On the Solution of Multi-Term Nonlinear Partial Derivative Delay Differential Equations

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Abstract: - In our paper, we used the Adomian decomposition method to solve multi-term nonlinear delay differential equations of partial derivative order, these types of equations are studied. When we used this method, the being of an exclusive solution will be provided, approximate analytics of this method applied to these kinds of equations will be disputed, and the maximum absolute briefed error of the Adomians series solution will be rated. A digital example is made ready to make clear the effectiveness of the offered method.

Key-Words:- Non-linear Delay Differential Equation, Integer Order, Adomian decomposition method, and convergence analysis.

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1 Introduction

Fractional differential equations are practiced to sample expansive space of physical problems including non-linear vacillation of earth shakes [4], fluid-dynamic passing (in 1999), [5], and hesitancy dependent on the waning behavior of many relativistic materials. Differential equations of integer orders and delay differential equations have many applications in engineering and science, including electrical networks, fluid flow, control theory, fractals theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology, visual and neural network systems with delayed feedback, see [14] and [15].

The authors in [1], [2] and the author in [3] examined the qualitative behavior of the delay differential equation of the form:

$$\acute{f}(x) = k\big(f(x) + f(qx)\big), \quad f(0) = f_0$$

Where k is a given function and q is greater than 0, and investigated the solution of this equation by expanding them into the Dirichlet series.

Delay differential equations were originally introduced in the 18th century by Laplace and Condorcet and the authors in [6] presented Lambert functions to obtain the complete solution for systems of delay differential equations.

The authors in [7] used the Legendre-Pseudospectra method to find the exact and approximate solutions of the fractional-order delay differential equations.

The author in [8] collected with the linear interpolation method to devise Adams Bash forth-Moulton method for non-linear fractional positive real number differential equations with well-established or time-changing delay, then employed this method to approximate the delayed fractional-order differential equations.

The authors in [9] converted the fractional delay differential equation to the fractional non-delay differential equation and then applied the Hermite wavelet method by utilizing the method of steps on the obtained fractional non-delay differential equation to find the solution.

The authors in [10] used the Laguerre Wavelets method and combined it with the stages method to solve linear and nonlinear delay differential equations of fractional order.

The authors in [11] gave the digital solution of A range of fractional delay differential equations, comparative between higher absolute errors for the suggested method and the results obtained by Haar wavelet.

The authors in [12] presented a new method (Gegenbauer Wavelets steps method) for solving non-linear fractional delay differential equations by using two methods: Gegenbauer polynomials and method of steps. They converted the fractional non-linear fractional delay differential equation into a fractional non-linear differential equation and applied the Gegenbauer wavelet method at each iteration of the fractional differential equation to find the solution.

In 2019, the authors in [13] used the spiritual collocation method for solving fractional order delay-differential equations by Chebyshev operational matrix.

2 Formulation of the Issue with the Solution Algorithm

Let

$$\xi_1, \xi_2, \ldots, \xi_n > 0,$$

and consider the nonlinear delay differential equation:

$$\frac{dx(t)}{dt} = f\big(t, x(t-\xi_1), x(t-\xi_2), \ldots, x(t-\xi_n)\big), \quad t > 0 \tag{1}$$

Where $\xi(t) \in C(J), J = [0,T]$ and f satisfies Lipschitz condition with Lipschitz constant k such as:

$$|f(t, x_1, x_2, \ldots, x_n) - f(t, y_1, y_2, \ldots, y_n)| \leq k \sum_{i=1}^{n} |x_i - y_i|. \tag{2}$$

which implies that,

$$|f(t, x(t-\xi_1), x(t-\xi_2), \ldots, x(t-\xi_n)) - f(t, y(t-\xi_1), y(t-\xi_2), \ldots, y(t-\xi_n))| \leq k \sum_{i=1}^{n} |x(t-\xi_i) - y(t-\xi_i)|. \tag{3}$$

Operating with L^{-1} to both sides of equation (1), where $L^{-1}(.) = \int_0^t (.)dt$, we get

$$x(t) = x_0 + \int_0^t f(\alpha, x(\alpha-\xi_1), \ldots, x(\alpha-\xi_n))d\alpha \tag{4}$$

The solution algorithm of equation (4) using the domain decomposition method is:

$$x_0(t) = x_0, \qquad x_q(t) = \int_0^t A_{q-1}(\alpha)d\alpha.$$

Where, Aq are Adomian polynomials of the non-linear term $f(t, x(t-\xi_1), x(t-\xi_2), \ldots, x(t-\xi_n))$ taken the form,

$$A_q = \frac{1}{q!}\left[\frac{d^q}{d\lambda^q} f\left(t, \sum_{i=0}^{\infty} \lambda^i x_i(t-\xi_1), \sum_{i=0}^{\infty} \lambda^i x_i(t-\xi_2), \ldots, \sum_{i=0}^{\infty} \lambda^i x_i(t-\xi_n)\right)\right]_{\lambda=0} \tag{5}$$

Now, the solution of the equations (1) and (2) will be,

$$x(t) = \sum_{i=0}^{\infty} x_i(t) \tag{6}$$

3 The Issue with the Partial Order Solution Algorithm

Let

$$\alpha_1\alpha_2, \ldots, \alpha_n > 0$$

and consider the Partial delay differential equation:

$$\frac{\prod_{k=1}^{n} \partial^{\sum_{j=1}^{n} i_j} x_k(t_k)}{\prod_{k=1}^{n} \partial t_k^{i_k}} = f\Big(t_1, x_1\big(t_1-\alpha_{11}\big), x_1(t_1-\alpha_{12}), \ldots, x_1(t_1-\alpha_{1n}), t_2, x_2\big(t_2-\alpha_{21}\big), x_2(t_2-\alpha_{22}), \ldots, x_2(t_2-\alpha_{2n}), \ldots, t_n, x_n(t_n-\alpha_{n1}), x_n(t_n-\alpha_{n2}), \ldots, x_n(t_n-\alpha_{nn})\big)\Big) \tag{7}$$

$$x(t) = x_0, \ \ t \leq 0, \tag{8}$$

Where f satisfies Lipschitz condition h and x(t) 2 [0; T] such as:

$$|f(t, x_1, x_2, \ldots, x_n) - f(t, y_2, y_2, \ldots, y_n)| \leq h \sum_{i=1}^{n} |x_i - y_i| \tag{9}$$

which implies that for,

$$\Big|f(t_1, x_1\left(t_1-\alpha_{11}\right), x_1(t_1-\alpha_{12}), \dots, x_1(t_1-\alpha_{1n}), t_2, x_2\left(t_2-\alpha_{21}\right), x_2(t_2-\alpha_{22}), \dots, x_2(t_2-\alpha_{2n}), \dots, t_n, x_n(t_n-\alpha_{n1}), x_n(t_n-\alpha_{n2}), \dots, x_n(t_n-\alpha_{nn})\big) - f(t_1, y_1\left(t_1-\alpha_{11}\right), y_1(t_1-\alpha_{12}), \dots, y_1(t_1-\alpha_{1n}), t_2, y_2\left(t_2-\alpha_{21}\right), y_2(t_2-\alpha_{22}), \dots, y_2(t_2-\alpha_{2n}), \dots, t_n, y_n\big)\big)(t_n-\alpha_{n1}), y_n(t_n-\alpha_{n2}), \dots, y_n(t_n-\alpha_{nn})\big)\Big| \le h\prod_{k=1}^{n}\sum_{i=1}^{n}|x_k(t_k-\alpha_{ki}) - y_k(t_k-\alpha_{ki})| \quad (10)$$

By using L^{-1} to both sides of the above first equation with the solution algorithm using the domain decomposition method, we get:

$$x_0(t) = x_0$$

$$x_k(t) = \int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n} A_{q-1}\left(\xi_1, \xi_{2,\dots,}\xi_n\right)d\xi_1 .. d\xi_n \quad (11)$$

Anywhere, Aq are Adomian polynomials of the non-linear idiom f (t, x(t – α₁), x(t – α₂),…, x(t – αₙ)) taken the form:

$$A_q = \frac{1}{q!}\left(\frac{d^q}{d\lambda^q} f(t_1, \sum_{i=0}^{\infty}\lambda^i x_{1i}(t_1-\alpha_{11}), \sum_{i=0}^{\infty}\lambda^i x_{1i}(t_1-\alpha_{12}), \dots, \sum_{i=0}^{\infty}\lambda^i x_{1i}(t_1-\alpha_{1n}), t_2, \sum_{i=0}^{\infty}\lambda^i x_{2i}(t_2-\alpha_{21}), \sum_{i=0}^{\infty}\lambda^i x_{2i}(t_2-\alpha_{22}), \dots, \sum_{i=0}^{\infty}\lambda^i x_{2i}(t_2-\alpha_{2n}), \dots, t_n, \sum_{i=0}^{\infty}\lambda^i x_{ni}(t_n-\alpha_{n1}), \sum_{i=0}^{\infty}\lambda^i x_{ni}\left(t_n-\alpha_{n2}\right), \dots, \sum_{i=0}^{\infty}\lambda^i x_{ni}\left(t_n-\alpha_{nn}\right)\right) \quad (12)$$

Now, the solution of the above first and second equations will be:

$$x(t_1, t_2, \dots, t_n) = \prod_{j=0}^{\infty}\sum_{i=0}^{\infty} x_{ij}(t_1, t_2, \dots, t_n) \quad (13)$$

4 Convergence analysis

Existence and uniqueness theorem

Define the mapping M: E →E where M is the Banach space

$$(C(J), \|.\|).$$

The space of all continuous functions on J with the norm $\| x \| = \max e^{-Nt}\left|x(t)\right|$, N> 0.

Theorem 4.1. Let f satisfy the

$$|f(t, x_1, x_2, \dots, x_n) - f(t, y_1, y_2, \dots, y_n)| \le k\sum_{i=1}^{n}|x_i - y_i|. \quad (14)$$

Then, the nonlinear partial derivative delay differential equation has a unique solution

$$x_i \in E.$$

for i=1,2,..,n. Where E is the Banach space

$$(C(J), \|.\|).$$

Proof:

Consider the mapping M: E →E is defined as

$$Mx(t_1, t_2, \dots, t_n) = x_0 + \int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n} f(\xi_1, x_1(\xi_1-\alpha_{11}), x_1(\xi_1-\alpha_{12}), \dots x_1(\xi_1-\alpha_{1n}), \xi_2, x_2(\xi_2-\alpha_{21}), x_2(\xi_2-\alpha_{22}), \dots, x_2(\xi_2-\alpha_{2n}), \dots, \xi_n, x_n(\xi_n-\alpha_{n1}), x_n(\xi_n-\alpha_{n2}), \dots, x_n(\xi_n-\alpha_{nn})d\xi_1, d\xi_2, \dots, d\xi_n. \quad (15)$$

Let

$$x(t_1, t_2, \dots, t_n), y(t_1, t_2, \dots, t_n) \in E$$

then,

$$Mx(t_1, t_2, \dots, t_n) - Mx(t_1, t_2, \dots, t_n) = \int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n} f(\xi_1, x_1(\xi_1-\alpha_{11}), x_1(\xi_1-\alpha_{12}), \dots x_1(\xi_1-\alpha_{1n}), \xi_2, x_2(\xi_2-\alpha_{21}), x_2(\xi_2-\alpha_{22}), \dots, x_2(\xi_2-\alpha_{2n}), \dots, \xi_n, x_n(\xi_n-\alpha_{n1}), x_n(\xi_n-\alpha_{n2}), \dots, x_n(\xi_n-\alpha_{nn})d\xi_1, d\xi_2, \dots, d\xi_n - \int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n} f(\xi_1, y_1(\xi_1-\alpha_{11}), y_1(\xi_1-\alpha_{12}), \dots y_1(\xi_1-\alpha_{1n}), \xi_2, y_2(\xi_2-\alpha_{21}), y_2(\xi_2-\alpha_{22}), \dots, y_2(\xi_2-\alpha_{2n}), \dots, \xi_n, y_n(\xi_n-\alpha_{n1}), x_n(\xi_n-\alpha_{n2}), \dots, y_n(\xi_n-\alpha_{nn})d\xi_1, d\xi_2, \dots, d\xi_n \quad (16)$$

This implies that:

$$|Mx(t_1,t_2,\ldots,t_n) - Mx(t_1,t_2,\ldots,t_n)| = \left|\int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_n} f(\xi_1, x_1(\xi_1-\alpha_{11}), x_1(\xi_1-\alpha_{12}),\ldots x_1(\xi_1-\alpha_{1n}),\xi_2, x_2(\xi_2-\alpha_{21}), x_2(\xi_2-\alpha_{22}),\ldots,x_2(\xi_2-\alpha_{2n}),\ldots,\xi_n,x_n(\xi_n-\alpha_{n1}),x_n(\xi_n-\alpha_{n2}),\ldots,x_n(\xi_n-\alpha_{nn})d\xi_1,d\xi_2,\ldots,d\xi_n - \int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_n} f(\xi_1, y_1(\xi_1-\alpha_{11}), y_1(\xi_1-\alpha_{12}),\ldots y_1(\xi_1-\alpha_{1n}),\xi_2, y_2(\xi_2-\alpha_{21}), y_2(\xi_2-\alpha_{22}),\ldots,y_2(\xi_2-\alpha_{2n}),\ldots,\xi_n,y_n(\xi_n-\alpha_{n1}),x_n(\xi_n-\alpha_{n2}),\ldots,y_n(\xi_n-\alpha_{nn})d\xi_1,d\xi_2,\ldots,d\xi_n\right| \quad (17)$$

$$e^{-\sum_{i=1}^{n}Nt_i}|Mx(t_1,t_2,\ldots,t_n) - Mx(t_1,t_2,\ldots,t_n)| \le e^{-\sum_{i=1}^{n}Nt_i}\left|\int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_n} f(\xi_1, x_1(\xi_1-\alpha_{11}), x_1(\xi_1-\alpha_{12}),\ldots x_1(\xi_1-\alpha_{1n}),\xi_2, x_2(\xi_2-\alpha_{21}), x_2(\xi_2-\alpha_{22}),\ldots,x_2(\xi_2-\alpha_{2n}),\ldots,\xi_n,x_n(\xi_n-\alpha_{n1}),x_n(\xi_n-\alpha_{n2}),\ldots,x_n(\xi_n-\alpha_{nn})d\xi_1,d\xi_2,\ldots,d\xi_n - \int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_n} f(\xi_1, y_1(\xi_1-\alpha_{11}), y_1(\xi_1-\alpha_{12}),\ldots y_1(\xi_1-\alpha_{1n}),\xi_2, y_2(\xi_2-\alpha_{21}), y_2(\xi_2-\alpha_{22}),\ldots,y_2(\xi_2-\alpha_{2n}),\ldots,\xi_n,y_n(\xi_n-\alpha_{n1}),x_n(\xi_n-\alpha_{n2}),\ldots,y_n(\xi_n-\alpha_{nn})d\xi_1,d\xi_2,\ldots,d\xi_n\right| \quad \ldots$$

$$\max_{t\in J} e^{-\sum_{i=1}^{n}Nt_i}|Mx(t_1,t_2,\ldots,t_n) - Mx(t_1,t_2,\ldots,t_n)| \le k\sum_{i=1}^{n}\max_{t\in J}\int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_n}\prod_{j=1}^{n} e^{-\sum_{i=1}^{n}N(t_i-\xi_i)} \quad (18)$$

$$e^{-\sum_{i=1}^{n}N\xi_i}\left|x_j(\xi_j-\alpha_{ij}) - y_j(\xi_j-\alpha_{ij})\right| d\xi_1,d\xi_2,\ldots,d\xi_n$$

$$\|\mathrm{M}_x - M_y\| \le nk\|x-y\|\int_0^t e^{-N(t-\xi)}\,d\xi$$

$$\le nk\left(\frac{1-e^{-Nt}}{N}\right)\|x-y\|$$

$$\le \frac{nk}{N}\|x-y\| \quad (19)$$

Where nk/N is less than 1, we get:

$$\|M_x - M_y\| \le \|x-y\| \quad (20)$$

Therefore, M there exists a unique solution and contraction.

Theorem 4.2. the series solution

$$\mathrm{x(t)} = \sum_{\mathrm{i}=0}^{\infty}\mathrm{x_i(t)}$$

of the nonlinear delay differential equation converges by using domain decomposition method if $|x_1(t)|$ is less than c, when c is any constant.

Proof

Define the sequence of partial sums from

$$\mathrm{f}(\mathrm{t},\mathrm{x}(\mathrm{t}-\alpha_1),\ldots,\mathrm{x}(\mathrm{t}-\alpha_n)) = \sum_{i=0}^{\infty}A_i(t) \quad (21)$$

So, we can write

$$\mathrm{f}\left(\mathrm{t}, S_p(\mathrm{t}-\alpha_1),\ldots,S_p(\mathrm{t}-\alpha_n)\right) = \sum_{i=0}^{p}A_i(t) \quad (22)$$

Where Ai are a domain polynomials of the non-linear idiom $f(t,S_p(t-\alpha_1),\ldots,S_p(t-\alpha_n))$
taken the form,

$$\sum_{i=0}^{\infty}x_i(t) = x_0 + \int_0^t (\sum_{i=0}^{\infty}A_{i-1}(\xi))d\xi \quad (23)$$

The sequences Sp and Sq be to despotic partial sums with p is greater than q, one can get

$$S_p = \sum_{i=0}^{p}x_i(t) = x_0 + \int_0^t \sum_{i=0}^{p}A_{i-1}(\xi))d\xi \quad (24)$$

We are going to verify that, the Cauchy sequence $\{S_p\}$ in Banach space E,

$$
\begin{aligned}
S_p(\mathrm{t}) - S_q(t) &= \int_0^t \left(\sum_{i=0}^{p} A_{i-1}(\xi) \right) d\xi - \int_0^t \left(\sum_{i=0}^{q} A_{i-1}(\xi) \right) d\xi \\
&= \int_0^t \left[\sum_{i=q+1}^{p} A_{i-1}(\xi)\, d\xi \right] \\
&= \int_0^t \left[\sum_{i=q}^{p-1} A_i(\xi)\, d\xi \right] \\
&= \int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n} f(\xi_1, S_{p-1}(\xi_1 - \alpha_{11}), S_{p-1}(\xi_1 - \alpha_{12}), \dots S_{p-1}(\xi_1 - \alpha_{1n}), \xi_2, S_{p_2-1}(\xi_2 - \alpha_{21}), S_{p_2-1}(\xi_2 - \alpha_{22}), \dots, S_{p_2-1}(\xi_2 - \alpha_{2n}), \dots, \xi_n, S_{p_n-1}(\xi_n - \alpha_{n1}), S_{p_n-1}(\xi_n - \alpha_{n2}), \dots, S_{p_n-1}(\xi_n - \alpha_{nn})) - \\
&\quad f(\xi_1, S_{q-1}(\xi_1 - \alpha_{11}), S_{q-1}(\xi_1 - \alpha_{12})_{q-1}(\xi_1 - \alpha_{1n}), \xi_2, S_{q_2-1}(\xi_2 - \alpha_{21}), S_{q_2-1}(\xi_2 - \alpha_{22}), \dots, S_{q_2-1}(\xi_2 - \alpha_{2n}), \dots, \xi_n, S_{q_n-1}(\xi_n - \alpha_{n1}), S_{q_n-1}(\xi_n - \alpha_{n2}), \dots, S_{q_n-1}(\xi_n - \alpha_{nn}))\, d\xi_1, d\xi_2, \dots, d\xi_n.
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&\left| S_p(t_1, t_2, \dots, t_n) - S_q(t_1, t_2, \dots, t_n) \right| \\
&= \Bigg| \int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n} f(\xi_1, S_{p-1}(\xi_1 - \alpha_{11}), S_{p-1}(\xi_1 - \alpha_{12}), \dots S_{p-1}(\xi_1 - \alpha_{1n}), \xi_2, S_{p_2-1}(\xi_2 - \alpha_{21}), S_{p_2-1}(\xi_2 - \alpha_{22}), \dots, S_{p_2-1}(\xi_2 - \alpha_{2n}), \dots, \xi_n, S_{p_n-1}(\xi_n - \alpha_{n1}), S_{p_n-1}(\xi_n - \alpha_{n2}), \dots, S_{p_n-1}(\xi_n - \alpha_{nn})) \\
&\quad - f(\xi_1, S_{q-1}(\xi_1 - \alpha_{11}), S_{q-1}(\xi_1 - \alpha_{12})_{q-1}(\xi_1 - \alpha_{1n}), \xi_2, S_{q_2-1}(\xi_2 - \alpha_{21}), S_{q_2-1}(\xi_2 - \alpha_{22}), \dots, S_{q_2-1}(\xi_2 - \alpha_{2n}), \dots, \xi_n, S_{q_n-1}(\xi_n - \alpha_{n1}), S_{q_n-1}(\xi_n - \alpha_{n2}), \dots, S_{q_n-1}(\xi_n - \alpha_{nn}))\, d\xi_1, d\xi_2, \dots, d\xi_n. \Bigg|
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&e^{-\sum_{i=1}^{n} N t_i} \left| S_p(t_1, t_2, \dots, t_n) - S_q(t_1, t_2, \dots, t_n) \right| \\
&\le e^{-\sum_{i=1}^{n} N t_i} \int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n} f(\xi_1, S_{p-1}(\xi_1 - \alpha_{11}), S_{p-1}(\xi_1 - \alpha_{12}), \dots S_{p-1}(\xi_1 - \alpha_{1n}), \xi_2, S_{p_2-1}(\xi_2 - \alpha_{21}), S_{p_2-1}(\xi_2 - \alpha_{22}), \dots, S_{p_2-1}(\xi_2 - \alpha_{2n}), \dots, \xi_n, S_{p_n-1}(\xi_n - \alpha_{n1}), S_{p_n-1}(\xi_n - \alpha_{n2}), \dots, S_{p_n-1}(\xi_n - \alpha_{nn})) \\
&\quad - f(\xi_1, S_{q-1}(\xi_1 - \alpha_{11}), S_{q-1}(\xi_1 - \alpha_{12})_{q-1}(\xi_1 - \alpha_{1n}), \xi_2, S_{q_2-1}(\xi_2 - \alpha_{21}), S_{q_2-1}(\xi_2 - \alpha_{22}), \dots, S_{q_2-1}(\xi_2 - \alpha_{2n}), \dots, \xi_n, S_{q_n-1}(\xi_n - \alpha_{n1}), S_{q_n-1}(\xi_n - \alpha_{n2}), \dots, S_{q_n-1}(\xi_n - \alpha_{nn}))\, d\xi_1 d\xi_2 \dots d\xi_n
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&\max_{t\in J} e^{-\sum_{i=1}^{n} N t_i} \left| S_p(t_1, t_2, \dots, t_n) - S_q(t_1, t_2, \dots, t_n) \right| \\
&\le k \sum_{i=1}^{n} \max_{t\in J} \int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n} \prod_{j=1}^{n} e^{-\sum_{i=1}^{n} N(t_i - \xi_i)} e^{-\sum_{i=1}^{n} N\xi_i} \left| S_{P_j-1}(\xi_j - \alpha_{ij}) - S_{q_j-1}(\xi_j - \alpha_{ij}) \right| d\xi_1, d\xi_2, \dots, d\xi_n \\
&\le \left\| S_p - S_q \right\| \le \beta \left\| S_{p-1} - S_{q-1} \right\|
\end{aligned}
\tag{28}
$$

When p=q+1, J=[0,T], β=nk/N, one can get:

$$
\left\| S_{q+1} - S_q \right\| \le \beta \left\| S_q - S_{q-1} \right\|
$$

$$\leq \beta^2 \| S_{q-1} - S_{q-2} \|$$
$$\leq \cdots \leq \beta^q \| S_1 - S_0 \| \quad (29)$$

Now, by triangle disparity, we can get:

$$\| S_p - S_q \|$$
$$\leq \| S_{q+1} - S_q \| + \| S_{q+2} - S_{q+1} \| + \cdots$$
$$+ \| S_p - S_{p-1} \|$$
$$\leq [\beta^q + \beta^{q+1} + \cdots + \beta^{p-1}] \quad \| S_1 - S_0 \|$$
$$\leq \beta^m \left[\frac{1-\beta^{p-q}}{1-\beta} \right] \| x_1 \| \quad (30)$$

Consequently,

$$\| S_p - S_q \| \leq \frac{\beta^q}{1-\beta} \| x_1 \|$$
$$\leq \frac{\beta^q}{1-\beta} \ (\max_{t \in J}) e^{-Nt} |x_1| \quad (31)$$

When $|x_1|$ less than costant $q \in \infty$ then,

$$\| S_p - S_q \| \to 0$$

And hence, $\{S_p\}$ is a Cauchy sequence in Banach space so, $\sum_{i=0}^{\infty} x_i(t)$ converges.

Theorem 4.3.
The maximum absolute error of the series solution

$$x(t) = \sum_{i=0}^{\infty} x_i(t)$$

to the nonlinear delay differential equation is conjectured to be,

$$\| x - \sum_{i=0}^{q} x_i \| \leq \frac{\beta^q}{1-\beta} \| x_1 \| \quad (32)$$

Proof
By the above theorem, we have

$$\| S_p(t) - S_q(t) \| \leq \frac{\beta^q}{1-\beta} \| x_1 \|$$
$$\leq \frac{\beta^q}{1-\beta} (\max_{t \in J}) \, e^{-Nt} \, |x_1(t)| \quad (33)$$

But,

$$S_p = \sum_{i=0}^{p} x_i, \quad (34)$$

When

$$P \to \infty, \quad S_p \to x(t),$$

So

$$\| x - S_q \| \leq \frac{\beta^q}{1-\beta} \| x_1 \| \quad (35)$$

so, the perfect error in J is

$$\left\| x - \sum_{i=0}^{q} x_i \right\| \leq \frac{\beta^q}{1-\beta} \| x_1 \| \quad (36)$$

Theorem 4.4. The solution of the nonlinear partial derivative delay differential equation
is uniformly stable

Proof
Let $x(t_1,t_2,\ldots,t_n)$ be a solution of

$$x(t_1, t_2, \ldots, t_n) = x_0 + \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_n} f(\xi_1, x_1(\xi_1 - \alpha_{11}), x_1(\xi_1 - \alpha_{12}), \ldots x_1(\xi_1 - \alpha_{1n}), \xi_2, x_2(\xi_2 - \alpha_{21}), x_2(\xi_2 - \alpha_{22}), \ldots, x_2(\xi_2 - \alpha_{2n}), \ldots, \xi_n, x_n(\xi_n - \alpha_{n1}), x_n(\xi_n - \alpha_{n2}), \ldots, x_n(\xi_n - \alpha_{nn}) d\xi_1, d\xi_2, \ldots, d\xi_n. \quad (37)$$

And let

$$\tilde{x}(t_1, t_2, \ldots, t_n)$$

be a solution to the above problem such that,

$$\tilde{x}(0) = \tilde{x}_0$$

Then

$$x(t_1, t_2, \ldots, t_n) - \tilde{x}(t_1, t_2, \ldots, t_n)$$
$$= x_0 - \tilde{x}_0 + \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_n} f(\xi_1, x_1(\xi_1 - \alpha_{11}), x_1(\xi_1 - \alpha_{12}), \ldots x_1(\xi_1 - \alpha_{1n}), \xi_2, x_2(\xi_2 - \alpha_{21}), x_2(\xi_2 - \alpha_{22}), \ldots, x_2(\xi_2 - \alpha_{2n}), \ldots, \xi_n, x_n(\xi_n - \alpha_{n1}), x_n(\xi_n - \alpha_{n2}), \ldots, x_n(\xi_n - \alpha_{nn}) - f(\xi_1, \tilde{x}_1(\xi_1 - \alpha_{11}), \tilde{x}_1(\xi_1 - \alpha_{12}), \ldots, \tilde{x}_1(\xi_1 - \alpha_{1n}), \xi_2, \tilde{x}_2(\xi_2 - \alpha_{21}), \tilde{x}_2(\xi_2 - \alpha_{22}), \ldots, \tilde{x}_2(\xi_2 - \alpha_{2n}), \ldots, \xi_n, \tilde{x}_n(\xi_n - \alpha_{n1}), \tilde{x}_n(\xi_n - \alpha_{n2}), \ldots, \tilde{x}_n(\xi_n - \alpha_{nn}) d\xi_1, d\xi_2, \ldots, d\xi_n \quad (38)$$

$$|x(t_1,t_2,\dots,t_n)-\tilde{x}(t_1,t_2,\dots,t_n)| = |x_0-\tilde{x}_0| + \left|\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n} f(\xi_1,x_1(\xi_1-\alpha_{11}),x_1(\xi_1-\alpha_{12}),\dots x_1(\xi_1-\alpha_{1n}),\xi_2,x_2(\xi_2-\alpha_{21}),x_2(\xi_2-\alpha_{22}),\dots,x_2(\xi_2-\alpha_{2n}),\dots,\xi_n,x_n(\xi_n-\alpha_{n1}),x_n(\xi_n-\alpha_{n2}),\dots,x_n(\xi_n-\alpha_{nn}) - f(\xi_1,\tilde{x}_1(\xi_1-\alpha_{11}),\tilde{x}_1(\xi_1-\alpha_{12}),\dots,\tilde{x}_1(\xi_1-\alpha_{1n}),\xi_2,\tilde{x}_2(\xi_2-\alpha_{21}),\tilde{x}_2(\xi_2-\alpha_{22}),\dots,\tilde{x}_2(\xi_2-\alpha_{2n}),\dots,\xi_n,\tilde{x}_n(\xi_n-\alpha_{n1}),\tilde{x}_n(\xi_n-\alpha_{n2}),\dots,\tilde{x}_n(\xi_n-\alpha_{nn})d\xi_1,d\xi_2,\dots,d\xi_n\right|$$

$$e^{-\sum_{i=1}^{n}Nt_i}|x(t_1,t_2,\dots,t_n)-\tilde{x}(t_1,t_2,\dots,t_n)| \le e^{-\sum_{i=1}^{n}Nt_i}|(x_0)-\widetilde{x_0}| + k\sum_{i=1}^{n}\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n}\prod_{j=1}^{n} e^{-\sum_{i=1}^{n}N(t_i-\xi_i)}e^{-\sum_{i=1}^{n}N\xi_i}\left|x_j(\xi_j-\alpha_{ij})-\tilde{x}_j(\xi_j-\alpha_{ij})\right|d\xi_1,d\xi_2,\dots,d\xi_n$$

$$\max_{t\in J} e^{-\sum_{i=1}^{n}Nt_i}|x(t_1,t_2,\dots,t_n)-\tilde{x}(t_1,t_2,\dots,t_n)| \le \max_{t\in J} e^{-\sum_{i=1}^{n}Nt_i}|(x_0)-\widetilde{x_0}| + k\sum_{i=1}^{n}\max_{t\in J}\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n}\prod_{j=1}^{n} e^{-\sum_{i=1}^{n}N(t_i-\xi_i)}e^{-\sum_{i=1}^{n}N\xi_i}$$

$$\left|x_j(\xi_j-\alpha_{ij})-\tilde{x}_j(\xi_j-\alpha_{ij})\right|d\xi_1,d\xi_2,\dots,d\xi_n$$

$$\|x-\tilde{x}\| \le |x_0-\tilde{x}_0| + nk\|x-\tilde{x}\| \int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_n}\prod_{j=1}^{n} e^{-\sum_{i=1}^{n}N(t_i-\xi_i)}d\xi_1,d\xi_2,\dots,d\xi_n \le |x_0-\tilde{x}_0| + \frac{nk}{N}\|x-\tilde{x}\| \le (1-\frac{nk}{N})^{-1}|x_0-\tilde{x}_0| \quad (39)$$

Therefore, if

$$|x_0-\tilde{x}_0|$$

Less than ϵ, then

$$\|x-\tilde{x}\| \quad (40)$$

Which completes the proof.

5 Numerical Examples

Example 5.1

Consider the following nonlinear delay differential equation,

$$\frac{dx(t)}{dt} = \left(1.2-\frac{t}{2}-\frac{t2}{4}\right)+\frac{1}{2}x(t-0.5) + \frac{`1}{4}\,x(t-0.1), t>0, \quad (41)$$

$$x(t)=0.1,\ t\le 0 \quad (42)$$

which has the exact solution (t+0.1).

Applying the Adomian decomposition method to above (41) and (42), we have:

$$x_0(t) = 0.1 + \int_0^t \left(1.2-\frac{\Im}{2}-\frac{\Im 2}{4}\right)d\Im,$$
$$x_i(t) = (\frac{1}{2})\int_0^t x_{i-1}(\Im-0.5)d\Im + \left(\frac{1}{4}\right)\int_0^t A_{i-1}(\Im)d\Im, i\ge 1. \quad (43)$$

Where A_i are Adomian polynomials of the nonlinear term $x^2(t-0.1)$. From the above equations, the solution of problem x(t)=0.1, t≤0 is:

$$x(t)=\sum_{i=0}^{m} x_i(t) \quad (44)$$

Table [1] shows the absolute error of Adomian decomposition method series solution at different values of m when t=1, while table [2] shows the maximum absolute truncated error using theorem 4.3, when t=1, N=2. Fig [1]: shows ADM and exact solutions (when m=20).

Table 1. Absolute Error

m	$\| x_{Exact} - x_{ADM} \|$
5	**0.000238942**
10	**3.03189×10^-6^**
15	**3.96516×10^-7^**
20	**2.929×10^-8^**

Table 2. Max. absolute error

m	max. error
5	**0.0136826**
10	**0.000688625**
15	**0.0000346574**
20	**1.74425×10^-6^**

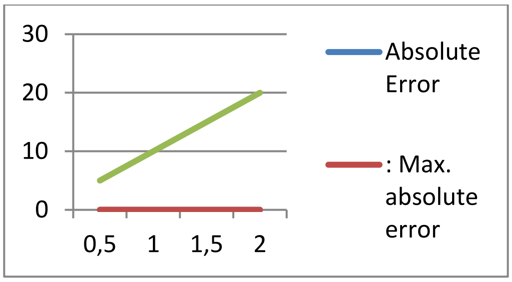

Fig. 1: ADM and exact solutions.

Example 5.2
Consider the logistic delay differential equation with two different delays,

$$\frac{dx(t)}{dt} = px(t-r_1)(1-x(t-r_2)), t, p, r_1 r_2 > 0 \quad (45)$$

$x(t)=x_2$, $t \leq 0$.

The above equation can be written as

$$dx(t)/dt = p[x(t-r_1) - x(t-r_1)x(t-r_2)], \quad (46)$$

Applying ADM to the above equation, we have

$$x_0(t) = x_0 \quad (47)$$

$$x_i(t) = p\int_0^t [x_{i-1}(\tau - r_1) - A_{i-1}(\tau)]d\tau, i \geq 1. \quad (48)$$

Where Ai are Adomian polynomials of the nonlinear term $x(t-r_1)x(t-r_2)$.
From equations (47) and (48), the first six terms of the series solution considering the following two cases:

Case 1:
(When $x_0 = \frac{1}{2}$, $p = \frac{1}{8}$, $r_1 = 0.1, r_2 = 0.3$):
$x(t)=0.5+0.03164+0.00002565t^2-0.00004218t^3+1.11262\times10^{-7}t^4+6.35783\times10^{-8}t^5$. (49)

Table [3] shows the maximum absolute truncated error at different values of m at t=10 and N=5,10.

Case 2:
(When $x_0 = 0.25$, $p = \frac{1}{50}$, $r_1 = 0.02, r_2 = 0.04$):
$x(t)=0.25+0.00374962t+0.0000187547t^2-3.11281\times10^{-8}t^3-7.80902\times10^{-10}t^4+2.03125\times10^{-12}t^5$. (50)

Table 3. Maximum absolute error

m	max. error (N=5)	max. error (N=10)
5	9.91686×10^{-7}	2.97291×10^{-8}
10	2.9009×10^{-12}	2.71763×10^{-15}
15	8.48579×10^{-18}	2.48427×10^{-22}
20	1.24114×10^{-23}	1.13548×10^{-29}

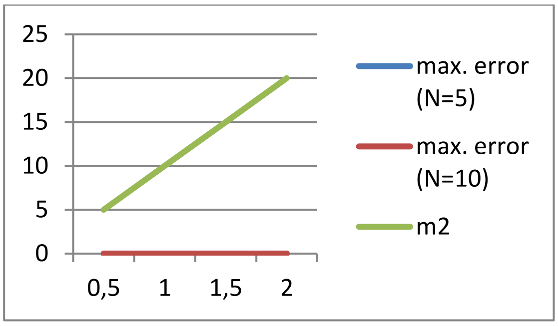

Fig. 2: ADM solutions.

6 Conclusion

In this paper, the Adomian decomposition method is an interesting and powerful tool when applied to different kinds of equations. Here we used it to solve the non-linear multi-term Partial derivative delay differential equation with new theorems introduced which give the sufficient conditions of existence, uniqueness, convergence, and estimations of the maximum absolute truncation error to Adomian decomposition method series solution when applied to these equations. Some numerical examples are discussed and solved by using the Adomian decomposition method.
The method has given an analytical solution that is still open for investigation, especially in Partial derivative delay differential equations with arbitrary orders.

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