the **wave equation**,

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \\ y(0, x) = y_0(x), \frac{\partial y}{\partial t}(0, x) = y_1(x), \\ y(t, 0) = 0, y(t, 1) = k(t), \end{cases}$$

for$0 \le t \le T$ and $0 \le x \le 1$. We apply here a **border control**, $k(t)$, at the right end, $x = 1$. We must, of course, specify all the functional spaces...

Control problem: find $k(t)$ such that, at time $t = T$,

$$y(T,x) = y_0^*(x), \frac{\partial y}{\partial t}\ T,x) = y_1^*(x).$$

2.1 Existence, uniqueness, causality, geometry

Is it possible to build such a control? It will be necessary to take into account:

- the functional spaces fory_0, y_1 and k;
- the minimum time for a wave to cross the string,$T \geq \frac{2}{c}$;
- the geometry (in 2 and 3 dimensions)

A robust and constructive way to find the solution is the **HUM** (Hilbert Uniqueness Method).

2.2 The control system

2.2.1 Notation

- an open domain, bounded $\Omega \subset \mathbb{R}^d$ with border$\Gamma$and time interval,$]0,T[$, $T > 0$,
- the space-time cylinder,$\Sigma =]0,T[\times \Gamma$
- the control border,$\Gamma_0 \subset \Gamma$ and $\Sigma_0 = 0,T[\times \Gamma_0$corresponding
- Note: in 1D,$\Omega$is the interval$]0,1[$and the edgecontrol$\Gamma_0$is the point $x = 1$.
- $y' = \frac{\partial y}{\partial t}and\Delta$ is the Laplacian operator

2.2.2 The control system

Consider the wave equation with a control over a part of the edge

$$y'' - \Delta y = 0 \ in \ Q = \Omega \times (0,T),$$

$$y(0,x) = y_0(x), y'(0,x) = y_1(x) \ in\ \Omega (U$$

$$y(t,x) = \begin{cases} k(t,x) \ on \ \Sigma_0 =]0,T[\times \Gamma_0, \\ 0 \ on \ \Sigma \setminus \Sigma_0 = \Gamma \setminus]0,T[\times \Gamma_0, \end{cases}$$

The problem of controllability by the edge is then:

for T, $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$given is what we can find a control$k \in L^2(\Sigma_0)$such as the solution of (U) verifies$y(T,x) = y'(T,x) = 0 \ in \ \Omega$?

The answer is "yes" if we take T large enough, and if we control on a set that is large enough and that satisfies certain geometric conditions.

2.3 Existence of a solution u

For all $(y_0, y_1) \in \tilde{\mathcal{E}}^* = L^2(\Omega) \times H^{-1}(\Omega)$and all$g \in \mathcal{B} = L^2(\Sigma_0)$, there is only one weak solution

$$(y, y') \in C([0,T]; \tilde{\mathcal{E}}^*)$$

and the application$\{y_0, y_1, k\} \mapsto \{y, y'\}$is linear; moreover, there exists a constant $c(T) > 0$ such that

$$(y, y')\|_{L^\infty(]0,T[;\tilde{\mathcal{E}}^*)} \leq c(T)(\|(y_0, y_1)\|_{\tilde{\mathcal{E}}^*} + \|k\|_{\mathcal{B}}).$$

Remark: The wave equation is reversible in time and the regularity is valid in both directions.

2.4 Types of Controllability

Is

$$R(T;\ (y_0, y_1) = \{(y(T,.), y'(T,.));\ y\ solution\ of\ (U)\}$$

set of reachable states with initial data$(y_0, y_1) \in \tilde{\mathcal{E}}^*$and control$k \in \mathcal{B}$.

Definition: The system is **exactly** controllable in time T if$R(T;\ (y_0, y_1)) = \tilde{\mathcal{E}}^*$ for all $y_0, y_1) \in \tilde{\mathcal{E}}^*$;**approximately** controllable in time T if $R\big(T;\ (y_0, y_1)\big)$ is dense in $\tilde{\mathcal{E}}^*$; **null** controllable if the state $(0,0) \in R\big(T;\ (y_0, y_1)\big)$.

Remark: For linear PDEs systems, null controllability and exact controllability are equivalent.

2.5 An auxiliary system

$$\psi'' - \Delta\psi = 0 \ \ in \ \ Q = \Omega \times (0, T),$$

$$\psi(T, x) = 0, \psi'(T, x) = 0 \ in \ \Omega (\mathrm{P})$$

$$\psi(t, x) = \begin{cases} k(t, x) \ on \ \Sigma_0 =]0, T[\times \Gamma_0, \\ 0 \ \ on \ \ \Sigma \setminus \Sigma_0 = \Gamma \setminus]0, T[\times \Gamma_0. \end{cases}$$

According to the existence theorem, $(\psi(t,.), \psi'(t,.)) \in \tilde{\mathcal{E}}^*$ and the problem of finding a k control that drives this system back to $(\psi(0,.), \psi'(0,.)) = (y_0, y_1)$ is **equivalent** to solving the original control problem.

2.6 The adjoint system

The HUM method is built on the relationship between the direct system (U), which is self-adjoint, and its **adjoint system**,

$$\phi'' - \Delta\phi = 0 \ \ in \ \ Q = \Omega \times (0, T),$$

$$\phi(0, x) = \phi_0(x), \phi'(0, x) = \phi_1(x) \ in \ \Omega (\mathrm{A})$$

$$\phi(t, x) = 0 \ \ on \ \ \Sigma = (0, T) \times \Gamma,$$

with initial data $(\phi_0, \phi_1) \in \mathcal{E} = H_0^1(\Omega) \times L^2(\Omega)$.

Theorem: For all $(\phi_0, \phi_1) \in \mathcal{E}$ the adjoint system admits a single weak solution

$$(\phi, \phi') \in C([0, T]; \mathcal{E}).$$

Furthermore $\frac{\partial \phi}{\partial \eta} \in L^2(\Sigma)$

and there is a constant $c(T) > 0$ such that

$$\|(\phi, \phi')\|_{L^\infty(]0,T[;\mathcal{E})} \leq c(T) \|(\phi_0, \phi_1)\|_{\mathcal{E}}.$$

Remark: The regularity-L^2 is stronger than the standard trace result (a half more ...) that could be obtained from $\phi(t,.) \in H_0^1(\Omega)$. This result is known as the "hidden regularity" of the wave equation.

3. Discrete Control

In order to calculate an approximate control, we must discretize the system ... We discretize the wave equation in 2 steps:

1. in space, which produces a semi-discrete model an ODE system!
2. in time, which produces the complete discrete system. We can use the control theory for linear ODE, but we lose the very rich Hilbertian structure, as well as the notion of control time, T.

3.1 HUM semi-discreet

- the wave equation is approximated by a system of N ordinary differential equations
- or X, of dimension $2N$, the approximation of space

$$\mathcal{E} = H_0^1(\Omega) \times L^2(\Omega)$$

and X^*, of dimension $2N$, the approximation of space

$$\mathcal{E}^* = H^{-1}(\Omega) \times L^2(\Omega)$$

- let $(y, z) \in \mathcal{E}, (u, v) \in \mathcal{E}^*$, and $[y, z]^T \in X$, $[u, v]^T \in X^*$ their approximations by vectors
- norms vectoriels are

$$\left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_X = \|y\|_1 + \|z\|_0$$

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{X^*} = \|u\|_{-1} + \|v\|_0$$

where $\|.\|_1, \|.\|_0, \|.\|_{-1}$, are approximations of the norms $H_0^1(\Omega), L^2(\Omega), H^{-1}(\Omega)$

- the product of duality$\langle .,. \rangle_{\mathcal{E}^*,\mathcal{E}}$is approached by

$$\left\langle \binom{u}{v}, \binom{y}{z} \right\rangle_{X,X^*} = \langle u, y\rangle_{-1,1} + \langle v, z\rangle_0$$

- discrete norms and their product of duality depending on the choice of semi-discretization (finite differences, finite elements, discontinuous elements, etc.)

3.2 Approximation

We introduce an approximation $(U_h)$of dimension $2N$the control system (U),

$$Y'(t) = L_h Y(t) + B_h k(t),\ \ t \in [0, T],$$
$$Y(0) = Y_0,$$

or

- the initial data (y_0, y_1) were projected onX^*giving$Y_0 = (y_0, y_1)^T$
- the function $k(t)$ is a border control applied at the right extremity of the domain
- the matrix L_h is an approximation of the spatial derivative
- matrix B_h affects the scalar boundary condition, $k(t)$ to the system

3.3 Semi-discrete control problem

We can define the problem of semi-discrete control: For $Y_0 \in X^*$given, find $k \in \mathcal{B}$ such that the discrete system (U_h) is led to zero at time $t = T$, that is to say, $Y(T) = 0$.

3.4 Adjoint system and semi-discrete observation

We introduce an approximation (A_h) of dimension$2N$ of the adjoint system (A),

$$W'(t) = L_h W(t),\ \ t \in [0, T],$$
$$W(0) = W_0,$$

for which the initial data$W_0 = (w_0, w_1)^T$correspond to the (ϕ_0, ϕ_1) of (A) and let $C_h W(t)$ be a discrete approximation of the normal derivative to $x = 1$,

$$\frac{\partial}{\partial \eta}\phi(t, 1) \simeq C_h W(t).$$

Finding the observation$C_h W(t)$ from the initial data W_0 is called **semi-discrete observation**.

Definition: Let $W_0 \in X$ be the initial data of the system (A_h) and let $W(t)$ be its solution. Calculating the output of Neumann $C_h W(t)$ is called the semi-discrete observation and the corresponding operator,

$$P^{sd}: X \longrightarrow \mathcal{B}$$

defined by

$$P^{sd}: W_0 \longrightarrow C_h W(t)$$

is called the operator semi-discrete observation “observation-sd (o-sd)”.

3.5 Retrograde system and semi-discrete reconstruction

We also define a semi-discrete version (R_h) of the auxiliary system (R)

$$Z'(t) = L_h Z(t) + B_h k(t),\ \ t \in [0, T],$$
$$Z(0) = 0,$$

which is resolved retrograde in time.

Definition: For a given function $k \in \mathcal{B}$, suppose that (R_h)admits a solution. Resolves $(R_h)$to get the output $[z'(0), -z(0)]^T \in X^*$ and call this operation the reconstruction-sd.

The corresponding operator,

$$R^{sd}: \mathcal{B} \longrightarrow X^*$$

defined by

$$R^{sd}: k \mapsto \begin{matrix} z'(0) \\ -z(0) \end{matrix}$$

is called the operator of reconstruction-sd.

3.6 Discrete control function

We are looking for a specific $k \in \mathcal{B}$ function, which checks the condition

$$R^{sd} k = \begin{pmatrix} y_1 \\ -y_0 \end{pmatrix}.$$

Such a function k will, by construction, solve the semi-discrete control system (U_h) and is therefore called a control. A control k that leads the semi-discrete system (U_h) to zero in time $t = T$ is called a HUM control, if it is calculated by

$$k = P^{sd} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix},$$

where$W_0 = [w_0, w_1]^T$ is a set of initial data for the semi-discrete adjoint (A_h).

3.7 Semi-discrete HUM operator

The equation for the semi-discrete HUM operator then becomes

$$L^{sd} \begin{matrix} w_0 \\ w_1 \end{matrix} = \begin{pmatrix} y_1 \\ -y_0 \end{pmatrix}, \qquad (*)$$

where the operator

$$L^{sd}: X \longrightarrow X^*$$

defined by

$$L^{sd} = R^{sd} P^{sd}$$

approach the continuous operator Λ,

$$\Lambda: \mathcal{E} \longrightarrow \mathcal{E} \quad such\ as \ \Lambda = \Psi \circ \Phi.$$

with the observation operator

$$\Phi: \mathcal{E} \longrightarrow \mathcal{B}$$

defined by

$$\Phi(\phi_0, \phi_1) = \frac{\partial \phi}{\partial \eta} \chi_{\Gamma_0}.$$

and the auxiliary system for is resolved backward in time. Introduce the reconstruction operator, Ψ, associated with this system

$$\Psi: \mathcal{B} \longrightarrow \mathcal{E}^*$$

defined by

$$\Psi: k \longmapsto (\psi'(0,.), -\psi(0,.)).$$

Thus$\Psi(k) = (y_1, -y_0)$.

3.8 Summary

1. The L^{sd} operator associates with the discrete initial data, $[w_0, w_1]^T$, the edge data of Neumann approached $C_h W(t)$.

2. Then, L^{sd} takes these data as a boundary condition of Dirichlet $k(t) = C_h W(t)$ and associates with it the state at $t = 0$, $z'(0), -z(0)]^T$.

3. If the solution $\overline{W}_0$of the semi-discrete HUM equation (*) exists, then it provides the desired control by $k = P^{sd} \overline{W}_0$.

3.9 Complete discretization of HUM

We introduce:

- uniform discretization in time of the interval $[0, T]$ by $t_m = m\Delta t$ for $m = 0, 1, \ldots, M-1$.
- the set of discrete operators that correspond to the choice of the integration scheme in time.

Definition 6: For an initial data $w(0), w'(0)]^T \in X$we define the discrete observation operator

$$P: X \longrightarrow \Upsilon$$

defined by

$$P: \begin{matrix} w_0 \\ w_1 \end{matrix} \longmapsto p,$$

where$p = [p(0), \ldots, p(M\Delta t)], \Upsilon = \mathbb{R}^M$with$p(m\Delta t) = C_h W(m\Delta t)$is the solution to (A_h) at the time step m.

Definition: For $k \in \Upsilon$ given, we define the discrete reconstruction operator

$$R: \Upsilon \longrightarrow X^*$$

defined by

$$PR: k \mapsto \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where$[z_1, -z_0]^T$ is the state of (R_h) at $t = 0$ after itsintegration of T to 0.

3.10 Discreet lambda

We can now define the discrete approximation of Λ,

$$L: X \longrightarrow X^*$$

defined by

$$L = RP.$$

As for the semi-discrete HUM operator, we introduce the equation for the discrete operator

$$L \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ -y_0 \end{pmatrix}. \qquad (**)$$

Its solution, $[\overline{w}_0, \overline{w}_1]^T$, if it exists, provides the control sought by

$$k^T = P \begin{pmatrix} \overline{w}_0 \\ \overline{w}_1 \end{pmatrix}.$$

The continuous operator Λ depends only on T (for Γ_0 fixed), but its approximation L also depends on:

1. The semi-discretization scheme L_h (element size h, order of approximation p).
2. The approximation of the normal derivative C_h.
3. The assignment of the Dirichlet condition with B_h.
4. Time integration: schema and Δt.

3.11 HUM numerical

The discrete HUM equation (**) can be solved directly by constructing L as a matrix, or iteratively.

Finding : The problem discreet and ill-posed! Indeed Stability + consistency $\not\Rightarrow$convergence.

Solutions:

0. Filtering by **two grids**.
1. Mixed finished elements.
2. Regulation of Tychonov.
3. Schemes uniformly controllable.

3.12 Iterative HUM by conjugate gradient

The works by [6], [7], [8], [9] proposed a preconditioned conjugate gradient algorithm to solve the HUM numerically. The conjugate gradient method is an **iterative algorithm** for solving the linear system

$$Ax = b,$$

where A is a matrix $(N \times N)$, symmetric, positive definite. This algorithm is the natural choice for HUM since the underlying operator,Λ, is self-adjoint and positive. We use a **preconditioning**, with a matrix M_p which is easy to reverse, so that the new problem,

$$M_p^{-1} A x = M_p^{-1} x$$

be easier to solve. The ideal preconditioning is$M_p = A^{-1}$.

3.13 Algorithm

For discrete initial conditions $[y_1, -y_0]^T$ given for the control problem (U_h), we aim to solve the preconditioned HUM problem,

$$\begin{pmatrix} M_p^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} L^1 & L^2 \\ L^3 & L^4 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} M_p^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y_1 \\ -y_0 \end{pmatrix}$$

where the preconditioned M_p is an approximation of Laplacian.

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