Inclined large-angle pendulum may produce endless linear motion of a cart when friction is negligible

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Abstract: - We present the mechanics for the oscillation of an inclined large-angle pendulum-drive attached to a cart which is allowed to perform translation in one direction only. Neglecting the overall friction, the application of Newton's second law shows that the oscillation of the pendulum is continuously converted into oscillating linear motion thus achieving a travel of infinite length. It is also shown that the frequency depends on the usual data of any pendulum plus the mass of the cart on which it is attached. After the determination of a novel effective pendulum length, a closed-form accurate analytical expression is presented for the amplitude of the pendulum, whereas semi-analytical formulas are provided for the period as well as the time-variation of the large azimuthal-like angle. Moreover, a simple expression was found for the position of the cart in terms of the azimuthal angle of the pendulum and the elapsed time. The extraction of the analytical formulas was facilitated by a computer model programmed in MATLAB®.

Keywords: - Large-angle pendulum, Motion conversion, Inertial drive, Frictionless model, Perpetual device

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1. Introduction

The term 'inertial propulsion' was introduced probably by the late Professor Eric Laithwaite (1921–1997), who is well known from his Christmas lectures on gyroscopes at Imperial College London in the United Kingdom, [1]. Although most of his ideas on inertial propulsion and gyroscopic thrust have been in advance rejected, the interest is still alive and some of the relevant mechanics are still covered by mystery. For example, Wayte, [2], conducted an experimental study in favor of Laithwaite while recently Provatidis, [3], revealed some of the associated mechanics and how a physicist may be cheated, [4].

Except for the aforementioned gyroscopes, contra-rotating eccentrics (the latter called Dean drive, [5]) as well as equivalent electromagnetic means have been also studied jointly by academics and the industry for possible suitability in alternative propulsion, [6]. Nevertheless, since the inertial forces are internal to the mechanical system, the resulting net thrust over a time period is null thus instead of propulsion at the best case we obtain a 'catapult' or a marching device like the bumper of a mobile phone. To become more specific, it is indisputable that under certain conditions an inertial drive may cause an *initial velocity* to an object so as the center of mass of the mechanical system moves as usual.

In this paper we study an alternative source of inertial propulsion, i.e. an inertial drive which is based on the oscillating motion of a pendulum and particularly in conjunction with large angles. The beneficiary characteristic is the very low friction which occurs in a pendulum and makes it superior (it takes much time to cease) compared to the motion of fully rotating eccentrics. Within this context, this study is probably the first publication of this kind on the use of the pendulum as an inertial drive.

From the pedagogical point of view, it is shown that the initial kinetic and potential energy of a pendulum can be easily converted into linear (translational) motion when it is attached to a movable cart. For the very theoretical assumption of negligible friction, the initial velocity of the center of mass causes the motion of the cart forever. Of course, the physical reality is different, thus after a few oscillations the appearance of friction on a real prototype, [7], makes the translational motion of the cart to cease. Therefore, this open system gives us the opportunity to discuss the subject of the conservation of energy and linear momentum, as well as the non-conservation of angular momentum, all of which are of major importance in the education of physics and mechanics.

2. Basic theory

Let us consider an inertial Cartesian coordinate system $OX_gY_gZ_g$ fixed to the Earth, in which the horizontal ground (on which the cart moves) is parallel to the X_gY_g-plane while Z_g is the vertical axis. For the sake of simplicity, at the initial time ($t = 0$) we consider a chassis (cart) as a concentrated mass exactly at the aforementioned point, O, so as the body-fixed coordinate system $O'xyz$ initially coincides with the fixed system $OX_gY_gZ_g$.

The mechanical system consists of the abovementioned chassis (cart with its wheels) on the horizontal ground (xy-plane) on which an inclined pivot is attached at the moving point O' (origin of body fixed axis on the cart, with global coordinates $X_{O',g} = X(t)$, $Y_{O',g}$=0 and $Z_{O',g} = 0$) while a large-angle pendulum rotates about the aforementioned body fixed axis $O'z'$ that is inclined by angle α with respect to the body fixed vertical axis $O'z$, as shown in Fig. 1a. In other words, the axis $O'z'$ is produced by rotating the initial global system $OX_gY_gZ_g$ about the axis OY_g by angle α, and then the resulting system is free to translate in the X_g-direction so as the pivot O' of the cart can shift from its initial position O to the final O'.

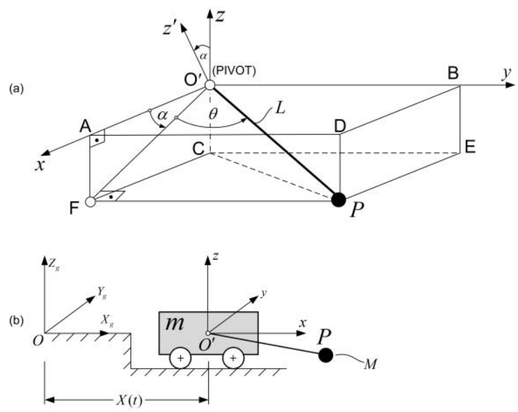

Fig. 1: (a) Coordinate system and (b) Set up of the inclined pendulum.

Obviously, with respect to the global (Earth fixed) system $OX_gY_gZ_g$, the upward unit vector $\vec{n}$ along the axis of rotation $O'z'$ will be given by:

$$(n_x, n_y, n_z) = (\sin\alpha, 0, \cos\alpha) \tag{1}$$

In accordance to the experiment, [7], the overall mechanical system is governed by two degrees of freedom (DOF), the former being the displacement, $X(t)$, of the chassis (cart) and the latter is the angle, $\theta(t)$, which is somehow related to the azimuthal angle but not exactly. Clearly, instead of the usual Euler angles, due to the constant inclination α ($> 0, slant\ angle$), in this manuscript we have reduced them in only one (see, Fig. 1), practically in the interval $-\pi/2 < \theta < \pi/2$, as explained below. Clearly, the position $\theta = -\pi/2$ corresponds to the extension of the axis $O'y$ in the negative direction, while the position $\theta = \pi/2$ corresponds exactly to it.

First of all, we divert the pendulum at an angle θ in absolute value larger than -90 degrees (at maximum in the negative extension of x-axis), and then we leave it rotate in the anti-clockwise direction due to its weight. Therefore, when the axis of the pendulum reaches the position $\theta = -\pi/2$ at time $t = 0$, the accelerated downward motion gives an initial angular velocity $\dot{\theta}_0 \neq 0$. At this position we may also assume that the cart is at rest, i.e., $X_0 = 0$ and $\dot{X}_0 = 0$, at $t = 0$.

Therefore, with respect to the fixed to the Earth global system $OXYZ$, the position of the oscillating mass, which is attached to the point $P(x,y,z)$ at the end of the bob, is given by:

$$\begin{aligned} x_b &= X(t) + L\cos\theta\cos\alpha, \\ y_b &= L\sin\theta, \\ z_b &= -L\cos\theta\sin\alpha, \end{aligned} \tag{2}$$

where L is the length (OP) while the subscript 'b' stands for the word 'bob'.

Let m and M be the masses of the cart and the bob, respectively. Between several alternative ways to derive the equations of motion, we consider the *center of mass* (subscript 'c') of the mechanical system "cart + bob", of which the coordinates are:

$$\begin{aligned} x_c &= \frac{mX + Mx_b}{m+M}, \\ y_c &= \frac{mY + My_b}{m+M}, \text{ with } Y \equiv 0, \\ z_c &= \frac{mZ + Mz_b}{m+M}, \text{ with } Z \equiv 0 \end{aligned} \tag{3}$$

with $(X,Y,Z) = (X,0,0)$ denoting the coordinates of the cart in the global inertial system fixed to the Earth, while (x_b, y_b, z_b) are the coordinates of the bob given by Eq. (2).

According to Newton's Second law, the sum of all external forces in the x-(horizontal) direction equals the total mass times the acceleration of the center of mass. Since the friction is zero (temporarily), there is no external force in this direction (otherwise is $-\mu F_{support,z}$), thus we have $(m+M)\ddot{x}_c = 0$. By virtue of Eq. (2) and (3), after rearrangement of the terms this equation eventually becomes:

$$(m+M)X + (ML\cos\alpha)(\cos\theta)^{gg} = 0\,. \tag{4}$$

Equation (4) is a relationship between the second temporal derivatives of the two DOF, $X(t)$ and $\theta(t)$, thus one of them may be eliminated. Although the purpose of this paper is to derive the cart displacement $X(t)$, it seems that the primary variable is the azimuthal-like angle $\theta(t)$. Therefore, after two successive integrations of Eq. (4) in time t, we eventually derive the general solution:

$$\begin{aligned} X = X_0 &+ A(\cos\theta_0 - \cos\theta) \\ &+ (X_0 - A\theta_0\sin\theta_0)\cdot t, \end{aligned} \tag{5a}$$

in which, for convenience, we have introduced the following constant A:

$$A = \frac{ML\cos\alpha}{m+M}\,. \tag{6}$$

Taking the first derivative of $X(t)$ with respect to time t, Eq. (5a) becomes:

$$X = X_0 + A(\theta\sin\theta - \theta_0\sin\theta_0). \tag{5b}$$

One may observe that Eq. (5a) includes the initial conditions $(X_0, \dot{X}_0)$ for the cart, as well as the initial conditions $(\theta_0, \dot{\theta}_0)$ for the angle $\theta(t)$ of the pendulum, and obviously satisfies the ODE (4). But the most interesting issue in Eq. (5a) is that $X(t)$ consists of a bounded term of amplitude $\pm 2A$ plus a linear term of the form at with the constant a being equal to $a = (X_0 - A\theta_0\sin\theta_0)$. This fact clearly shows that the cart will perform an oscillatory motion of amplitude $2A$ but it continuously moves in the positive X-direction.

Obviously, the above remark regarding the continuous motion $X(t)$ of the cart is not peculiar and this happens simply because the center of mass has an *initial velocity*, which due to the absence of friction is preserved forever. From the practical point of view, as also can be noticed in a video of a prototype device, [7], the pendulum is diverted (lifted) at an angle θ_{lift}, at which (according to Eq. (2)) the potential energy of the bob mass will be equal to:

$$E_{pot}^{lift} = Mgz_b = -MgL\cos\theta\sin\alpha \tag{7a}$$

Setting in Eq. (7a) the technically maximum possible value (most upper position of the pendulum) $\theta_{lift} = -\pi$ at which the cosine becomes equal to (-1), the aforementioned initial potential energy (produced from the maximum possible rotation of the pendulum on the slant plane) becomes:

$$\left(E_{pot}^{lift}\right)_{\max} = MgL\sin\alpha \tag{7b}$$

Therefore, keeping the cart *firmly fixed*, as the pendulum rotates in the anti-clockwise direction by lowering the height z_b and thus increasing its angular velocity $\dot{\theta}(t)$, the energy conservation for the pendulum gives:

$$\begin{aligned} &-(MgL\sin\alpha)\cos\theta_{lift} = \\ &\quad -(MgL\sin\alpha)\cos\theta + \frac{1}{2}(ML^2)(\theta_0)^2 \end{aligned} \tag{8a}$$

Setting $\theta_{lift} = -\pi$ and $\theta_0 = -\pi/2$, Eq. (8a) gives the maximum possible initial angular velocity at $\theta_0 = -\pi/2$, given by:

$$(\theta_0)_{\max} = \sqrt{\frac{2g\sin\alpha}{L}} \tag{8b}$$

Now, exactly when the pendulum passes through the position $\theta_0 = -\pi/2$ at any given initial angular velocity $\dot{\theta}_0$, not necessarily equal to that maximum of Eq. (8b), the cart is *suddenly left free to move* according to the dominating physical laws. Henceforth, the problem is to determine the function $\theta(t)$ and then, applying Eq. (5a), to determine the position $X(t)$ of the cart. In other words, despite the motion of the cart, the problem decoupled as we have to solve only the large angle oscillation of the pendulum. Nevertheless, it does not fit the well-known forms in physics (mechanics).

3. Solution of the mechanical model

For the sake of conservatism, we derive the equation of motion considering the equation of *energy conservation*. The mechanical system includes two kinetic energies for the cart (of mass m) and the bob (of mass M), respectively, and also the potential energy of the bob mass.

In more detail, the kinetic energy of the cart is simply given by:

$$E_{kin}^{cart} = \frac{1}{2}mX^2, \tag{9}$$

Also, according to Eq. (2) the velocity components of the bob will be:

$$\begin{aligned} x_b &= X - L\theta\sin\theta\cos\alpha, \\ y_b &= L\theta\cos\theta, \\ z_b &= L\theta\sin\theta\sin\alpha, \end{aligned} \tag{10}$$

thus the kinetic energy of the bob mass, with respect to an inertial system fixed to the Earth is given by:

$$E_{kin}^{bob} = \frac{1}{2}M(x_b^2 + y_b^2 + z_b^2) \tag{11a}$$

Substitution of Eq. (10) into Eq. (11a), after manipulation we eventually receive:

$$E_{kin}^{bob} = \frac{1}{2}M(X^2 + L^2\theta^2 - 2LX\theta\sin\theta\cos\alpha) \tag{11b}$$

In addition, the potential energy of the bob mass is given by:

$$\begin{aligned} E_{pot}^{bob} &= Mgz_{bob} \\ &= Mg(0 - L\cos\theta\sin\alpha) \end{aligned} \tag{12}$$

Summing Eqs. (9), (11b) and (12), after rearrangement of the terms the total mechanical energy of the system takes the form:

$$\begin{aligned} E_{total} &= \frac{1}{2}(m+M)X^2 + \frac{1}{2}(ML^2)\theta^2 \\ &\quad -(ML\cos\alpha)X\theta\sin\theta \\ &\quad -(MgL\sin\alpha)\cos\theta. \end{aligned} \tag{13}$$

Substituting Eq. (5a) into (13), after elaboration the latter takes the form:

$$\boxed{B\theta^2 + C = E_0} \tag{14}$$

with the variables B and C being functions of only the degree of freedom θ (and the initial conditions as well), as follows:

$$B = -\frac{1}{2}(m+M)A^2\sin^2\theta + \frac{1}{2}ML^2 \tag{15}$$

and

$$C = \frac{1}{2}(m+M)(X_0 - A\theta_0 \sin\theta_0)^2 - (MgL\sin\alpha)\cos\theta \tag{16}$$

Also, E_0 in Eq. (14) is the initial value of the total energy, which according to (13) equals to:

$$\begin{aligned} E_0 &= \frac{1}{2}(m+M){X_0}^2 + \frac{1}{2}(ML^2){\theta_0}^2 \\ &- (ML\cos\alpha)X_0\theta_0\sin\theta_0 \\ &- (MgL\sin\alpha)\cos\theta_0 \end{aligned} \tag{17}$$

Below we present two alternative equations of motion, of first and second order respectively, which have to be numerically solved.

3.1 First-order differential equation

The energy conservation Eq. (14) is directly solved in $\dot{\theta}$ thus we receive:

$$\theta = \pm\sqrt{\frac{(E_0 - C)}{B}}\,, \tag{18}$$

Equation (18) is a first-order ordinary differential equation of $\theta(t)$ in t , thus can be easily solved in a numerical way, for example implementing the Runge-Kutta algorithm. The only difficulty with Eq. (18) is the sign $\pm$ which is related to whether the bob keeps going in a certain direction. In more detail, considering the definition of the azimuthal angle according to Fig. 1a, when the bob rotates in the anti-clockwise direction (thus increasing the function $\theta(t)$) the proper sign is $(+)$ whereas when rotates in the clockwise direction (thus decreasing the function $\theta(t)$) the suitable sign is $(-)$.

To control the sign in the above mentioned procedure, it is useful to determine the two positions at which the condition $\dot{\theta} = 0$ is met, called "*turn points*" (extreme oscillation points), and correspond to the maximum possible angle θ_{max} (amplitude $\theta_a = 2\theta_{max}$). Therefore, setting in Eq. (18) the condition $\dot{\theta} = 0$, it simplifies to $C = E_0$, and then we obtain:

$$\boxed{\cos\theta_{\max} = \frac{\frac{1}{2}(m+M)(X_0 - A\theta_0\sin\theta_0)^2 - E_0}{(MgL\sin\alpha)}} \tag{19a}$$

Then, the maximum value (half-amplitude) is:

$$\theta_{\max} = \cos^{-1}\left(\frac{L\theta_0^2}{2g\sin\alpha}\left\lfloor\frac{M\cos^2\alpha}{(m+M)} - 1\right\rfloor\right) \tag{20a}$$

Therefore, in the first half-period (angle interval $[-\theta_{max}, \theta_{max}]$) the sign is (+), in the second half-period the sign is (–), in the third half-period the sign is (+), and so on.

3.2 Second-order differential equation

Taking the first derivative in time of both parts in Eq. (14), we derive $\dot{B}\dot{\theta}^2 + 2B\dot{\theta}\ddot{\theta} + \dot{C} = 0$, whence we can solve in $\ddot{\theta}$:

$$\theta = -\frac{B\theta}{2B} - \frac{C}{2B\theta} \tag{21}$$

After substitution of Eq. (15) and Eq. (16) in Eq.(21), and then replacing the constant A according to Eq. (6), we eventually obtain the desired 2nd-order ODE:

$$\boxed{\theta = \frac{\sin\theta\left\lfloor g(M+m)\sin\alpha - \theta^2 LM\cos^2\alpha\cos\theta\right\rfloor}{L(M\cos^2\alpha\sin^2\theta - M - m)}} \tag{22}$$

The advantage of Eq. (22) compared to Eq. (18) is that it requires no book-keeping on the sign in front of the square root, while the disadvantage is that its numerical solution does not ensure energy conservation. Of course, since Eq. (22) was derived from the energy conservation after differentiation, the initial energy is lost and is merely substituted by the initial conditions $(\theta_0, \dot{\theta}_0)$ that ensure E_0, according to Eq. (17). In any case, the degree in which the numerical solution ensures the energy conservation is a safe criterion to measure and judge its quality.

4. Estimation of the time period

4.1 A primitive model

One may observe that the equation of motion, Eq. (22), differs from the following standard nonlinear equation that characterizes large-angle pendulum oscillations:

$$\theta + \frac{g}{L}\sin\theta(t) \approx 0\,, \tag{23}$$

As a very primitive approximation, we may set the term $\sin^2\theta$ in the denominator of Eq. (22) equal to unity thus the $\cos\theta$ in the numerator will be equal to zero. Then, transferring all terms in the left-hand side we get the approximation:

$$\theta + g\left\lfloor \frac{(M+m)\sin\alpha}{L(-M\cos^2\alpha + M + m)} \right\rfloor \sin\theta \approx 0 \tag{24}$$

Comparing the very approximate Eq. (24) with the golden standard Eq. (23), one may observe that the model of this paper is related to an effective length given by:

$$L_{eff} = L\frac{(-M\cos^2\alpha + M + m)}{(M+m)\sin\alpha} \tag{25}$$

Having obtained the above efficient (equivalent) length L_{eff}, according to the state-of-the-art the small angle oscillation formula $T = 2\pi\sqrt{L_{eff}/g}$ is replaced by one of the following alternative formulas, (see, [8], [9]):

$$\begin{aligned} T_1 &= 2\pi\sqrt{L_{eff}/g}\left(1 + \frac{\theta_{\max}^2}{16}\right) \\ T_2 &= 2\pi\sqrt{L_{eff}/g}\left(\cos\left(\frac{\theta_{\max}}{2}\right)\right)^{-\frac{1}{2}} \\ T_3 &= 2\pi\sqrt{L_{eff}/g}\left(\frac{\sin\theta_{\max}}{\theta_{\max}}\right)^{-\frac{3}{8}} \end{aligned} \tag{26}$$

4.2 A more accurate approximate model

Starting from the definition of the instantaneous angular velocity, $\omega = d\theta/dt$, we can derive $dt = d\theta/\omega$, after integrating in the angle interval $[-\theta_{max}, \theta_{max}]$ we can derive the half-period $T/2$, thus for the entire period T we get

$$T = 2\int_{-\theta_{\max}}^{\theta_{\max}} \frac{d\theta}{\omega(\theta)} \tag{27}$$

Working for the first half-period in which the sign of Eq. (18) is positive, thus considering that $\omega = \sqrt{(E_0 - C)/B} = \omega(\theta)$, the angular velocity can be written in terms of four constants as follows:

$$\omega(\theta) = \sqrt{\frac{a + b\cos\theta}{c + d\sin^2\theta}}\,, \tag{28}$$

with

$$\begin{aligned} a &= L\dot{\theta}_0^2(m + M - M\cos^2\alpha) \\ b &= 2g(m+M)\sin\alpha \\ c &= L(m+M) \\ d &= -LM\cos^2\alpha \end{aligned} \tag{29}$$

The infinite integral of Eq. (27) in conjunction with Eq. (28) cannot be analytically found, however instead one can easily apply either the Simpson's trapezoidal rule or Gauss numerical integration.

Alternatively, we can resort to the substitution of the involved trigonometric functions by:

$$\begin{aligned} \cos\theta &= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{720} + \cdots \\ \sin\theta &= \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \frac{\theta^7}{5040} + \cdots, \end{aligned} \tag{30}$$

but then only a lengthy expression in the form of the Taylor series may be derived using symbol manipulation software such as MATHEMATICA®.

Whatever method is applied to determine the period T, having determined the half-amplitude θ_{max} using Eq. (20a), and knowing that at the initial time $t = 0$ corresponds to a position that is a little *after* the initial turn point, a closed-form analytical solution could be:

$$\theta(t) \approx -\left|\theta_{\max}\right|\cos(\bar{\omega}t + \bar{\alpha})\,, \tag{31}$$

with $\bar{\omega} = 2\pi/T$ denoting the average angular frequency.

Applying Eq. (31) for $t = 0$, we receive $\theta_0 = -\left|\theta_{\max}\right|\cos\bar{\alpha}$, thus we can derive the unknown phase difference $\bar{\alpha}$ from the formula:

$$\bar{\alpha} = \cos^{-1}\left(-\frac{\theta_0}{\left|\theta_{\max}\right|}\right) \tag{32}$$

Combining (31) and (32) we obtain the following approximation:

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$$\theta(t) \approx -|\theta_{\max}| \cos\left[\bar{\omega} t + \cos^{-1}\left(-\frac{\theta_0}{|\theta_{\max}|}\right)\right] \qquad (33)$$

5. Numerical application

Based on the experimental prototype, [7], the parameters of the device under consideration are:

M = 0.445 kg (bob mass)

m = 2.500 kg (chassis, i.e. cart without bob)

α =0.203 radians (11.65 degrees)

L = 0.225 m (shaft length)

g= 9.81 m/s^2, gravitational acceleration

The initial conditions for the two variables at $t = 0$ are:

$X_0 = 0$ (initial position of the cart)

$\dot{X}_0 = 0$ (initial velocity of the cart)

$\theta_0 = -\frac{\pi}{2}$ (90 degrees), initial azimuthal angle

$\dot{\theta}_0 = 0.1$ (rad/s), initial azimuthal angular velocity.

5.1 Time response

It is noted that, according to Eq. (8b), the maximum possible initial angular velocity is $(\dot{\theta}_0)_{max} = 4.1929\ rad/sec$. Therefore, the selected value $\dot{\theta}_0 = 0.1$ is a rather small one.

For the particular conditions, ($\theta_0 = \pi/2$, $\dot{\theta}_0 = 0$, as well as $X_0 = 0$ and $\dot{X}_0 = 0$), Eq. (19a) provides:

$$\cos\theta_{\max} = 0.999999881725856 \qquad (19b)$$

thus:

$$\begin{aligned}\theta_{\max} &= 1.571282689119088 \text{ radians,} \\ &\quad 90.027866508490348 \text{ degrees}\end{aligned} \qquad (20b)$$

One may observe that Eq. (20b) provides two angles, one close to the initial position $(-\pi/2)$ and another close to the final position $(+\pi/2)$ of the half oscillation. From the computational point of view, these two values are valuable to control the sign in Eq. (18) when the first-order formulation is used, so that when they are met the sign immediately changes into its opposite value than the previous one. In contrast, the second-order formulation given by Eq. (22) recognizes the reversal of motion (turn points) automatically, because the unknown variable $\dot{\theta}$ is a part of the solution.

For $0 < t < 6.5\ sec$, the obtained results are shown from Fig. 2, Fig. 2, Fig. 3, Fig. 4, Fig. 5, and Fig. 6. One may observe that the overall behavior of the results is very similar to other inertial drives, i.e., it constitutes an oscillating motion around a linearly increased (with respect to time) displacement (according to Eq. (5a), (5b)). In other words, the cart, of mass $m = 2.5\ kg$, is progressively displaced to the x-direction.

The absolute velocity of the cart is according to the derivative of $X(t)$ described by Eq. (5b). Therefore, for the particular initial conditions, ($X_0 = 0$, $\dot{X}_0 = 0$, and $\theta_0 = -\pi/2$), it is given as $\dot{X} = A(\theta\sin\theta - \theta_0\sin\theta_0)$, thus the absolute velocity of the cart oscillates in accordance to a repeated pattern, as shown in Fig. 3: .

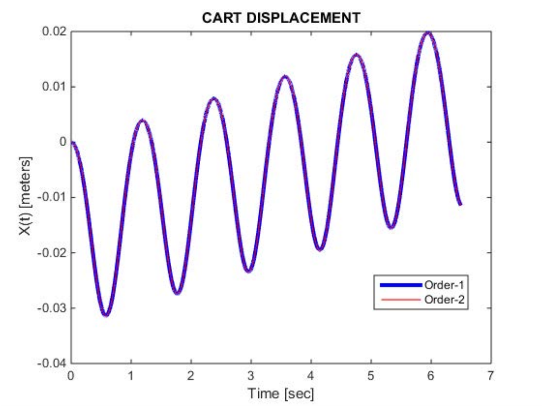

Fig. 2: Cart displacement.

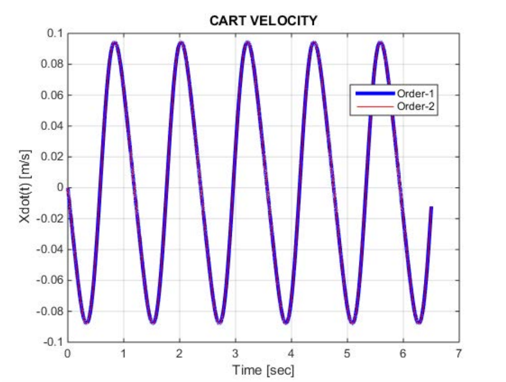

Fig. 3: Cart velocity.

Next, the calculated "azimuthal" (longitudinal) angle $\theta(t)$ is harmonic as shown in Fig. 4, and again, the two formulations (first- and second-order ODEs) visually coincide with one another.

Also, the 'azimuthal' angular velocity $\omega = \dot{\theta}(t)$ is harmonic, as shown in Fig. 5. Interestingly, in contrast to the well known contra-rotating drives which are based on an almost constant rotation ω_0, [10], in the case of this paper the function of the instantaneous cyclic frequency $\omega(t)$ is harmonic.

Moreover, the altitude $z_b(t)$ of the bob mass is shown in Fig. 6, and depicts that the bob mass is always below the horizontal plane.

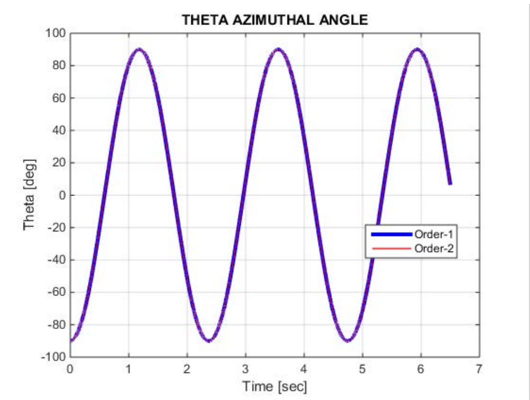

Fig. 4: "Azimuthal" angle $\theta(t)$.

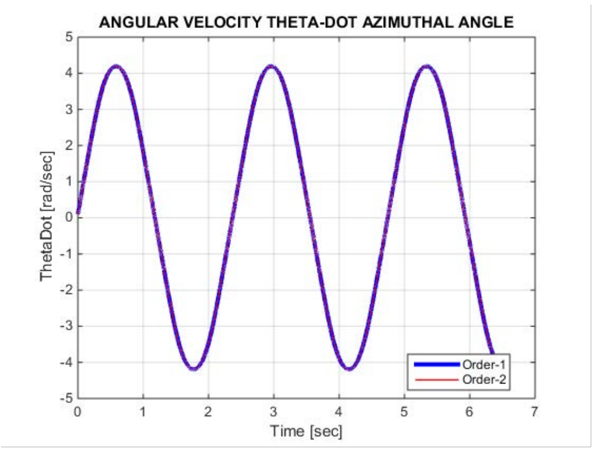

Fig. 5: "Azimuthal" angular velocity $\omega = \dot{\theta}(t)$.

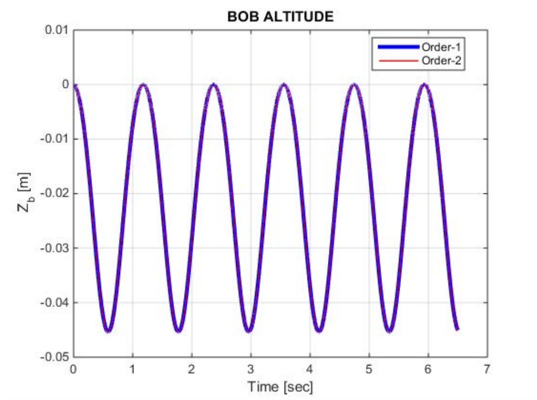

Fig. 6: Altitude of bob mass (z-coordinate).

5.2 Energy Breakdown

In the first formulation (first-order ODE) the energy conservation is ensured *per se* since the governing equation, Eq. (18), is coming from it.

In the second formulation (second-order ODE, i.e., Eq. (22)) the conservation of total energy works as a criterion to judge the quality of the numerical solution.

Actually, while in both formulations the total energy is practically preserved at the level of 1.1264×10^{-4} Joule, Fig. 7 shows that in the second formulation it slightly decreases.

Regarding the energy breakdown in kinetic and potential energy, Fig. 8 shows that the kinetic energy of the cart is very small compared to the kinetic energy and potential energy of the bob. Note that, although due to the scale the total energy seems close to zero, actually it is equal to 1.1264×10^{-4} Joule, as also mentioned above.

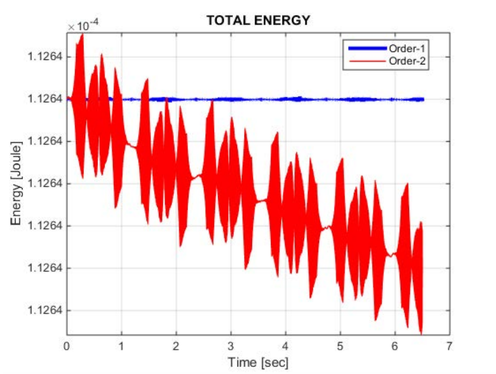

Fig. 7: Energy conservation.

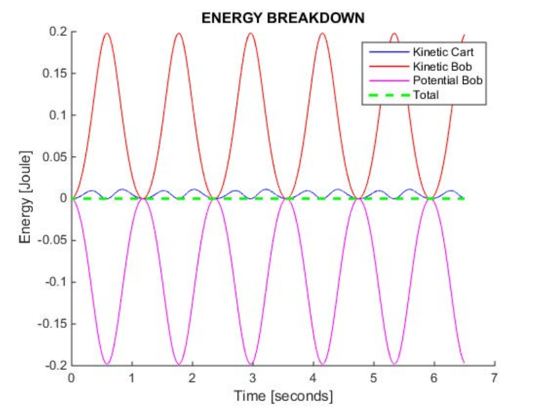

Fig. 8: Energy breakdown.

Now, we present results regarding the analytical estimation of the time period T of the oscillation using Eq. (26), as well as the possible approximation of the azimuthal angle function $\theta(t)$. Considering the time period $T_{RK} = 2.3769\, sec$ coming from the peak-to-peak graph in the Runge-Kutta solution (in our case we have: $\theta_0 = -\pi/2, \dot{\theta}_0 = 0.1\, rad/sec$), the comparison of the latter with the approximate values (Eq. (26)) is adequately satisfactory, as shown in Table 1.

Table 1. Ratio of the estimated period over the accurate T_{RK} (Runge-Kutta) value

Normalized period Eq. (26)	**Value**
T_1/T_{RK}	0.9517
T_2/T_{RK}	0.9806
T_3/T_{RK}	0.9767

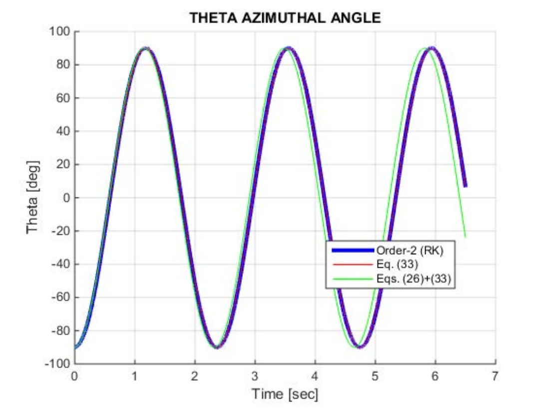

Fig. 9: Approximation of the angle $\theta(t)$.

It is noted that when using the first two terms of Eq. (30) and then calculate the integral of Eq. (27) through a series expansion (powers of $x, x^3, x^5, x^7, x^9, x^{11}$) using MATHEMATICA®, the calculated value is not of adequate accuracy ($T/T_{RK} = 0.9265$), that is worse than all those shown in Table 1.

Then, in Fig. 9 we present the estimation of the azimuthal angle $\theta(t)$, in two ways. The former is based on the actual period $T_{RK} = 2.3769 sec$ [in the legend of the graph is labeled as 'Eq. (33)'], while the latter is based on the best estimation in Table 1 [in the legend of the graph is labeled as 'Eqs. (26)+(33)']. One may observe that the former visually coincides with the Runge-Kutta solution whether in the progress of time the latter (with $T/T_{RK} = 0.9806$) appears a small shift to the left.

In any case, having a closed-form analytical solution for $\theta(t)$ given by Eq. (33), we can immediately apply Eq. (5a) and thus determine the cart position $X(t)$. The conservation of Linear Momentumn is presented in Fig. 10.

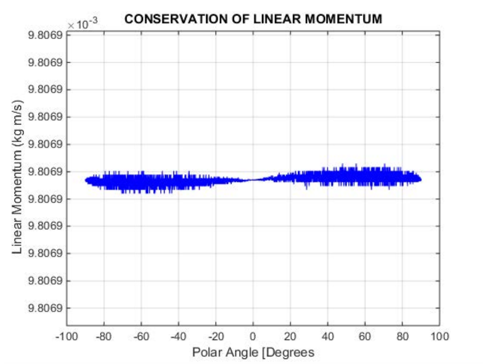

Fig. 10: Conservation of Linear Momentum.

6. Conservation of linear and angular momentum

6.1 Linear momentum

Regarding the conservation of the linear momentum in the system, Fig. 1 shows that it is preserved at the level of $P_{total} = 9.8069\ kg\ m/s$.

From the theoretical point of view, the total momentum equals that of the center of mass which possesses a *constant* velocity (the initial one) because no external force is exerted in the x-direction. Therefore, by virtue of the first equality in Eqs. (3) we have:

$$P_{total} = mX + Mx_b = (m+M)x_{cm}. \quad (34)$$

And since the initial velocity of the cart is $\dot{X}_0 = 0$ whereas that of the bob is $(\dot{x}_b)_0 = \dot{X}_0 - L\dot{\theta}_0 \sin\theta_0 \cos\alpha$, the initial linear momentum P_0 of the system will be:

$$\begin{aligned} P_0 &= mX_0 + M\left(X_0 - L\theta_0 \sin\theta_0 \cos\alpha\right) \\ &= (m+M)X_0 - (ML\cos\alpha)\theta_0 \sin\theta_0. \end{aligned} \quad (35)$$

Therefore, for the particular initial conditions adopted in Section V (i.e., $\theta_0 = -\pi/2$ and $\dot{\theta}_0 = 0.1\ rad/sec$), we receive:

$$\begin{aligned} P_0 &= -(0.445 \times 0.225 \times \cos 0.203) \times 0.1 \times (-1) \\ &= 9.8069 \times 10^{-3}\ \text{kg m/s}. \end{aligned}$$

Then we shall show the 'mechanism' according to which the momentum of the bob mass is transferred to the cart. Actually, when the pendulum reaches the most forward position $\theta = 0$ (almost at the end of the first quarter of the oscillation period), the first component of the velocity (i.e., $\dot{x}_b = \dot{X} - L\dot{\theta}\sin\theta\cos\alpha$) of the bob mass becomes $(\dot{x}_b)_{\theta=0} = \dot{X}$, that means it shares the same velocity with the cart, thus the total momentum is $(m+M)\left(\dot{X}\right)_{\theta=0}$, whence $\left(\dot{X}\right)_{\theta=0} = P_0/(m+M) = 0.0033\ m/s$ and this point corresponds to the intersection of the two branches in the center of the eight-shaped curve shown in Fig. 11.

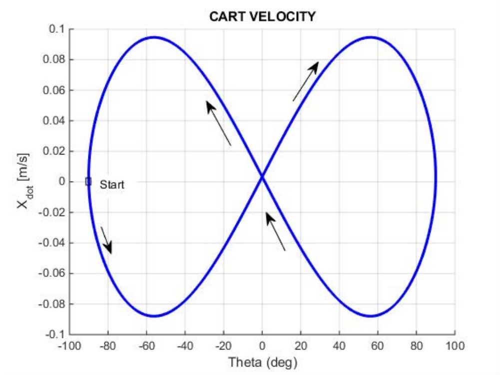

Fig. 11: Cart velocity versus azimuthal angle θ.

As the bob mass keeps moving toward the position about $\theta = \pi/2$, it loses momentum (it becomes negative) which is gained by the cart thus increasing the travel length X. The whole procedure is shown in Fig. 12, with respect to both time and angle parameters.

6.2 Angular momentum

Regarding the angular momentum, first of all it is interesting to note that the rigid rod $O'P$ is always perpendicular to the unit vector $\vec{n}$ that is normal to the slant plane, and the same happens with the relative velocities of the bob and cart mass with respect to the center of mass of the system. In other words, these two masses rotate about an axis of constant direction (i.e., parallel to the $\vec{n}$-vector) given by Eq. (1), which (axis) always passes through the moving center of mass.

The position of the center of mass is given by the distances L_1 and L_2 from the cart and the bob mass, respectively, given by:

$$L_1 = \frac{M}{m+M}L, \text{ and } L_2 = \frac{m}{m+M}L \quad (36)$$

Therefore the total second moment of inertia is

$$I = mL_1^2 + ML_2^2 = \frac{mM}{(m+M)}L^2. \quad (37)$$

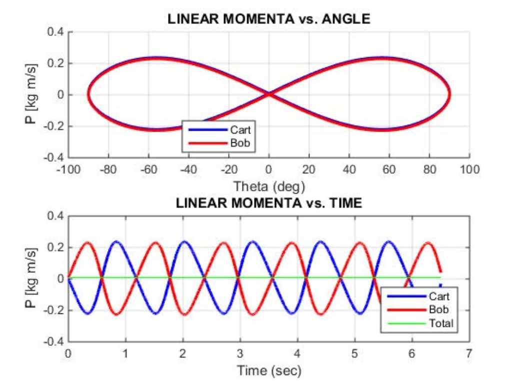

Fig. 12: Exchange of linear momentum between pendulum and cart.

With respect to the inclined $\vec{n}$-axis of rotation (passing through the center of mass), the equation of motion (Newton's second law for rotation) is written as follows (see, [11], [12]):

$$\boxed{\frac{\mathrm{d}\vec{\mathbf{G}}}{\mathrm{d}t} = I\frac{\mathrm{d}^2\vec{\theta}}{\mathrm{d}t^2} = \vec{\boldsymbol{\tau}}_{ext}} \tag{38}$$

with $\vec{\theta} = \theta\vec{n}$ and $\vec{\boldsymbol{\tau}}_{ext} = \tau_{ext}\vec{n}$.

The key factor is the careful consideration of the external forces based on Newton's second law for translation, which for the two point masses are according to Table 2.

To determine the algebraic value of the external torque, τ_{ext}, we need to find the sum of torques due to external forces with respect to the center of mass, and then to perpendicularly project it on the $\vec{n}$-axis. Therefore, we find:

$$\tau_{ext} = (\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 F_2)\cdot\vec{n} . \tag{39}$$

Obviously, the fact that the system is constrained (supported on the ground) is the cause that the total angular momentum $\vec{\boldsymbol{G}}$ is *not* preserved, but instead Eq. (38) shows that its time derivative ($d\vec{\boldsymbol{G}}/dt = I\ddot{\theta}$) equals to the sum of the *nonzero* external torques. In other words, the fact that the total angular momentum is not preserved is fully consistent with Newton's second law for rotation.

Table 2. External forces (F) and position vectors (r)

Cart mass (m): Mass No.1	Bob mass (M): Mass No.2
$F_{1x} = 0$	$F_{2x} = 0$
$F_{1y} = M\ddot{y}_b$	$F_{2y} = 0$
$F_{1z} = M(\ddot{z}_b + g)$	$F_{2z} = -Mg$
$\vec{r}_1 = [(X - x_c), (0 - y_c), (0 - z_c)]$	$\vec{r}_2 = [(x_b - x_c), (y_b - y_c), (z_b - z_c)]$

Actually, using a computer program the interested reader may see that the two parts of Eq. (38) are the same, as clearly is shown in Fig. 13.

The last important issue is to start from Eq. (38) and eventually to derive Eq. (22), without passing through the condition of energy conservation (on which Eq. (22) was based). This is accomplished as follows. The left part of Eq. (38) remains as is, i.e., $I\ddot{\theta}$ (although one could forget the first paragraph of this subsection regarding the second moment of inertia and, alternatively, could blindly derive it simply by differentiating the general expression $\vec{\boldsymbol{G}} = \vec{r}_1 \times m\vec{V}_{cart} + \vec{r}_2 \times M\vec{V}_{bob}$). Moreover, the first term in the right part of Eq. (39), i.e., $\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$, is replaced according to Table 2. Then, the involved variables (x_b, y_b, z_b) are replaced by Eq. (2), while the required second derivatives $(\ddot{y}_b, \ddot{z}_b)$ by differentiating Eq. (10) thus receiving:

$$\begin{aligned} y_b &= L(\theta\cos\theta - \theta \ \sin\theta), \\ z_b &= L\sin\alpha(\theta\sin\theta + \theta^2\cos\theta). \end{aligned} \tag{40}$$

Also, the coordinates (x_c, y_c, z_c) of the center of mass are given by Eq. (3).

Substituting Eq. (2), (3) and (40) into the right-hand side of Eq. (39) using also Eq. (1), we derive a certain expression which includes $\ddot{y}_b$ and $\ddot{z}_b$ thus depends on the parameter $\ddot{\theta}$ (see, Eq. (40)). Therefore, $\ddot{\theta}$ appears in both parts of Eq. (38) and after extensive manipulation and rearrangement, when solving in $\ddot{\theta}$, Eq. (22) is eventually obtained again.

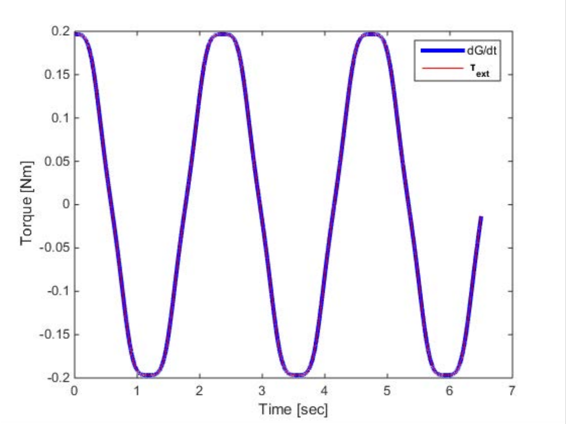

Fig. 13: Coincidence between the left and right parts of Eq. (38).

7. Discussion

In this study, the friction has been entirely neglected on purpose, just to test the conservation of energy and momentum as well as the overall feasibility of the prototype mechanism and the model. Actually, while the conservation of energy and linear momentum are fulfilled, the angular momentum is not preserved but its change in time equals to the external torque.

In other words, the system 'cart + pendulum' is an *open* system with the characteristic that it undertakes no external forces in the x-direction of motion (thus linear momentum is preserved) and has no energy loses (thus energy is preserved). Regarding the abovementioned angular momentum, the three-dimensional motion of the bob mass causes inertial forces in all the three directions (x, y, z) but only those in the (y, z) directions are transferred to the wheels as support forces (see, Table 2) thus causing external torque in addition to the dead weights. We recall that the absence of support force in the x-direction is due to the lack of friction.

A weakness is that the model of this study has not considered the possibility of the cart either to move toward the y-direction or to overturn, because both facts would be inconsistent with the experiment, [7], in which the friction with the ground plays a significant role. From a different point of view, we could additionally assume that the wheels are forced to roll on rails toward the x-direction and this resolves the supposed weakness of our model.

Regarding the angular momentum, for instructive purposes it is worthy to point out that while a similar finding (i.e., the change of angular momentum equals the external torque) is valid for a fixed pivot spinning top where the torque is taken with respect to the fixed point, in the present paper we had to implement Newton's second law (for rotation) with respect to the *center of mass* of the mechanical system.

It was clearly shown that an inclined large-angle pendulum attached on a cart in the form of a 'winding' clock (where the winding is replaced by the initial potential energy), may feed it with linear momentum thus it can travel at an infinite distance, provided the friction between the wheels of the cart and the ground as well as the friction at the pivot O' is negligible, otherwise cart's motion progressively ceases. The advantage of the pendulum over other inertia drives is that it actually requires much time to decay due to friction at the pivoting point O'. In addition, we only need to offer an initial potential energy to the bob mass by elevating the pendulum setting it at best parallel to the desired direction of motion, and then leaving it free to oscillate.

In simple words, the key factor for the continuous motion of the cart is the conservation of the linear momentum in the x-direction. Actually, the corresponding velocity of the bob mass [see, Eq. (10)] is given by $\dot{x}_b = \dot{X} - L\dot{\theta}\sin\theta\cos\alpha$, which means that with respect to an inertial observer at the pivot point O', the product $(-\dot{\theta}\sin\theta)$ plays an important role. The quantity $P_x = m\dot{X} + M\dot{x}_b = (m + M)\dot{X} - LM\cos\alpha(\dot{\theta}\sin\theta)$ remains constant, which means that at the minimum value of the product $(-\dot{\theta}\sin\theta)$ we have the maximum value of $\dot{X}$ and vice versa. Moreover, regarding the absolute position of the cart given by $X(t) = A(-\cos\theta + \dot{\theta}_0 t)$, for the particular initial conditions of our test case, it is a simple superposition of a constant velocity $(A\dot{\theta}_0)$ and a parasitic cosine term. The whole procedure was explained by details in Section 6.

Interestingly, while the amplitude in the oscillation of the linear momentum is relatively high (see, Fig. 12), the magnitude of the constant total linear momentum is rather small.

In Section 5, results were presented for a relatively low initial angular velocity $\dot{\theta}_0 = 0.1\, rad/sec$, thus consequently the calculated displacement of the cart was rather small. If the aforementioned $\dot{\theta}_0$ increases up to its maximum allowable value given by Eq. (8b), that is close to the value $\dot{\theta}_0 = 4\, rad/sec$, in the same time interval, $0 \le t \le 6.5\, sec$, the displacement increases a lot, as shown in Fig. 14. One may observe that, the higher the initial angular velocity the longer the travelled length in the same time interval.

It is obvious that in any large-angle oscillation, the angular velocity vanishes at the extreme points. For example, if a small initial velocity such as $\dot{\theta}_0 = 0.1 rad/s$ had been given at an initial angle of $\theta_0 = -45^0$ degrees, the zero value of the angular velocity would happen earlier at -45.0394^0 degrees, while the other extreme point would happen later at $+45.0394^0$ degrees. Similarly, now that in the present study we have chosen the initial value be $\dot{\theta}_0 = 0.1 rad/s$ at $\theta_0 = -90^0$ degrees, it has been shown that the half-amplitude $\theta_{\max}$ of the 'azimuthal' angle is slightly larger than $\theta_{\max} = 90^0$ (i.e., 90.0279^0 degrees in absolute value). According to Eq. (20a), the inverse cosine of $\theta_{\max}$ is proportional to the term $\dot{\theta}_0^2$ thus the higher the initial angular velocity the higher the half-amplitude ($\theta_{\max}$) is.

Interestingly, although the system has two degrees of freedom, the azimuthal angle θ is the primary variable and even it satisfies either a first-order (due to energy itself) or a second-order (due to the differentiation of the total energy in time) differential equation. For didactic purposes, the same second-order equation was derived following a much more difficult way, by considering the change of the angular momentum. A fourth straightforward formulation is to implement Lagrange's equations, which include the difference between the kinetic and potential energies (see Appendix B).

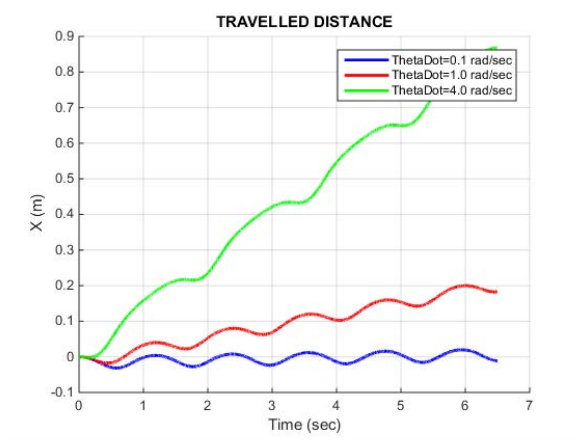

Fig. 14: Cart's displacement for various initial angular velocities $\dot{\theta}_0 = 0.1, 1,$ and 4 rad/sec.

APPENDIX A: Derivatives in time *t*

$$(\cos\theta)^{\cdot} = -\sin\theta \cdot \dot{\theta} \qquad \text{(A-1)}$$

$$(\sin\theta)^{\cdot} = \cos\theta \cdot \dot{\theta} \qquad \text{(A-2)}$$

$$(\cos\theta)^{\cdot\cdot} = -\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \qquad \text{(A-3)}$$

$$(\sin\theta)^{\cdot\cdot} = +\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) \qquad \text{(A-4)}$$

APPENDIX B: Lagrange's equations

Lagrange equations are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad q_1 = X, q_2 = \theta \quad \text{(B-1)}$$

$$\text{and} \qquad L = E_{kin} - E_{pot} \qquad \text{(B-2)}$$

Setting the kinetic energy E_{kin} as the sum of Eq. (9) and Eq. (11b), while the potential energy E_{pot} is according to Eq. (12), the Lagrangian in (B-2) becomes:

$$\begin{aligned} L &= \frac{1}{2} m\dot{X}^2 \\ &+ \frac{1}{2} M(\dot{X}^2 + L^2\dot{\theta}^2 - 2L\dot{X}\dot{\theta}\sin\theta\cos\alpha) \\ &+ MgL\cos\theta\sin\alpha \end{aligned} \qquad \text{(B-3)}$$

The first Lagrange equation (with $q_1 = X$) leads to:

$$(m+M)\ddot{X} + (ML\cos\alpha)(\cos\theta)^{\cdot\cdot} = 0, \qquad \text{(B-4)}$$

which coincides with Eq. (4). The second equation (with $q_1 = \theta$) leads to:

$$ML^2\ddot{\theta} - (ML\cos\alpha)\ddot{X}\sin\theta + MgL\sin\alpha\sin\theta = 0 \tag{B-5}$$

Eliminating the quantity $\ddot{X}$ between Eq. (B-4) and Eq. (B-5), we eventually obtain Eq. (22).

Conflict of interest: "The authors have no conflicts to disclose."

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