Local Interpolation Splines and Solution of Integro-Differential Equations of Mechanic's Problems

I. G. BUROVA
Department of Computational Mathematics,
St. Petersburg State University,
RUSSIA

Abstract: - Integro-differential equations are encountered when solving various problems of mechanics. Although Integro-Differential equations are encountered frequently in mathematical analysis of mechanical problems, very few of these equations will ever give us analytic solutions in a closed form. So that construction of numerical methods is the only way to find the approximate solution. This paper discusses the calculation schemes for solving integro-differential equations using local polynomial spline approximations of the Lagrangian type of the fourth and fifth orders of approximation. The features of solving integro-differential equations with the first derivative and the Fredholm and Volterra integrals of the second kind are discussed. Using the proposed spline approximations, formulas for numerical differentiation are obtained. These formulas are used to approximate the first derivative of a function. The numerical experiments are presented.

Key-Words: - Fredholm integro-differential equations, Volterra-Fredholm integro-differential equations, local polynomial splines, problems of mechanics, numerical solution

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1 Introduction

The history of the development of the theory of integro-differential equations (i.e. integral equations relating an unknown function and its derivatives) began with the work of Volterra [1]. The investigation of the theory of elasticity was the beginning of Volterra's theory of integro-differential equations. In 1909, Volterra published two papers in which he suggested that the deformation is a linear functional of pressure. In this case, the system of linear integro-differential equations is the main one, by solving which it is possible to determine the deformation from a known force and pressure.

Integro-differential equations are encountered in solving various problems of mechanics. Of the problems that lead to the solution of integro-differential equations, we can cite: the Proctor problem on the equilibrium of an elastic beam, the Volterra problem of torsional vibrations, the Prandtl problem for calculating an aircraft wing. Integro-differential equations with hinged boundary conditions are used to study the vibrations of suspension bridges.

For an approximate solution of integro-differential equations, one can use various representations of functions in the form of series, in particular, power series. It should also be noted that the simplest approach to solving the integro differential equation is to replace the definite integral with an approximating summation of a finite number of suitably weighted discrete values of approximate solution of an unknown function.

Integro-differential equations arise when solving various problems of mechanics. As it is generally known, obtaining an analytical solution for some integro-differential equations is not possible. In this regard, various numerical methods have been developed for finding approximate solutions to such equations. For example, for an approximate solution of integro-differential equations, one can use various representations of functions in the form of series, in particular, power series. The improvement of numerical methods for solving such problems is a very important area of computational mathematics.

Let us list a few works that have recently been published. Numerical solutions of the Fredholm integro-differential equations of the second kind have been considered in many papers (see, for example, [2]-[10]. In paper [2] four numerical methods are compared, namely, the Laplace decomposition method (LDM), the Wavelet–Galerkin method (WGM), the Laplace decomposition method with the Pade approximant (LD–PA) and the homotopy perturbation method (HPM). In paper [3] superconvergent versions for the numerical solution of a class of linear Fredholm integro-differential equations of the second kind are discussed. In paper [4] the Hermite wavelet method (HWM) is applied to approximate the solution of the integro-differential equations. In paper [5] the B-splines-least-square method and weight function of B-splines, were proposed for solving integro-differential equations.

The following methods are noted in these papers: the Legendre method, Bernoulli polynomials [6], pseudospectral methods, piecewise linear approximation, polynomial approximation, rational approximation [7], exponential spline [8], differential transformation [9], and Schauder bases [10]. The existence, uniqueness, and stability of solutions for a class of systems of non-linear complex Integro-differential equations on complex planes were investigated in [11]. The Abel integral equation of the second kind was investigated in paper [12].

Local polynomial splines have good approximation properties and are easy to use. The application splines of the fifth and fourth order of approximation to the construction of the solution of Fredholm integral equation was considered in the author's paper [13]. In this paper, we explore the application of the local polynomial splines to the construction of the solution of integro-differential equations with the first derivative in more detail. Using the proposed spline approximations, formulas for numerical differentiation are obtained. These formulas are used to approximate the first derivative of a function. In Section 2 we consider the polynomial cubic splines of the fourth order of approximation. In Section 3 we consider the polynomial splines of the fifth order of approximation and the use of them for solving the Fredholm and Volterra integro-differential equations of the second kind. A comparison of the results of applying splines of the fourth and fifth orders of approximation with the results of applying the methods are considered in paper [2].

2 Local Splines of the Fourth order of Approximation and Applications

Let a, b be real and n be integer. Let the values of the function $u(x)$ be known at the nodes of the grid $\{t_i\}$: $a = t_0 < t_1 < \cdots < t_n = b$. Denote $u_i = u(t_i)$. Recall that the approximation by local polynomial splines is built separately on each grid interval $[x_j, x_{j+1}]$. This approximation has the form of the product of the function values at the grid nodes and the basis functions. Basis functions are determined by solving a system of equations. Prof. S.G. Mikhlin called this system of equations a system of approximation relations. Note that the basic splines are calculated in advance once, and then they are used in solving various problems, including interpolation, solving differential and integral problems by variational methods, constructing grid schemes, etc. When applied on a finite interval $[a, b]$, the left, the right and the middle splines have to be applied. This is due to the fact that it is necessary to use grid nodes only on the given interval $[a, b]$.

Each basis function has support consisting of s grid intervals. The theory of approximation by minimal interpolation splines was built in Yu.K. Demyanovich and I.G.Burova's works. Approximation theorems by interpolating polynomial splines were obtained earlier by the authors. Assume that a uniform grid of nodes is built and the length of the interval $[t_j, t_{j+1}]$ is equal to h.

Features of the use of the polynomial cubic splines of the fourth order of approximation, and polynomial splines of the fifth order of approximation are noted in the author's paper [13]. Now we recall the approximation properties of these splines. First consider the use of polynomial cubic splines. Approximation is constructed separately on each grid interval $[t_j, t_{j+1}] \subset [a, b]$ as a sum of products of function values at grid nodes and basis splines. Approximations differ at the beginning, in the middle and at the end of the interval $[a, b]$.

The approximation with the right polynomial splines is used at the beginning of the interval $[a, b]$ and can be written in the form:

$$U_j^R(x) = u(t_j)\omega_j^R(x) + u(t_{j+1})\omega_{j+1}^R(x) + u(t_{j+2})\omega_{j+2}^R(x) + u(t_{j+3})\omega_{j+3}^R(x), \; x \in [t_j, t_{j+1}],$$

where

$$\omega_j^R(x) = \frac{(x - t_{j+1})(x - t_{j+2})(x - t_{j+3})}{(t_j - t_{j+1})(t_j - t_{j+2})(t_j - t_{j+3})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_{j+1}^R(x) = \frac{(x - t_j)(x - t_{j+2})(x - t_{j+3})}{(t_{j+1} - t_j)(t_{j+1} - t_{j+2})(t_{j+1} - t_{j+3})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_{j+2}^R(x) = \frac{(x - t_j)(x - t_{j+1})(x - t_{j+3})}{(t_{j+2} - t_j)(t_{j+2} - t_{j+1})(t_{j+2} - t_{j+3})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_{j+3}^R(x) = \frac{(x-t_j)(x-t_{j+1})(x-t_{j+2})}{(t_{j+3}-t_j)(t_{j+3}-t_{j+1})(t_{j+3}-t_{j+2})},$$
$$x \in [t_j, t_{j+1}].$$

Note that the derivative of the function satisfies the relation:

$$(U_j^R(x))' = u(t_j)(\omega_j^R(x))' + u(t_{j+1})(\omega_{j+1}^R(x))' + u(t_{j+2})(\omega_{j+2}^R(x))' + u(t_{j+3})(\omega_{j+3}^R(x))'.$$

The formulae for the first derivative of the basis splines on a uniform grid of nodes with step h takes the form:

$$(\omega_j^R(x_j+th))' = \frac{-3t^2+12t-11}{6h},$$

$$(\omega_{j+1}^R(x_j+th))' = \frac{3t^2-10t+6}{2h},$$

$$(\omega_{j+2}^R(x_j+th))' = \frac{-3t^2+8t-3}{2h},$$

$$(\omega_{j+3}^R(x_j+th))' = \frac{3t^2-6t+2}{6h}.$$

Let us denote

$$\| u^{(q)} \|_{[a,b]} = \max_{[a,b]} |u^{(q)}(x)|,$$
$$R_0^R = \max_{[a,b]} |u(x) - U_j^R(x)|$$
$$R_1^R = \max_{[a,b]} |u'(x) - (U_j^R(x))'|.$$

Table 1 shows the maximum errors in the approximation of functions and also the maximum errors in the approximation of their first derivative when the right splines were used on a uniform grid with a grid step $h = 0.01$. The grid of knots were extended to the right of the interval $[a, b]$ by two nodes: t_{n+1}, t_{n+2}. It was assumed that the function values at these additional nodes are known. To calculate the maximum error, each grid interval $[t_j, t_{j+1}]$ was divided into 100 parts. At each division point, an approximation with the cubic splines of the function u was calculated (the calculations were done in Maple, $Digits =$ 15).

Table 1. The maximum errors in absolute values in the approximation of functions and of their first derivative

$u(x)$	R_0^R	R_1^R
$\frac{1}{1+25x^2}$	$0.6184 \cdot 10^{-5}$	$0.3713 \cdot 10^{-2}$
x^3	0	$0.1 \cdot 10^{-12}$
x^4	$0.9999 \cdot 10^{-8}$	$0.6 \cdot 10^{-5}$
x^5	$0.4962 \cdot 10^{-7}$	$0.3 \cdot 10^{-4}$

The graph of the error of the approximation of Runge function $\frac{1}{1+25x^2}$ with the right cubic splines is shown in Fig.1. The graph of the error of the approximation of the first derivative of Runge function $\frac{1}{1+25x^2}$ with the right cubic basis splines is shown in Fig.2.

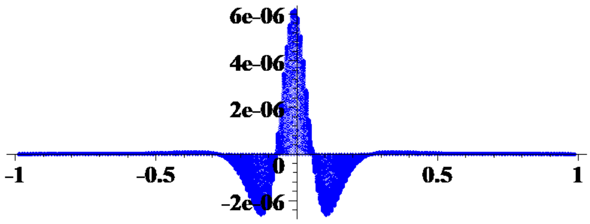

Fig. 1: The graph of the error of the approximation of Runge function $\frac{1}{1+25x^2}$ with the right cubic splines

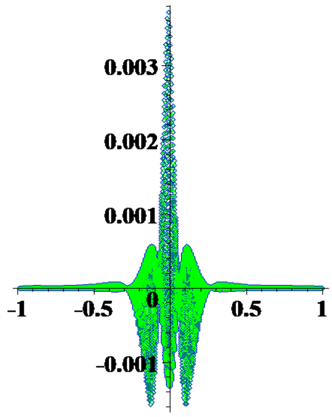

Fig. 2: The graph of the error of the approximation of the first derivative of Runge function $\frac{1}{1+25x^2}$ with the right cubic splines

Note that the formula for approximating the function by right splines implies the formula for approximating the first derivative on a uniform grid of nodes with step h

$$u'_j \approx \frac{-11u_j + 18u_{j+1} - 9u_{j+2} + 2u_{j+3}}{6h}.$$

The continuous polynomial approximation $U_j^L(x)$ near the right end of the interval $[a,b]$ uses the left basis spline $\omega_j^L(x)$ of the form:

$$\omega_{j-2}^L(x) = \frac{(x-t_{j-1})(x-t_j)(x-t_{j+1})}{(t_{j-2}-t_{j-1})(t_{j-2}-t_j)(t_{j-2}-t_{j+1})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_{j-1}^L(x) = \frac{(x-t_{j-2})(x-t_j)(x-t_{j+1})}{(t_{j-1}-t_{j-2})(t_{j-1}-t_j)(t_{j-1}-t_{j+1})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_j^L(x) = \frac{(x-t_{j-2})(x-t_{j-1})(x-t_{j+1})}{(t_j-t_{j-2})(t_j-t_{j-1})(t_j-t_{j+1})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_{j+1}^L(x) = \frac{(x-t_{j-2})(x-t_{j-1})(x-t_j)}{(t_{j+1}-t_{j-2})(t_{j+1}-t_{j-1})(t_{j+1}-t_j)}, \quad x \in [t_j, t_{j+1}].$$

Note that the derivative of the function satisfies the relation:

$$\left(U_j^L(x)\right)' = u(t_{j-2})(\omega_{j-2}^L(x))' + u(t_{j-1})(\omega_{j-1}^L(x))' + u(t_j)(\omega_j^L(x))' + u(t_{j+1})(\omega_{j+1}^L(x))', \quad x \in [t_j, t_{j+1}].$$

Let $t \in [0,1]$. The formulae for the first derivative of the basis splines on a uniform grid of nodes with step h take the form:

$$(\omega_j^L(x_j+th))' = \frac{-3t^2-4t+1}{2h},$$

$$(\omega_{j+1}^L(x_j+th))' = \frac{3t^2+6t+2}{6h},$$

$$(\omega_{j-2}^L(x_j+th))' = \frac{-3t^2+1}{6h},$$

$$(\omega_{j-1}^L(x_j+th))' = \frac{3t^2+2t-2}{2h}.$$

The graph of the error of the approximation of Runge function $\frac{1}{1+25\,x^2}$ with the left cubic splines is shown in Fig.3. The graph of the error of the approximation of the first derivative of Runge function $\frac{1}{1+25\,x^2}$ with the left cubic splines is shown in Fig.4.

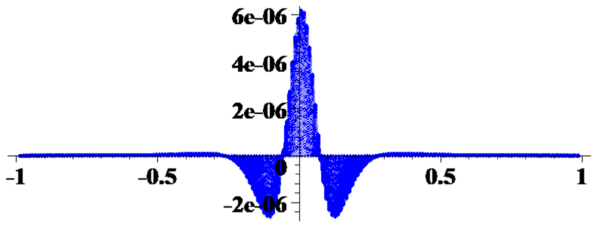

Fig. 3: The graph of the error of the approximation of Runge function $\frac{1}{1+25\,x^2}$ with the left cubic splines

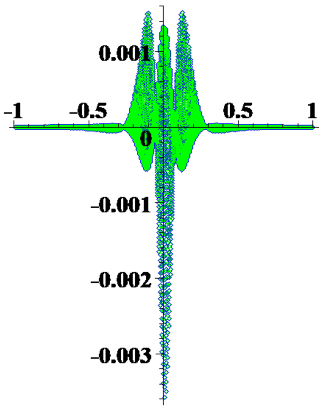

Fig. 4: The graph of the error of the approximation of the first derivative of Runge function $\frac{1}{1+25\,x^2}$ with the left cubic splines

The errors in the approximation errors of the Runge function (Figs. 1, 3) and the derivative of the Runge function (Figs. 2, 4) confirm the theoretical results presented in the Theorem and Tables 1, 2. In addition, we should remember that we should not approximate the Runge function with the Lagrange interpolation polynomials on a uniform grid on the interval [-1,1]. The problem is that the norm of the error in approximating the Runge function by interpolation polynomials tends to grow infinitely as the degree of the interpolation polynomial increases. In our case, when we apply spline approximations, we obtain a completely satisfactory result. Here, when the uniform grid is refined, the approximation error decreases. This follows from the theoretical results formulated in the theorems.

In addition, looking at Figures 1-4, we can see that the use of an appropriate non-uniform grid can give a solution with a smaller error.

In earlier works of the author, also the middle splines were considered. With application of the middle splines, we get a smaller approximation error. In this paper, we will not dwell on approximations by the middle splines.

Let us denote

$$R_0^L = \max_{[a,b]} |u(x) - U_j^L(x)|,$$

$$R_1^L = \max_{[a,b]} |u'(x) - (U_j^L(x))'|.$$

Table 2 shows the maximum errors in the approximation of functions and their first derivative with the left splines on a uniform grid with a grid step $h = 0.01$. The grid of knots has been extended to the right by two nodes: t_{-1}, t_{-2}.

Table 2. The maximum errors in absolute values in the approximation of functions and of their first derivative

$u(x)$	R_0^L	R_1^L
$\frac{1}{1+25x^2}$	$0.6184 \cdot 10^{-5}$	$0.3578 \cdot 10^{-2}$
x^3	0	$0.1 \cdot 10^{-11}$
x^4	$0.9999 \cdot 10^{-8}$	$0.5782 \cdot 10^{-5}$
x^5	$0.4962 \cdot 10^{-7}$	$0.2868 \cdot 10^{-4}$

Note that the formula for approximating the function by right splines implies the formula for approximating the first derivative on a uniform grid of nodes with step h:

$$u'_j \approx \frac{3u_j + 2u_{j+1} - 6u_{j-1} + u_{j-2}}{6h}.$$

The following Theorem is true.

Theorem 1. Let $u \in C^4[a,b]$. $t_j = a + jh, j = 0,1,\dots,n, h = \frac{b-a}{n}, n \geq 3.$ To approximate the function $u(x)$, $x \in [x_j, x_{j+1}]$, with the left and right splines, the following inequalities are valid:

$$|u(x) - U_j^L(x)| \leq Kh^4 \parallel u^{(4)} \parallel_{[t_{j-2},t_{j+1}]}, K = 1.$$

$$|u(x) - U_j^R(x)| \leq Kh^4 \parallel u^{(4)} \parallel_{[t_j,t_{j+3}]}, \ K = 1.$$

The proof can be found in paper [13].

Consequence. Let the values of the function be given at the grid nodes with step h. For an approximate calculation of the first derivative of a function, the following equalities are valid.

$$u'_j \approx \frac{3u_j + 2u_{j+1} - 6u_{j-1} + u_{j-2}}{6h},$$

$$u'_j \approx \frac{-11u_j + 18u_{j+1} - 9u_{j+2} + 2u_{j+3}}{6h}.$$

Let us dwell on the case of the presence in the equation of the derivative of both the first and second orders. We will replace these derivatives both with the help of known numerical differentiation formulas and with the help of formulas obtained using cubic splines.

Problem 1. Consider the integro-differential equation

$$u'(x) - 1 + \frac{u(x)}{3} - \int_0^1 K(x,t)u(t)dt = 0$$

with the $K(x,t) = x\,t$ and the exact solution is the next: $u(x) = x$.

The function $u(t)$ under the integral sign is approximated by the cubic splines. For the approximation $u'(x)$ we use the formulae from the Consequence to Theorem 1.

The error of the solution of *Problem 1* obtained with cubic splines at 9 grid nodes ($n = 8$) is shown in Figs. 5, 6. Fig. 5 shows us the solution when $Digits = 18$, Fig. 6 shows us the solution when $Digits = 20$.

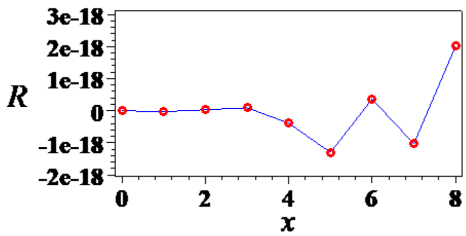

Fig. 5: The plot of the error of the solution of *Problem 1* obtained when $n = 8$.

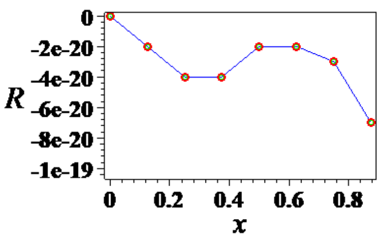

Fig. 6: The plot of the error of the solution of *Problem 1* obtained when $n = 9$.

3 Local Splines of the Fifth order of Approximation and Applications

Next, we are interested in comparing the results of when we apply the fifth-order polynomial splines [13]. Denote $u_i = u(t_i)$. In what follows, we will use the following types of approximations of the function $u(t)$ on interval $[t_i, t_{i+1}]$. At the beginning of the interval $[a,b]$, we apply the approximation with the right splines

$$U_{R5}^{i}(x)=\sum_{j=i}^{i+4}u_j w_j(x),\ x\in[t_i,t_{i+1}],$$

where u_j, $j=0,\dots,n$, are the values of the function in nodes t_j the basis splines $w_i(x)$ are the next:

$$w_i(x)=\frac{(x-t_{i+1})(x-t_{i+2})(x-t_{i+3})(x-t_{i+4})}{(t_i-t_{i+1})(t_i-t_{i+2})(t_i-t_{i+3})(t_i-t_{i+4})},$$

$$w_{i+1}(x)=\frac{(x-t_i)(x-t_{i+2})(x-t_{i+3})(x-t_{i+4})}{(t_{i+1}-t_i)(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})(t_{i+1}-t_{i+4})},$$

$$w_{i+2}(x)=\frac{(x-t_i)(x-t_{i+1})(x-t_{i+3})(x-t_{i+4})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+3})(t_{i+2}-t_{i+4})},$$

$$w_{i+3}(x)=\frac{(x-t_i)(x-t_{i+1})(x-t_{i+2})(x-t_{i+4})}{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})(t_{i+3}-t_{i+4})},$$

$$w_{i+4}(x)=\frac{(x-t_i)(x-t_{i+1})(x-t_{i+2})(x-t_{i+3})}{(t_{i+4}-t_i)(t_{i+4}-t_{i+1})(t_{i+4}-t_{i+2})(t_{i+4}-t_{i+3})}.$$

In the middle of the interval $[a,b]$, we apply the approximation with the middle splines:

$$U_{S5}^{i}(x)=\sum_{j=i-2}^{i+2}u_j w_j^s(x),\ x\in[t_i,t_{i+1}],$$

where

$$w_{i-2}^s(x)=\frac{(x-t_{i-1})(x-t_i)(x-t_{i+1})(x-t_{i+2})}{(t_{i-2}-t_{i-1})(t_{i-2}-t_i)(t_{i-2}-t_{i+1})(t_{i-2}-t_{i+2})},$$

$$w_{i-1}^s(x)=\frac{(x-t_{i-2})(x-t_i)(x-t_{i+1})(x-t_{i+2})}{(t_{i-1}-t_{i-2})(t_{i-1}-t_i)(t_{i-1}-t_{i+1})(t_{i-1}-t_{i+2})},$$

$$w_i^s(x)=\frac{(x-t_{i-2})(x-t_{i-1})(x-t_{i+1})(x-t_{i+2})}{(t_i-t_{i-2})(t_i-t_{i-1})(t_i-t_{i+1})(t_i-t_{i+2})},$$

$$w_{i+1}^s(x)=\frac{(x-t_{i-2})(x-t_{i-1})(x-t_i)(x-t_{i+2})}{(t_{i+1}-t_{i-2})(t_{i+1}-t_{i-1})(t_{i+1}-t_i)(t_i-t_{i+2})},$$

$$w_{i+2}^s(x)=\frac{(x-t_{i-2})(x-t_{i-1})(x-t_i)(x-t_{i+1})}{(t_{i+2}-t_{i-2})(t_{i+2}-t_{i-1})(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}$$

At the end of the interval $[a,b]$, we apply the approximation with the left splines:

$$U_{L5}^{i}(x)=\sum_{j=i-3}^{i+1}u_j w_j(t),\ t\in[t_i,t_{i+1}],$$

where the basis splines are the following:

$$w_{i-3}(x)=\frac{(x-t_{i-2})(x-t_{i-1})(x-t_i)(x-t_{i+1})}{(t_{i-3}-t_{i-2})(t_{i-3}-t_{i-1})(t_{i-3}-t_i)(t_{i-3}-t_{i+1})},$$

$$w_{i-2}(x)=\frac{(x-t_{i-3})(x-t_{i-1})(x-t_i)(x-t_{i+1})}{(t_{i-2}-t_{i-3})(t_{i-2}-t_{i-1})(t_{i-2}-t_i)(t_{i-2}-t_{i+1})},$$

$$w_{i-1}(x)=\frac{(x-t_{i-3})(x-t_{i-2})(x-t_i)(x-t_{i+1})}{(t_{i-1}-t_{i-3})(t_{i-1}-t_{i-2})(t_{i-1}-t_i)(t_{i-1}-t_{i+1})},$$

$$w_i(x)=\frac{(x-t_{i-3})(x-t_{i-2})(x-t_{i-1})(x-t_{i+1})}{(t_i-t_{i-3})(t_i-t_{i-2})(t_i-t_{i-1})(t_i-t_{i+1})},$$

$$w_{i+1}(x)=\frac{(x-t_{i-3})(x-t_{i-2})(x-t_{i-1})(x-t_i)}{(t_{i+1}-t_{i-3})(t_{i+1}-t_{i-2})(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}.$$

Table 3 shows the maximum errors in the approximation of functions and also the maximum errors in the approximation of their first derivative when the right splines were used on a uniform grid with a grid step $h=0.01$. The grid of knots has been extended to the right by three nodes: $t_{n+1}, t_{n+2},\ t_{n+3}$.

Table 3. The maximum errors in absolute values in the approximation of functions and of their first derivative

$u(x)$	R_0	R_1
$\frac{1}{1+25\,x^2}$	$0.9354\ \cdot 10^{-6}$	$0.6179\ \cdot 10^{-3}$
x^3	0	$0.1\ \cdot 10^{-11}$
x^4	0.	$0.8\ \cdot 10^{-12}$
x^5	$0.3631\ \cdot 10^{-9}$	$0.24\ \cdot 10^{-6}$

The graph of the error of the approximation of the Runge function $\frac{1}{1+25\,x^2}$ with the right splines of the fifth order of approximations shown in Fig.7.

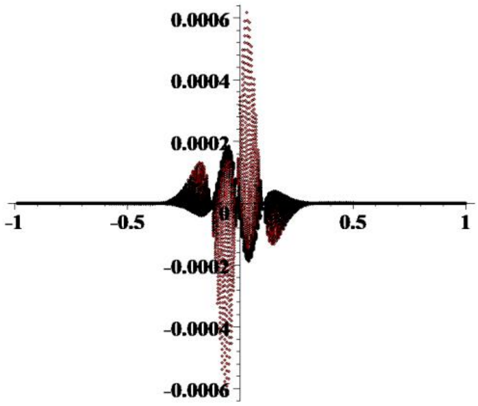

Fig. 7: The graph of the error of the approximation of the first derivative of the Runge function $\frac{1}{1+25\,x^2}$ with the right splines of the fifth order of approximation

Theorem 2. Let $u \in C^5[a,b]$. $t_j = a + jh, j = 0,1,\ldots,n,\ h = \frac{b-a}{n}, n \geq 4$.
To approximate the function $u(x)$, $x \in [t_i, t_{i+1}]$, with the left and right splines, the following inequalities are valid:

$$\left|u(x) - U_{L5}^{i}(x)\right| \leq Kh^5 \parallel u^{(5)} \parallel_{[t_{i-3},t_{i+1}]}, K = 3.63/5!.$$

$$\left|u(x) - U_{R5}^{i}(x)\right| \leq Kh^5 \parallel u^{(5)} \parallel_{[t_i,t_{i+4}]},\ K = 3.63/5!.$$

To approximate the function $u(x)$, $x \in [t_i, t_{i+1}]$, with the middle splines, the following inequality is valid:

$$\left|u(x) - U_{S5}^{i}(x)\right| \leq Kh^5 \parallel u^{(5)} \parallel_{[t_{i-2},t_{i+2}]},\ K = 1.42/5!.$$

Consequence. Let the values of the function be given at the grid nodes with step h. For an approximate calculation of the first derivative of a function, the following equalities are valid:

$$u'_j \approx \frac{25u_j - 48u_{j+1} + 36u_{j+2} - 16u_{j+3} + 3u_{j+4}}{12\,h}.$$

Next, we present the results of solving several integro-differential equations.

Problem 2. Consider the following equation:

$$u'(x) - u(x) - \exp(x) + x - \int_0^1 K(x,t)u(t)dt = 0\,.$$

The kernel of this equation is the next: $K(x,t) = x$, and the exact solution is the following: $u = x\exp(x)$. To solve this equation, we use polynomial splines of the fifth order of approximation. Let us take 8 nodes. The graph of the solution error is shown in Fig. 8.

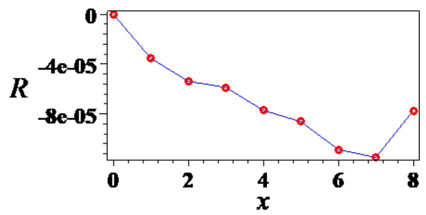

Fig. 8: The plot of the solution error of *Problem* 2.

Problem 3. Now, consider the following equation from paper [2].

$$u'(x) = 1 + \int_0^x u(t)u'(t)dt\,,\ 0 \leq x \leq 1,$$

where $u(0) = 0$, with the exact solution $u(x) = \sqrt{2}\tan\left(\frac{x}{\sqrt{2}}\right)$. The interval [0,1] was divided into 32 subintervals. The grid nodes are renumbered from 0 to 32.Fig. 9 shows the plot of the solution errors that was obtained using polynomial splines of the fifth order of approximation. Fig. 10 shows the plot of the solution errors that were obtained using cubic splines of the fourth order of approximation.

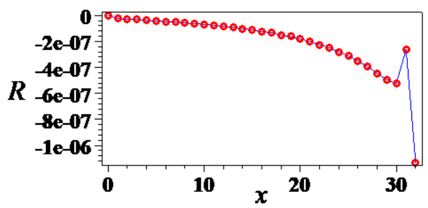

Fig. 9: The plot of the solution error of *Problem 3* (splines of the fifth order of approximation) *Digits=18*

As noted earlier, in paper [2] the next numerical methods were compared, namely, the Laplace decomposition method (LDM), the Wavelet-Galerkin method (WGM). Figures 11-15 show plots of the error of solution obtained with the cubic splines and WGM-method, with the cubic splines and LDM-method, with the splines of the fifth order of approximation and the LDM-method, with the splines of the fifth order of approximation and the WGM-method.

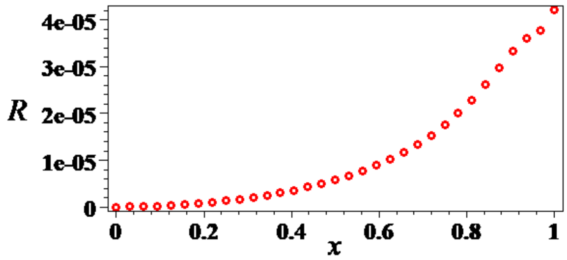

Fig. 10: The plot of the solution error of *Problem* 3 (cubic splines of the fourth order of approximation) *Digits=18*

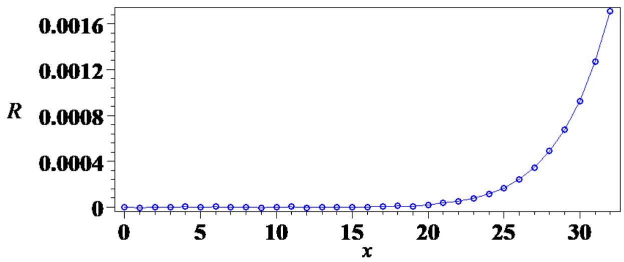

Fig. 11: The plot of the solution error of *Problem* 3 obtained with the LDM-method (blue) (paper [2])

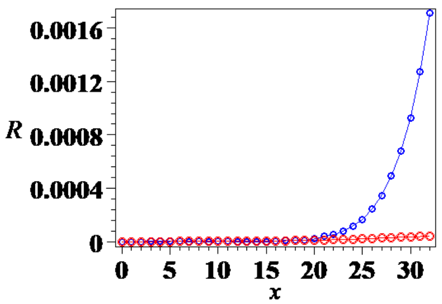

Fig. 12: The plot of the solution error of *Problem* 3 obtained with the cubic splines (red) and the LDM-method (blue)

Table 4 presents the errors in solving this equation obtained using polynomial splines of the fourth order of approximation (column 2), and using polynomial splines of the fifth order of approximation (column 3). Column 1 lists the node numbers.

Table 4. The errors in solving *Problem 3* are obtained when using polynomial splines of the fourth order of approximation.

Number of	Cubic spline of	Splines of the

node	the 4th order of approximation	5th order of approximation
0	0	0
1	$0.5086 \cdot 10^{-7}$	$0.2047 \cdot 10^{-7}$
2	$0.1323 \cdot 10^{-6}$	$0.3001 \cdot 10^{-7}$
3	$0.2452 \cdot 10^{-6}$	$0.3148 \cdot 10^{-7}$
4	$0.3908 \cdot 10^{-6}$	$0.3882 \cdot 10^{-7}$
5	$0.5707 \cdot 10^{-6}$	$0.4033 \cdot 10^{-7}$
6	$0.7870 \cdot 10^{-6}$	$0.4805 \cdot 10^{-7}$
7	$0.1042 \cdot 10^{-5}$	$0.5011 \cdot 10^{-7}$
8	$0.1339 \cdot 10^{-5}$	$0.5844 \cdot 10^{-7}$
9	$0.1682 \cdot 10^{-5}$	$0.6128 \cdot 10^{-7}$
10	$0.2073 \cdot 10^{-5}$	$0.7047 \cdot 10^{-7}$
11	$0.2519 \cdot 10^{-5}$	$0.7438 \cdot 10^{-7}$
12	$0.3025 \cdot 10^{-5}$	$0.8473 \cdot 10^{-7}$
13	$0.3597 \cdot 10^{-5}$	$0.9005 \cdot 10^{-7}$
14	$0.4242 \cdot 10^{-5}$	$0.1019 \cdot 10^{-6}$
15	$0.4969 \cdot 10^{-5}$	$0.1091 \cdot 10^{-6}$
16	$0.5789 \cdot 10^{-5}$	$0.1231 \cdot 10^{-6}$
17	$0.6711 \cdot 10^{-5}$	$0.1327 \cdot 10^{-6}$
18	$0.7750 \cdot 10^{-5}$	$0.1494 \cdot 10^{-6}$
19	$0.8922 \cdot 10^{-5}$	$0.1623 \cdot 10^{-6}$
20	$0.1024 \cdot 10^{-4}$	$0.1826 \cdot 10^{-6}$
21	$0.1173 \cdot 10^{-4}$	$0.1998 \cdot 10^{-6}$
22	$0.1341 \cdot 10^{-4}$	$0.2249 \cdot 10^{-6}$
23	$0.1532 \cdot 10^{-4}$	$0.2479 \cdot 10^{-6}$
24	$0.1749 \cdot 10^{-4}$	$0.2594 \cdot 10^{-6}$
25	$0.1998 \cdot 10^{-4}$	$0.2796 \cdot 10^{-6}$
26	$0.2284 \cdot 10^{-4}$	$0.3104 \cdot 10^{-6}$
27	$0.2612 \cdot 10^{-4}$	$0.3510 \cdot 10^{-6}$
28	$0.2974 \cdot 10^{-4}$	$0.3923 \cdot 10^{-6}$
29	$0.3328 \cdot 10^{-4}$	$0.4446 \cdot 10^{-6}$
30	$0.3601 \cdot 10^{-4}$	$0.4953 \cdot 10^{-6}$
31	$0.3778 \cdot 10^{-4}$	$0.5222 \cdot 10^{-6}$
32	$0.4220 \cdot 10^{-4}$	$0.1134 \cdot 10^{-5}$

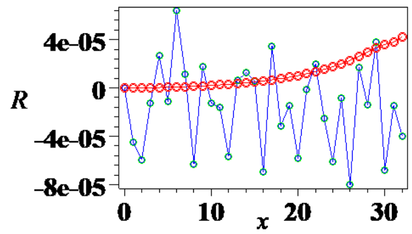

Fig. 13: The plot of the solution error of *Problem* 3 obtained with the cubic splines (red) and the WGM -method (green points)

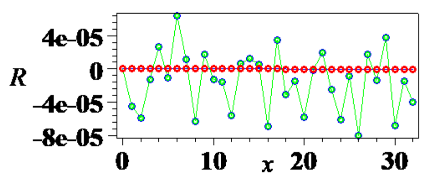

Fig. 14: The plot of the solution error of *Problem* 3 obtained with the splines of the fifth order of approximation and the WGM -method

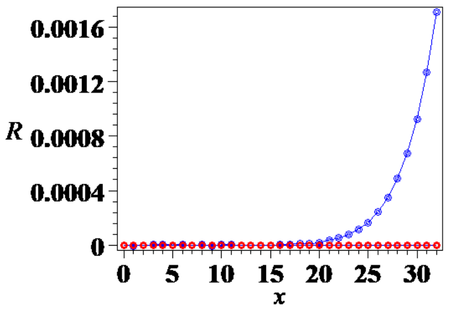

Fig. 15: The plot of the solution error of *Problem* 3 obtained with the splines of the fifth order of approximation (red) and the LDM-method (blue)

In Figures 11-15, the solution at the grid nodes, obtained using splines of the fourth and fifth approximation orders, is marked with red circles, the solutions obtained by other methods of paper [2] are marked by green and blue circles. Thus, it can be seen that for this equation, the error of the solution with the splines turned out to be no worse than when using the WGM-method, and the LDM-method.

Problem 4. Consider the following integro-differential equation:

$$u'(x) = -1 + \int_0^x u^2(t)dt\,, \qquad 0 \le x \le 1,$$

with the initial condition $u(0) = 0$.

The solution of this integro-differential equation cannot be represented explicitly. To solve this nonlinear integro-differential equation, we will first use splines of the fourth order of approximation, then splines of the fifth order of approximation. To approximate the first derivative, we will use numerical differentiation formulas obtained on the basis of the corresponding splines. The interval [0,1] was divided into 32 subintervals. The grid nodes are renumbered from 0 to 32. Table 5 presents the solutions of this equation obtained using polynomial splines of the fourth order of approximation (column 2), and using polynomial splines of the fifth order of approximation (column 3).

Table 5. The solutions of *Problem 4*

Number of node	Cubic spline of the 4^{th} order of approximation	Splines of the 5^{th} order of approximation
0	0	0
1	-0.3125$\cdot 10^{-1}$	-0.3125$\cdot 10^{-1}$
2	-0.6250$\cdot 10^{-1}$	-0.6250$\cdot 10^{-1}$
3	-0.9374$\cdot 10^{-1}$	-0.9374$\cdot 10^{-1}$
4	-0.1250	-0.1250
5	0.1562	0.1562
6	0.1874	0.1874
7	0.2186	0.2186
8	0.2497	0.2497
9	0.2807	0.2807
10	0.3117	0.3117
11	0.3426	0.3426
12	0.3734	0.3734
13	0.4040	0.4040
14	0.4345	0.4345
15	0.4647	0.4647
16	0.4948	0.4948
17	0.5247	0.5247
18	0.5542	0.5542
19	0.5835	0.5835
20	0.6124	0.6124
21	0.6410	0.6410
22	0.6692	0.6692
23	0.6969	0.6969
24	0.7242	0.7242
25	0.7509	0.7509
26	0.7771	0.7771
27	0.8027	0.8027
28	0.8277	0.8277
29	0.8520	0.8520
30	0.8756	0.8756
31	0.8984	0.8984
32	0.9205	0.9205

Column 1 of Table 5 lists the node numbers. It should be noted that in the paper [2] the results of numerical experiments are obtained based on the application of four different methods (the Laplace decomposition method (LDM), the Wavelet–Galerkin method (WGM), the Laplace decomposition method with the Pade approximant and the homotopy perturbation method (HPM)). The results obtained using the method based on the use of splines coincide with the results of applying the methods of paper [2].

4 Conclusion

As is known, in the numerical solution of equations and systems of equations, several different solution methods are usually used. This is necessary for a verification of the result. If the kernel, coefficients and the right side of the equation are sufficiently smooth, then the proposed method can give a solution with a smaller error.

This paper shows the results of applying local interpolation splines of the fourth and fifth order of approximation for solving integro-differential equations with Fredholm and Volterra integrals of the second kind. The main focus was on the equations with the first derivative.

Comparisons of the results of applying local splines with the use of other methods for solving integro-differential equations are shown. It is shown that in some cases the application of the approach to solving integral equations based on splines gives a smaller error for the same number of nodes. In addition, the approach based on spline approximations is quite simple to implement and gives a reliable result.

We emphasize once again that the advantage of the spline approach is the simple implementation of the algorithm. As a result of applying this approach, we have to solve a system of equations (linear or nonlinear). As a result, we obtain an approximation to the solution of the original integro-differential equation in the form of grid function values at the grid nodes.

To obtain an approximate solution at points between the grid nodes, it is convenient to use the same spline approximations. The result is a continuous line.

To obtain a twice continuously differentiable approximate solution, a special method considered by the author earlier can be used. In this case, it is necessary to solve a system of linear algebraic equations additionally. The matrix of this system of equations will have a tape form.

In the future, other types of integro-differential equations and systems of equations will be considered.

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