On the estimates in various spaces to the control function of the extremum problem for parabolic equation

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Abstract: For the minimization problem with pointwise observation governed by a one-dimensional parabolic equation with a free convection term and a depletion potential, we formulate a result on the existence and uniqueness of a minimizer from a prescribed set. We use a weight quadratic cost functional showing the temperature deviation. We obtain estimates for the norm of control functions in terms of the value of the quality functional in different functional spaces. It gives us a possibility to estimate the required internal energy of the system. To prove these results we establish the positivity principle.

Key–Words: Parabolic equation, pointwise observation, extremum problem, control function, estimates.

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1 Introduction

We consider the extremum problem with weighted integral cost functional for the following parabolic mixed problem

$$u_t = (a(x,t)u_x)_x + b(x,t)u_x + d(x,t)u, \qquad (1)$$
$$(x,t) \in Q_T = (0,1) \times (0,T), \quad T > 0,$$
$$u(0,t) = \varphi(t), \quad u_x(1,t) = \psi(t), \qquad (2)$$
$$0 < t < T,$$
$$u(x,0) = \xi(x), \quad 0 < x < 1, \qquad (3)$$

where the real functions a, b and d are smooth in $\overline{Q}_T$,

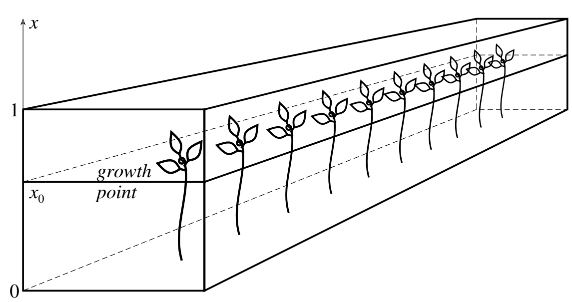
$$0 < a_0 \le a(x,t) \le a_1 < \infty, \qquad (4)$$

$\varphi \in W_2^1(0,T)$, $\psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$. Here $W_2^1(0,T)$ is Sobolev space of weakly differentiable functions with the norm

$$\|u\|_{W_2^1(0,T)} = \left(\int_0^T (|u'|^2 + |u|^2)dt\right)^{1/2}. \qquad (5)$$

We study the control problem with a point observation: by controlling the temperature φ at the left end of the segment (the functions ψ and ξ are assumed to be fixed), we try to make at some point $x_0 \in (0,1)$ the temperature $u(x_0,t)$ close to the given function $z(t)$ over the entire time interval $(0,T)$.

This problem arises in the model of climate control in industrial greenhouses [1] – [2].



Note that extremal problems for parabolic equations were considered in [10] – [13] (as usual, problems with final or distributed observation). But the results and methods of investigation are not similar to our methods.

The proposed paper develops and generalizes the authors' results of [1] – [8]. We study here a more general equation with a variable diffusion coefficient a, a convection coefficient b, and a potential d, called the depletion potential, and obtain lower estimates of control function φ with the help of the value of a quality functional in various spaces.

Some methods of study the parabolic control problems is used in [14], but there were no results similar to presented in our article (for example, estimates of control functions or positivity principle).

Definition 1. ([9], p. 6.) *We denote by $V_2^{1,0}(Q_T)$ the Banach space of functions $u \in W_2^{1,0}(Q_T)$ with the finite norm*

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0\le t\le T}\|u(x,t)\|_{L_2(0,1)} + \|u_x\|_{L_2(Q_T)} \tag{6}$$

such that $t \mapsto u(\cdot,t)$ is a continuous mapping $[0,T] \to L_2(0,1)$.

We denote by $\widetilde{W}_2^1(Q_T)$ the set of functions $\eta \in W_2^1(Q_T)$ satisfying the conditions $\eta(x,T) = 0$, $\eta(0,t) = 0$.

Definition 2. *A weak solution to problem (1) — (3) is a function $u \in V_2^{1,0}(Q_T)$ satisfying the condition $u(0,t) = \varphi(t)$ and the equality*

$$\int_{Q_T} (a(x,t)u_x\eta_x - b(x,t)u_x\eta - d(x,t)u\eta - u\eta_t)\,dx\,dt = \int_0^1 \xi(x)\eta(x,0)\,dx + \int_0^T a(1,t)\psi(t)\,\eta(1,t)\,dt \tag{7}$$

for all $\eta \in \widetilde{W}_2^1(Q_T)$.

2 Main Results

Theorem 3. ([7] – [8]) *The problem (1) — (3) has a unique weak solution $u \in V_2^{1,0}(Q_T)$, and it satisfies the inequality*

$$\|u\|_{V_2^{1,0}(Q_T)} \le C_1\big(\|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)} + \|\xi\|_{L_2(0,1)}\big), \tag{8}$$

with some constant C_1 independent of φ, ψ and ξ.

Corollary 4. *The solution u of the problem (1) — (3) continuously depends on the initial datum ξ and boundary data φ, ψ from $L_2(0,1) \times W_2^1(0,T) \times W_2^1(0,T)$ to $V_2^{1,0}(Q_T)$.*

Note that a numerical solution of a parabolic Dirichlet boundary value problem is obtained in [15].

To obtain the next estimate we need the following *positivity principle.*

Theorem 5. *Let u be a solution of the problem (1) — (3) with the boundary and initial functions φ, ψ and ξ satisfying the conditions* $\operatorname{ess}\inf_{t\in(0,T)} \varphi \ge 0$, $\operatorname{ess}\inf_{t\in(0,T)} \psi \ge 0$, $t \in [0,T]$, $\operatorname{ess}\inf_{t\in(0,1)} \xi \ge 0$, $x \in [0,1]$. *Then the solution u is nonnegative, too:*

$$\operatorname{ess}\inf_{(x,t)\in Q_T} u \ge 0. \tag{9}$$

Using Theorem 5, in the case of nonnegative φ, ψ and ξ, we have the following estimate.

Theorem 6. *Let the functions a, b, d satisfy the conditions $a_t \ge 0$, $b_x - d \ge 0$, $(x,t) \in Q_T$; $b \ge 0$, $(x,t) \in [0,x_0] \times [0,T]$, $x_0 \in (0,1]$; $b(1,t) \le 0$, $t \in [0,T]$; $\varphi \ge 0$, $\psi \ge 0$, $\xi \ge 0$. Then for the solution u of the problem (1) — (3) the inequality*

$$\|u(x_0,t)\|_{L_1(0,T)} \le \|\varphi\|_{L_1(0,T)} + \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right), \tag{10}$$

holds.

Corollary 7. *Let a, b, d satisfy the conditions of Theorem 6, $\varphi \ge 0$, $\psi = 0$, $\xi = 0$. Then, for solution of the problem (1) — (3), the following inequality holds:*

$$\|u(x_0,t)\|_{L_1(0,T)} \le \|\varphi\|_{L_1(0,T)}. \tag{11}$$

We denote by $\Phi \subset W_2^1(0,T)$ the set of control functions φ and by $Z \subset L_2(0,T)$ the set of objective functions z. We further suppose that Φ is nonempty, closed, convex and bounded set. Consider the weighted integral cost functional

$$J[z,\rho,\varphi] = \int_0^T |u_\varphi(x_0,t) - z(t)|^2\rho(t)dt, \tag{12}$$
$$x_0 \in (0,1), \quad \varphi \in \Phi, \quad z \in Z,$$

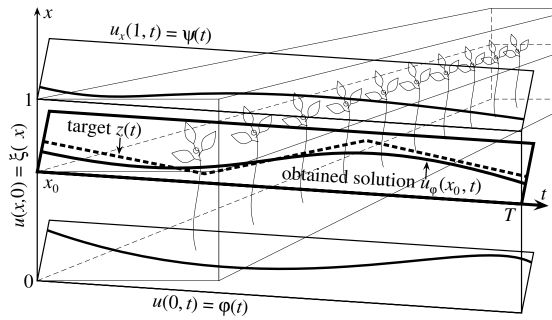
where $u_\varphi \in V_2^{1,0}(Q_T)$ is the solution to the problem (1) — (3) with the given control function φ. Here $\rho \in L_\infty(0,T)$ is a real-valued weight function such that

$$0 < \rho_1 = \operatorname{ess}\inf_{t\in(0,T)} \rho(t), \tag{13}$$

$$\rho_2 = \operatorname{ess}\sup_{t\in(0,T)} \rho(t). \tag{14}$$

Assuming the functions z and ρ to be fixed, consider the minimization problem

$$m[z,\rho,\Phi] = \inf_{\varphi\in\Phi} J[z,\rho,\varphi]. \tag{15}$$



In ([7] – [8]) the following result is obtained.

Theorem 8. *For any $z \in L_2(0,T)$ there exists a unique function $\varphi_0 \in \Phi$ such that*

$$m[z,\rho,\Phi] = J[z,\rho,\varphi_0].$$

Theorem 6 implies a lower estimate for the norm of control functions in terms of the value of the quality functional, as you can see from the following theorem.

Theorem 9. *Let x_0 and the functions a, b, d, φ, ψ, ξ satisfy the conditions of Theorem 6. Then the following inequality holds:*

$$\|\varphi\|_{L_1(0,T)} \tag{16}$$
$$\geq \max\left\{0, \|z\|_{L_1(0,T)} - \left(\frac{TJ[\varphi,\rho,z]}{\rho_1}\right)^{1/2}\right.$$
$$\left. - \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right)\right\}.$$

The resulting estimate (16) allows us to obtain information on control functions. It is useful for estimating the internal energy of the system necessary to achieve a required value of the functional J.

Corollary 10. *Let x_0 and the functions a, b, d, φ satisfy the conditions of Theorem 6. Suppose $\psi = \xi = 0$. Then the following inequality holds:*

$$\|\varphi\|_{L_1(0,T)} \tag{17}$$
$$\geq \max\left\{0, \|z\|_{L_1(0,T)} - \left(\frac{TJ[\varphi,\rho,z]}{\rho_1}\right)^{1/2}\right\}.$$

We can also give lower estimates to the various norms of the control function φ.

Theorem 11. *Let x_0 and the functions a, b, d, φ, ψ, ξ satisfy the conditions of Theorem 6. Then the following inequality holds:*

$$\|\varphi\|_{L_2(0,T)} \geq \max\left\{0, \frac{1}{T^{1/2}}\|z\|_{L_1(0,T)}\right. \tag{18}$$
$$- \left(\frac{J[\varphi,\rho,z]}{\rho_1}\right)^{1/2}$$
$$\left. - \frac{x_0}{a_1 T^{1/2}}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right)\right\}.$$

Theorem 12. *Let $x_0 \in (0,1]$ and the functions a, b, d, φ, ψ and ξ satisfy the conditions of Theorem 6. Then the following inequality holds:*

$$\|\varphi\|_{W_2^1(0,T)} \geq \max\left\{0, \frac{1}{T^{1/2}}\|z\|_{L_1(0,T)}\right. \tag{19}$$
$$- \left(\frac{J[\varphi,\rho,z]}{\rho_1}\right)^{1/2}$$
$$\left. - \frac{x_0}{a_1 T^{1/2}}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right)\right\}.$$

3 Proofs

Proof of theorem 6. At first, let us suppose, that the boundary and initial data of the problem (1) — (3) are smooth and the solution u is smooth, too. Integrating equation (1) by x over the interval $(z,1)$, $z \in (0,1)$ and using the boundary condition (2), we obtain the following equality:

$$\int_z^1 u_t(x,t)dx + a(z,t)u_x(z,t) - a(1,t)\psi(t) \tag{20}$$
$$- b(1,t)u(1,t) + b(z,t)u(z,t)$$
$$+ \int_z^1 (b_x(x,t) - d(x,t))u(x,t)dx.$$

Now, we divide the equality (20) by the positive function $a(z,t)$ and integrate over the integral $(0,x_0)$, where $x_0 \in (0,1)$ is a given point. Therefore,

$$\int_0^{x_0} \frac{dz}{a(z,t)} \int_z^1 u_t(x,t)dx + u(x_0,t) - \varphi(t) \tag{21}$$
$$- a(1,t)\psi(t)\int_0^{x_0}\frac{dz}{a(z,t)}$$
$$- b(1,t)u(1,t)\int_0^{x_0}\frac{dz}{a(z,t)} + \int_0^{x_0}\frac{b(z,t)u(z,t)dz}{a(z,t)}$$
$$+ \int_0^{x_0}\frac{dz}{a(z,t)}\int_z^1 (b_x(x,t) - d(x,t))u(x,t)dx.$$

The next step of proof is the integration of the resulting equality (21) by t over the interval $(0,T)$. We obtain

$$\begin{aligned}&\int_0^T dt\int_0^{x_0}\frac{dz}{a(z,t)}\int_z^1 u_t(x,t)dx \qquad (22)\\&+\int_0^T u(x_0,t)dt-\int_0^T\varphi(t)dt\\&-\int_0^T a(1,t)\psi(t)dt\int_0^{x_0}\frac{dz}{a(z,t)}\\&-\int_0^T b(1,t)u(1,t)dt\int_0^{x_0}\frac{dz}{a(z,t)}\\&+\int_0^T dt\int_0^{x_0}\frac{b(z,t)u(z,t)dz}{a(z,t)}\\&+\int_0^T dt\int_0^{x_0}\frac{dz}{a(z,t)}\times\\&\times\int_z^1(b_x(x,t)-d(x,t))u(x,t)dx.\end{aligned}$$

Integration by parts of (22) gives us the following relation

$$\begin{aligned}&\int_0^T u(x_0,t)dt+\int_0^{x_0}\frac{dz}{a(z,T)}\int_z^1 u(x,T)dx \qquad (23)\\&+\int_0^T dt\int_0^{x_0}\frac{a_t(z,t)}{a^2(z,t)}dz\int_z^1 u(x,t)dx\\&-\int_0^T b(1,t)u(1,t)dt\int_0^{x_0}\frac{dz}{a(z,t)}\\&+\int_0^T dt\int_0^{x_0}\frac{b(z,t)u(z,t)dz}{a(z,t)}\\&+\int_0^T dt\int_0^{x_0}\frac{dz}{a(z,t)}\times\\&\times\int_z^1(b_x(x,t)-d(x,t))u(x,t)dx\\&\le\int_0^T\varphi(t)dt+\int_0^T a(1,t)\psi(t)dt\int_0^{x_0}\frac{dz}{a(z,t)}\\&+\int_0^{x_0}\frac{dz}{a(z,0)}\int_z^1\xi(x)dx.\end{aligned}$$

Using Theorem 5, we obtain the non-negativity of the function u in $\overline{Q}_T$. Therefore, from equality (23), using the conditions $a_t\ge 0$, $b_x-d\ge 0$, $(x,t)\in Q_T$; $b\ge 0$, $(x,t)\in[0,x_0]\times[0,T]$, $x_0\in(0,1]$; $b(1,t)\le 0$, $t\in[0,T]$; $\varphi\ge 0$, $\psi\ge 0$, $\xi\ge 0$ and the inequality $u\ge 0$, we get the following inequality:

$$\begin{aligned}&\int_0^T u(x_0,t)dt \qquad (24)\\&\le\int_0^T\varphi(t)dt+\int_0^T a(1,t)\psi(t)dt\int_0^{x_0}\frac{dz}{a(z,t)}\\&+\int_0^{x_0}\frac{dz}{a(z,0)}\int_z^1\xi(x)dx.\end{aligned}$$

Combining (4) and (24), we obtain

$$\begin{aligned}&\int_0^T u(x_0,t)dt \qquad (25)\\&\le\int_0^T\varphi(t)dt+\int_0^T a_2\psi(t)dt\int_0^{x_0}\frac{dz}{a_1}\\&+\int_0^{x_0}\frac{dz}{a_1}\int_z^1\xi(x)dx\\&\le\int_0^T\varphi(t)dt\\&+\frac{a_2x_0}{a_1}\int_0^T\psi(t)dt+\frac{x_0}{a_1}\int_0^1\xi(x)dx.\end{aligned}$$

For non-negative functions, inequality (25) implies the inequality

$$\begin{aligned}&\|u(x_0,t)\|_{L_1(0,T)}\le\|\varphi\|_{L_1(0,T)}\\&+\frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)}+\|\xi\|_{L_1(0,1)}\right).\end{aligned}$$

Hence, we prove (10) for smooth data and smooth solutions. To prove it for all prescribed boundary and initial functions, we have to take the sequences of smooth functions φ_j, ψ_j and ξ_j such that $\|\varphi_j\|_{W_2^1(0,T)}\to 0$, $\|\psi_j\|_{W_2^1(0,T)}\to 0$, and $\|\xi_j\|_{L_2(0,1)}\to 0$, $j\to\infty$. Now, we have the estimate (10) for the corresponding solutions u_j, and obtain, by the inequality (8), the estimate (10) for the limit function u. Proof of Theorem 5 is complete. □

For further considerations we need the following elementary inequality.

Lemma 13. *Let B be a Banach space with the norm $\|\cdot\|_B$. Then for all $g\in B$, $h\in B$ we have*

$$\|g\|_B\ge\max\{0,\|h\|_B-\|h-g\|_B\}. \qquad (26)$$

Proof of lemma 13. The application of the triangle inequality $\|p+q\|_B\le\|p\|_B+\|q\|_B$ with $p=g$, $q=h-g$ yields

$$\|h\|_B\le\|g\|_B+\|h-g\|_B. \qquad (27)$$

Combining (27) and the relation $\|g\|\ge 0$, we obtain the inequality (26). □

Proof of theorem 9. Applying lemma 13 for $B=L_1(0,T)$, $g=u(x_0,t)$, $h=z(t)$, we get

$$\begin{aligned}&\|u_\varphi(x_0,\cdot)\|_{L_1(0,T)} \qquad (28)\\&\ge\max\left\{0,\|z(\cdot)\|_{L_1(0,T)}\right.\\&\left.-\|u_\varphi(x_0,\cdot)-z(\cdot)\|_{L_1(0,T)}\right\}.\end{aligned}$$

Now, It follows from Holder inequality and (13) that

$$\|u_\varphi(x_0,\cdot)-z(\cdot)\|_{L_1(0,T)} \le T^{1/2}\|u_\varphi(x_0,\cdot)-z(\cdot)\|_{L_2(0,T)} \le \left(\frac{TJ[\varphi,\rho,z]}{\rho_1}\right)^{1/2} \tag{29}$$

By (10) and (29) we obtain

$$\|\varphi\|_{L_1(0,T)} \ge \|u_\varphi(x_0,\cdot)\|_{L_1(0,T)} - \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)}+\|\xi\|_{L_1(0,1)}\right) \ge \|z(t)\|_{L_1(0,T)} - \left(\frac{TJ[\varphi,\rho,z]}{\rho_1}\right)^{1/2} - \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)}+\|\xi\|_{L_1(0,1)}\right). \tag{30}$$

□

Proof of theorem 11. By Holder inequality we have

$$\|\varphi\|_{L_1(0,T)} \le T^{1/2}\|\varphi\|_{L_2(0,T)}. \tag{31}$$

Combining (16) and (31) we get the estimate

$$T^{1/2}\|\varphi\|_{L_2(0,T)} \ge \max\left\{0, \|z\|_{L_1(0,T)} - \left(\frac{T^{1/2}J[\varphi,\rho,z]}{\rho_1}\right)^{1/2} - \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)}+\|\xi\|_{L_1(0,1)}\right)\right\}.$$

This completes the proof of theorem 11. □

Proof of theorem 12. By (5) we have

$$\|\varphi\|^2_{W_2^1(0,T)} = \|\varphi'\|^2_{L_2(0,T)} + \|\varphi\|^2_{L_2(0,T)} \ge \|\varphi\|^2_{L_2(0,T)}. \tag{32}$$

so

$$\|\varphi\|_{W_2^1(0,T)} \ge \|\varphi\|_{L_2(0,T)}. \tag{33}$$

Now, by (29) we obtain

$$\|\varphi\|_{L_2(0,T)} \ge \max\left\{0, \frac{1}{T^{1/2}}\|z\|_{L_1(0,T)} - \left(\frac{J[\varphi,\rho,z]}{\rho_1}\right)^{1/2} - \frac{x_0}{a_1T^{1/2}}\left(a_2\|\psi\|_{L_1(0,T)}+\|\xi\|_{L_1(0,1)}\right)\right\}. \tag{34}$$

So, from (33) and (34) it follows that

$$\|\varphi\|_{W_2^1(0,T)} \ge \max\left\{0, \frac{1}{T^{1/2}}\|z\|_{L_1(0,T)} - \left(\frac{J[\varphi,\rho,z]}{\rho_1}\right)^{1/2} - \frac{x_0}{a_1T^{1/2}}\left(a_2\|\psi\|_{L_1(0,T)}+\|\xi\|_{L_1(0,1)}\right)\right\}.$$

□

4 Open problems

1. It is interesting to obtain upper estimates in different spaces for control function φ similar to lower estimates (16), (18), (19).
2. An important problem is to prove that $m[z,\rho,W_2^1(0,T)] = 0$ for any $z \in L_2(0,T)$ and $\rho \in L_\infty(0,T)$ to the equation (1) with general type variable coefficients $a(x,t)$, $b(x,t)$, $d(x,t)$. Now this result is proved for equations with coefficients independent of t (see [4, 5, 6]).

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