Trigonometric Splines of the Third Order of Approximation and Interval Estimation

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Abstract: It is useful to apply interval estimates to improve the evaluation of reliability results of calculations, and therefore the evaluation of the reliability of mechanical structures. In this paper, interval estimates are used to establish the range of variation of a function and its derivatives As is known, the problem of the simultaneous approximation of a function and its derivatives cannot be solved using classical interpolation polynomials. In this paper, we consider the approximation of a function and its first derivative by using polynomial and trigonometric splines with the third order of approximation. In this case, the approximation of the first derivative turns out to be discontinuous at the nodes of the grid. The values of the constants in the estimates of the errors of approximation with the trigonometric and polynomial splines of the third order are given. It is shown that these constants cannot be reduced. To solve practical problems, it is often important not to calculate the values of the function and its derivatives in a number of nodes on the grid interval, but to estimate the range of change of the function on this interval. For the interval estimation of the approximation of function or its first derivative, we use the technique of working with real intervals from interval analysis. The algorithms for constructing the variation domain of the approximation of the function and the first derivative of this function are described. The results of the numerical experiments are given.

Key–Words: trigonometric splines, polynomial splines, interval estimation

1 Introduction

In this paper, we consider local trigonometric and polynomial splines suitable for solving many problems of applied mechanics. For example, the considered splines can be used to automatically generate the tool path associated with a certain geometry of the pocket profile [1] for light alloy aerospace parts. These splines can be successfully used in calculating the orbits of the planets, cloud of particles and trajectories of asteroids [2]. Another area of application of the splines is the flight control of unmanned aircrafts [3]. Our splines can be used in the Global Positioning System (GPS) that is a satellite navigation system which allows the users to determine 3D positioning and the time with high precision [4]. To improve the evaluation of reliability, a non-probabilistic reliability method was used in [5] to analyse the resonance of fluid-filled pipeline systems. In this paper the uncertain parameters of structures were described by both ellipsoidal modes and interval parameters.

It is useful to determine the lower and upper bounds of the values of functions, eigenvalues of operators, solutions of systems of linear and nonlinear equations without calculating a detailed numerical solution of the corresponding problems. The solution to these problems has been considered in many papers, which has recently been published.

In paper [6] a new approach for solving non linear systems of equations was proposed. This approach is based on Interval-Newton and Interval-Krawczyk operators and B-splines. The proposed algorithm is making great benefits of the geometric properties of B-spline functions to avoid unnecessary computations. For eigenvalue problems of self-adjoint differential operators, a universal framework is proposed to give explicit lower and upper bounds for the eigenvalues (see [7]).

To improve the calculation accuracy and reduce the computational cost, the interval analysis technique and radial point interpolation method are adopted in [8] to obtain the approximate frequency response characteristics for each focal element, and the corresponding formulations of structural-acoustic system for interval response analysis are deduced.

In 1964 Schoenberg introduced trigonometric spline functions [9].

This paper continues the series of papers on approximation by local polynomial and non-polynomial splines and interval estimation (see [12, 13, 14]).

For constructing the interval extension of the approximation of the function or its first derivative, we use techniques from interval analysis.

This paper focuses on polynomial and trigonometric splines of the third order approximation. It should be noted that Yu.K.Demyanovich devotes a lot of attention to the study of quadratic polynomial splines of the Lagrangian type (see [11]).

In some cases, the use of the trigonometric approximations is preferable to the polynomial approximations. Here we compare these two types of approximation. To approximate functions on a finite grid of nodes, we will use the left and right splines.

2 Approximation with the Left Splines with the Third Order of Approximation

We will apply left splines near the right end of the finite interval $[a, b]$. Right splines will be applied near the left end of the finite interval $[a, b]$. Suppose a, b be real numbers. Let the set of nodes x_j be such that $a < \ldots < x_{j-1} < x_j < x_{j+1} < \ldots < b$.

We construct an approximation $F(x)$ of function $f(x)$, $f \in C^{(3)}[a, b]$ with local splines, in which the support consists of three adjacent intervals. When approximating a function on a finite interval near the left and right boundaries of the interval $[a, b]$ we will use the approximation with the left or the right continuous splines. The set of interpolation local left and right splines are called boundary minimal splines. Near the right end of the finite interval $[a, b]$ we use the left splines.

First, suppose that basis spline $w_j(x)$ is such that $supp\ w_j = [x_{j-1}, x_{j+2}]$, and $w_j(x) = 0$ if $x \notin [x_{j-1}, x_{j+2}]$.

The approximation with the left polynomial or non-polynomial splines of the third approximation order can be written as $x \in [x_j, x_{j+1}]$ in the form:

$$F^L(x) = f(x_{j-1})w_{j-1}(x) + f(x_j)w_j(x)+ f(x_{j+1})w_{j+1}(x). \qquad (1)$$

Suppose that the functions $\varphi_0, \varphi_1, \varphi_2$ form the Chebyshev system and the determinant

$$det \begin{pmatrix} \varphi_0(x_{j-1}) & \varphi_0(x_j) & \varphi_0(x_{j+1}) \\ \varphi_1(x_{j-1}) & \varphi_1(x_j) & \varphi_1(x_{j+1}) \\ \varphi_2(x_{j-1}) & \varphi_2(x_j) & \varphi_2(x_{j+1}) \end{pmatrix}$$

is non-zero.

We obtain the basic functions $w_{j-1}(x)$, $w_j(x)$, $w_{j+1}(x)$, $x \in [x_j, x_{j+1}]$, solving the following system:

$$\begin{gathered} \varphi_0(x_{j-1})w_{j-1}(x) + \varphi_0(X_j)w_j(x)+ \\ \varphi_0(x_{j+1})w_{j+1}(x) = \varphi_0(x), \\ \varphi_1(x_{j-1})w_{j-1}(x) + \varphi_1(x_j)w_j(x)+ \\ \varphi_1(x_{j+1})w_{j+1}(x) = \varphi_1(x), \\ \varphi_2(x_{j-1})w_{j-1}(x) + \varphi_2(x_j)w_j(x)+ \\ \varphi_2(x_{j+1})w_{j+1}(x)= \varphi_2(x). \end{gathered} \qquad (2)$$

By the assumption that the spline support consists of three adjacent intervals $supp\ w_j = [x_{j-1}, x_{j+2}]$. The formulas defining the basis spline w_j can be found by solving two additional systems of equations. When $x \in [x_{j-1}, x_j]$ we obtain the basis spline w_j by solving the system of equations:

$$\begin{gathered} \varphi_0(x_{j-2})w_{j-2}(x) + \varphi_0(x_{j-1})w_{j-1}(x)+ \\ \varphi_0(x_j)w_j(x) = \varphi_0(x), \\ \varphi_1(x_{j-2})w_{j-2}(x) + \varphi_1(x_{j-1})w_{j-1}(x)+ \\ \varphi_1(x_j)w_j(x) = \varphi_1(x), \\ \varphi_2(x_{j-2})w_{j-2}(x) + \varphi_2(x_{j-1})w_{j-1}(x)+ \\ \varphi_2(x_j)w_j(x) = \varphi_2(x), \end{gathered}$$

When $x \in [x_{j+1}, x_{j+2}]$ we obtain the basis spline w_j by solving the system of equations:

$$\begin{gathered} \varphi_0(x_j)w_j(x) + \varphi_0(x_{j+1})w_{j+1}(x)+ \\ \varphi_0(x_{j+2})w_{j+2}(x) = \varphi_0(x), \\ \varphi_1(x_j)w_j(x) + \varphi_1(X_{j+1})w_{j+1}(x)+ \\ \varphi_1(x_{j+2})w_{j+2}(x) = \varphi_1(x), \\ \varphi_2(x_j)w_j(x) + \varphi_2(x_{j+1})w_{j+1}(x)+ \\ \varphi_2(x_{j+2})w_{j+2}(x) = \varphi_2(x). \end{gathered}$$

It can easily be shown that
$w_j(x_j) = 1,\ w_j(x_{j+2}) = 0,\ w_{j+1}(x_j) = 0,$
$w_{j-1}(x_j) = 0,\ \ w_j(x_{j+1}) = w_j(x_{j-1}) = 0,$
$w_{j+1}(x_{j+1}) = 1,\ \ w_{j+1}(x_{j-1}) = 0,$
$w_{j-1}(x_{j-1}) = 1, w_{j-1}(x_{j+1}) = 0.$

The approximation $F^L(x)$ is only a continuous one. But we can get the first derivative of the basis functions and construct the approximation of the first derivative of the function (everywhere except nodes) in the form:

$$(F^L)'(x) = f(x_{j-1})w'_{j-1}(x) + f(x_j)w'_j(x)+ f(x_{j+1})w'_{j+1}(x),\ x \in [x_j, x_{j+1}]. \qquad (3)$$

2.1 Left Trigonometric Splines

First, we consider the approximation of a function $f(x)$ with the left trigonometric splines (see [13]). In this case $\varphi_0(x) = 1$, $\varphi_1(x) = \sin(x)$, $\varphi_2(x) = \cos(x)$. We obtain the basis functions $w_{j-1}(x)$, $w_j(x)$, $w_{j+1}(x)$, $x \in [x_j, x_{j+1}]$, solving system (2). Now it has the form:

$$\begin{gathered} w_{j-1}(x) + w_j(x) + w_{j+1}(x) = 1, \\ \sin(X_{j-1})w_{j-1}(x) + \sin(X_j)w_j(x) + \\ \sin(X_{j+1})w_{j+1}(x) = \sin(x), \qquad (4) \\ \cos(X_{j-1})w_{j-1}(x) + \cos(X_j)w_j(x) + \\ \cos(X_{j+1})w_{j+1}(x) = \cos(x). \end{gathered}$$

The solution of this system can be written as follows:

$$\begin{gathered} w_j(x) = (\sin(x_{j+1} - x) - \sin(X_{j+1} - x_{j-1}) + \\ \sin(x - x_{j-1}))/Z_j, \\ w_{j+1}(x) = (\sin(x - x_j) + \sin(x_j - x_{j-1}) - \\ \sin(x - x_{j-1}))/Z_j, \\ w_{j-1}(x) = (\sin(X_j - x) + \sin(x_{j+1} - x_j) - \\ \sin(x_{j+1} - x))/Z_j, \end{gathered}$$

where $Z_j = \sin(x_j - x_{j-1}) - \sin(x_{j+1} - x_{j-1}) - \sin(x_j - x_{j+1})$.

It is not difficult to see that the solution of the system (4) can be written as follows:

$$w_j(x) = \frac{\sin((x - x_{j-1})/2)\sin((x_{j+1} - x)/2))}{\sin((x_j - x_{j-1})/2)\sin((x_{j+1} - x_j)/2)},$$

$$w_{j+1}(x) = \frac{\sin((x - x_{j-1})/2)\sin((x - x_j)/2)}{\sin((x_{j+1} - x_{j-1})/2)\sin((x_{j+1} - x_j)/2)},$$

$$w_{j-1}(x) = \frac{\sin((x - x_j)/2)\sin((x - x_{j+1})/2)}{\sin((x_{j-1} - x_j)/2)\sin((x_{j-1} - x_{j+1})/2)}.$$

Under the assumption $h = x_{j+1} - x_j$, $x_j - x_{j-1} = Ah$, $A > 0$, $x = x_j + th$, $t \in [0, 1]$, the basis splines $w_{j-1}(x)$, $w_j(x)$, $w_{j+1}(x)$, $x \in [x_j, x_{j+1}]$, can now also be written as follows:

$$w_j(x_j + th) = \frac{-\sin(th/2 + Ah/2)\sin(th/2 - h/2)}{\sin(Ah/2)/\sin(h/2)},$$

$$w_{j+1}(x_j + th) = \frac{\sin(th/2 + Ah/2)\sin(th/2)}{\sin(Ah/2 + h/2)/\sin(h/2)},$$

$$w_{j-1}(x_j + th) = \frac{\sin(th/2)\sin(-h/2 + th/2)}{\sin(Ah/2)/\sin(Ah/2 + h/2)}.$$

This form of the basis splines will be used to construct the approximation.

It is not difficult to see that the solution of the system (4) can also be written as follows:

$$w_j(x) = \frac{\cos(x - \frac{x_{j-1}}{2} - \frac{x_{j+1}}{2}) - \cos(\frac{x_{j-1}}{2} - \frac{x_{j+1}}{2})}{2\sin(\frac{x_j}{2} - \frac{x_{j-1}}{2})\sin(\frac{x_{j+1}}{2} - \frac{x_j}{2})},$$

$$w_{j+1}(x) = \frac{\cos(\frac{x_j}{2} - \frac{x_{j-1}}{2}) - \cos(\frac{x_j}{2} + \frac{x_{j-1}}{2} - x)}{2\sin(\frac{x_{j+1}}{2} - \frac{x_{j-1}}{2})\sin(\frac{x_{j+1}}{2} - \frac{x_j}{2})}, \quad (5)$$

$$w_{j-1}(x) = \frac{\cos(\frac{x_j}{2} - \frac{x_{j+1}}{2}) - (\cos(\frac{x_j}{2} + \frac{x_{j+1}}{2} - x)}{2\sin(\frac{x_{j-1}}{2} - \frac{x_j}{2})\sin(\frac{x_{j-1}}{2} - \frac{x_{j+1}}{2})}.$$

The last form of the basis splines will be used to construct interval estimation of the approximation.

The next form of the left trigonometric basis splines will be used to construct interval estimation of the first derivative of the approximation:

$$w'_j(x) = -\frac{\sin(x - \frac{x_{j-1}}{2} - \frac{x_{j+1}}{2})}{2\sin(\frac{x_j}{2} - \frac{x_{j-1}}{2})\sin(\frac{x_{j+1}}{2} - \frac{x_j}{2})},$$

$$w'_{j+1}(x) = \frac{\sin(x - \frac{x_j}{2} - \frac{x_{j-1}}{2})}{2\sin(\frac{x_{j+1}}{2} - \frac{x_{j-1}}{2})\sin(\frac{x_{j+1}}{2} - \frac{x_j}{2})}, \quad (6)$$

$$w'_{j-1}(x) = \frac{\sin(x - \frac{x_j}{2} - \frac{x_{j+1}}{2})}{2\sin(\frac{x_{j-1}}{2} - \frac{x_j}{2})\sin(\frac{x_{j-1}}{2} - \frac{x_{j+1}}{2})}.$$

The plots of the left trigonometric basis spline w_j and its first derivative are given in Fig. 1. The plot of the approximation of the function $\sin(3x)\cos(2x)$ with the left trigonometric basis splines and the approximation of the first derivative of this function are given in Fig. 2. The plots of the error of approximation of the function $f(x) = \sin(3x)\cos(2x)$ with the left trigonometric basis splines, and the error of the approximation of the first derivative of this function, are given in Fig. 3.

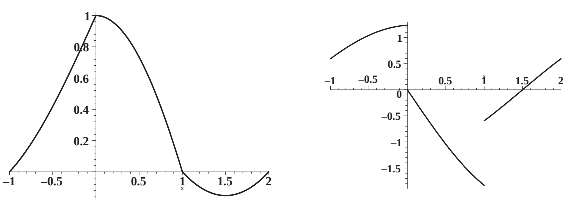

Figure 1: The plots of the left trigonometric basis spline (left) and its first derivative (right).

In the next section, to compare the quality of the approximation, we present the basic information about the left polynomial splines with the third order of approximation.

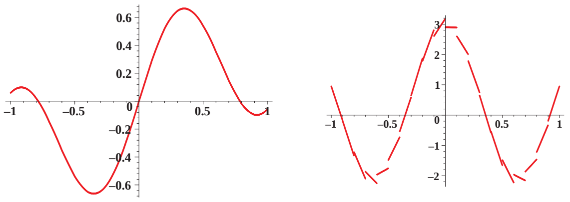

Figure 2: The plots of the approximation of the function $f(x) = \sin(3x)\cos(2x)$ (left) and the approximation of its first derivative (right) with the left trigonometric splines.

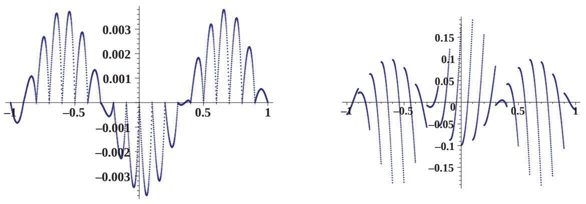

Figure 3: The plots of the error of approximation of the function $f(x) = \sin(3x)\cos(2x)$ (left) and the error of approximation of its first derivative (right) with the left trigonometric splines.

2.2 Comparison with the left polynomial splines

In case of polynomial splines we use $\varphi_0(x) = 1$, $\varphi_1(x) = x$, $\varphi_2(x) = x^2$ (see [13]). We use the following approximation:

$$G^L(x) = f(x_{j-1})\omega_{j-1}(x) + f(x_j)\omega_j(x)+$$

$$f(x_{j+1})\omega_{j+1}(x),\ x \in [x_j, x_{j+1}].$$

We obtain basic functions $\omega_{j-1}(x)$, $\omega_j(x)$, $\omega_{j+1}(x)$, $x \in [x_j, x_{j+1}]$, solving the following system:

$$\omega_{j-1}(x) + \omega_j(x) + \omega_{j+1}(x) = 1.$$

$$x_{j-1}\omega_{j-1}(x) + x_j\omega_j(x) + x_{j+1}\omega_{j+1}(x) = x,$$

$$x_{j-1}^2\omega_{j-1}(x) + x_j^2\omega_j(x) + x_{j+1}^2\omega_{j+1}(x) = x^2.$$

The solution of this system is the following:

$$\omega_j(x) = \frac{(x - x_{j-1})(x - x_{j+1})}{(x_j - x_{j-1})(x_j - x_{j+1})},$$

$$\omega_{j+1}(x) = \frac{(x - x_j)(x - x_{j-1})}{(x_{j+1} - x_j)(x_{j+1} - x_{j-1})},$$

$$\omega_{j-1}(x) = \frac{(x - x_j)(x - x_{j+1})}{(x_{j-1} - x_{j+1})(x_{j-1} - x_j)}.$$

Using the notation $x = x_j + th$, $x_{j+1} = x_j + h$, $x_{j-1} = x_j - h$ we get

$$\omega_j(x_j + th) = -(t - 1)(t + 1),$$

$$\omega_{j+1}(x_j + th) = t(t + 1)/2, \qquad (7)$$

$$\omega_{j-1}(x_j + th) = t(t - 1)/2.$$

It can easily be shown that there are relations between trigonometrical and polynomial splines:

$w_j(x_j + th) = \omega_j(x_j + th) + O(h^2)$,

$w_{j+1}(x_j + th) = \omega_{j+1}(x_j + th) + O(h^2)$,

$w_{j-1}(x_j + th) = \omega_{j-1}(x_j + th) + O(h^2)$.

The plots of the left polynomial basis spline ω_j and its first derivative are given in Fig. 4. The plot of the approximation of the function $\sin(3x)\cos(2x)$ with the left polynomial basis splines and the approximation of the first derivative of this function are given in Fig. 5. The plots of the error of approximation of the function $f(x) = \sin(3x)\cos(2x)$ with the left polynomial basis splines and the error of the approximation of the first derivative of this function are given in Fig. 6.

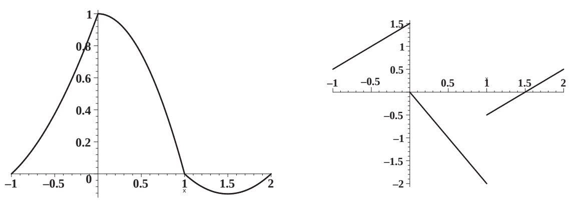

Figure 4: The plots of the left polynomial basis function ω_j (left) and its first derivative of the basis function (right).

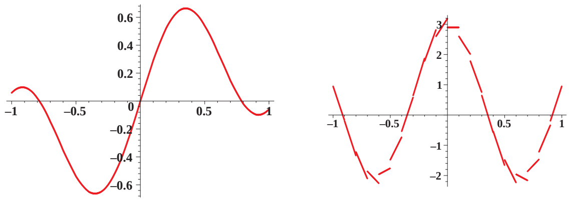

Figure 5: The plots of approximation of the function $f(x) = \sin(3x)\cos(2x)$ (left) and its its first derivative (right) with the left polynomial splines.

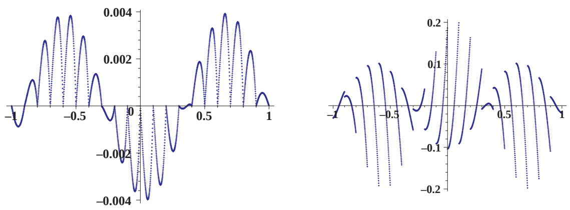

Figure 6: The plots of the error of approximation of the function $f(x) = \sin(3x)\cos(2x)$ (left) and its first derivative (right) with the left polynomial splines.

3 The Approximation with the Right Splines

Now we consider the approximation of a function $f(x)$, $x \in [x_j, x_{j+1}]$, with the right trigonometric splines. The right trigonometric splines we apply near the left border of the finite interval $[a, b]$. Suppose that the functions φ_0, φ_1, φ_2 form the Chebyshev system and the determinant

$$det \begin{pmatrix} \varphi_0(x_j) & \varphi_0(x_{j+1}) & \varphi_0(x_{j+2}) \\ \varphi_1(x_j) & \varphi_1(x_{j+1}) & \varphi_1(x_{j+2}) \\ \varphi_2(x_j) & \varphi_2(x_{j+1}) & \varphi_2(x_{j+2}) \end{pmatrix}$$

is non-zero.

We can construct the approximation of function $f(x)$, in the form:

$$F^R(x) = f(x_j)W_j(x) + f(x_{j+1})W_{j+1}(x)+ f(x_{j+2})W_{j+2}(x). \quad (8)$$

We construct the approximation of the first derivative of the function $f(x)$ in the form:

$$(F^R)'(x) = f(x_j)W_j'(x) + f(x_{j+1})W_{j+1}'(x)+ f(x_{j+2})W_{j+2}'(x). \quad (9)$$

We obtain basic functions $w_j(x)$, $w_{j+1}(x)$, $w_{j+2}(x)$ from the following system:

$$\begin{aligned} &\sin(x_j)W_j(x) + \sin(x_{j+1})W_{j+1}(x)+ \\ &\sin(x_{j+2})W_{j+2}(x) = \sin(x), \\ &\cos(x_j)W_j(x) + \cos(x_{j+1})W_{j+1}(x)+ \\ &\cos(x_{j+2})W_{j+2}(x) = \cos(x), \quad (10) \\ &W_j(x) + W_{j+1}(x) + W_{j+2}(x) = 1. \end{aligned}$$

The solution of this system can be written as follows:

$$\begin{aligned} W_j(x) &= (-\sin(x_{j+1} - x_{j+2}) - \sin(x_{j+2} - x)+ \\ &\sin(x_{j+1} - x))/S_j, \\ W_{j+1}(x) &= (\sin(x_j - x_{j+2}) - \sin(x_j - x)+ \\ &\sin(x_{j+2} - x))/S_j \\ W_{j+2}(x) &= (\sin(x_j - x) + \sin(x_{j+1} - x_j)- \\ &\sin(x_{j+1} - x))/S_j, \end{aligned}$$

where

$S_j = (\sin(x_j - x_{j+2}) + \sin(x_{j+1} - x_j) - \sin(x_{j+1} - x_{j+2}))$.

It is not difficult to see that the solution of system (10) can be written as follows:

$$W_j(x) = \frac{\sin(\frac{x}{2} - \frac{x_{j+2}}{2})\sin(\frac{x}{2} - \frac{x_{j+1}}{2})}{\sin(\frac{x_j}{2} - \frac{x_{j+2}}{2})\sin(\frac{x_j}{2} - \frac{x_{j+1}}{2})},$$

$$W_{j+1}(x) = \frac{\sin(\frac{x}{2} - \frac{x_{j+2}}{2)}\sin(\frac{x}{2} - \frac{x_j}{2})}{\sin(\frac{x_{j+1}}{2} - \frac{x_{j+2}}{2})\sin(\frac{xj+1}{2} - \frac{x_j}{2})},$$

$$W_{j+2}(x) = \frac{\sin(\frac{x}{2} - \frac{x_j}{2})\sin(\frac{x}{2} - \frac{x_{j+1}}{2})}{\sin(\frac{x_{j+2}}{2} - \frac{x_j}{2})\sin(\frac{x_{j+2}}{2} - \frac{x_{j+1}}{2})}.$$

It is not difficult to see that the solution of the system can also be written as follows:

$$W_j(x) = \frac{\cos(\frac{x_{j+1}}{2} - \frac{x_{j+2}}{2}) - \cos(x - \frac{x_{j+2}}{2} - \frac{x_{j+1}}{2})}{2\sin(\frac{x_j}{2} - \frac{x_{j+2}}{2})\sin(\frac{x_j}{2} - \frac{x_{j+1}}{2})},$$

$$W_{j+1}(x) = \frac{\cos(x - \frac{x_{j+2}}{2} - \frac{x_j}{2}) - \cos(\frac{x_j}{2} - \frac{x_{j+2}}{2})}{2\sin(\frac{x_{j+1}}{2} - \frac{x_{j+2}}{2})\sin(\frac{x_j}{2} - \frac{x_{j+1}}{2})}, \quad (11)$$

$$W_{j+2}(x) = \frac{\cos(\frac{x_j}{2} - \frac{x_{j+1}}{2}) - \cos(x - \frac{x_j}{2} - \frac{x_{j+1}}{2})}{2\sin(\frac{x_j}{2} - \frac{x_{j+2}}{2})\sin(\frac{x_{j+1}}{2} - \frac{x_{j+2}}{2})}.$$

The last form of the basis splines will be used to construct interval estimation of the approximation.

The next form of the basis splines will be used to construct interval estimation of the first derivative of the approximation:

$$W_j'(x) = \frac{\sin(x - \frac{x_{j+2}}{2} - \frac{x_{j+1}}{2})}{2\sin(\frac{x_j}{2} - \frac{x_{j+2}}{2})\sin(\frac{x_j}{2} - \frac{x_{j+1}}{2})},$$

$$W_{j+1}'(x) = -\frac{\sin(x - \frac{x_{j+2}}{2} - \frac{x_j}{2})}{2\sin(\frac{x_{j+1}}{2} - \frac{x_{j+2}}{2})\sin(\frac{x_j}{2} - \frac{x_{j+1}}{2})}, \quad (12)$$

$$W_{j+2}'(x) = \frac{\sin(x - \frac{x_j}{2} - \frac{x_{j+1}}{2})}{2\sin(\frac{x_j}{2} - \frac{x_{j+2}}{2})\sin(\frac{x_{j+1}}{2} - \frac{x_{j+2}}{2})}.$$

The plots of the right trigonometric basis spline W_j and its first derivative are given in Fig. 7. The plot of the approximation of the function $\sin(3x)\cos(2x)$ with the right trigonometric basis splines and the approximation of the first derivative of this function are given in Fig. 8. The plots of the error of approximation of the function $f(x) = \sin(3x)\cos(2x)$ with the right trigonometric basis splines, and the error of the approximation of the first derivative of this function, are given in Fig. 9.

Similar to what was done earlier, we find formulas for the right polynomial basis splines $v_j(x)$, $v_{j+1}(x)$, $v_{j+2}(x)$ and their first derivatives.

$$v_j(x) = \frac{(x - x_{j+1})(x - x_j)}{(x_j - x_{j+2})(x_{j+1} - x_{j+2})},$$

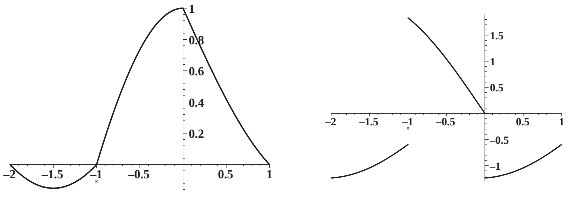

Figure 7: The plots of the right basis trigonometric spline (left) and its first derivative (right).

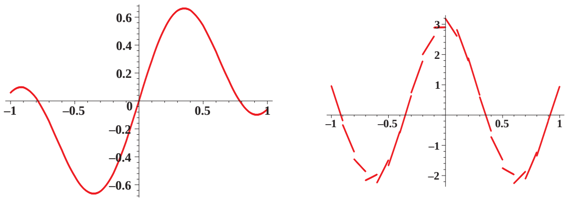

Figure 8: The plots of the approximation of the function $f(x) = \sin(3x)\cos(2x)$ (left) and the approximation of its first derivative (right) with the right trigonometric splines.

$$v_{j+1}(x) = -\frac{(x - x_{j+2})(x - x_j)}{(x_j - x_{j+1})(x_{j+1} - x_{j+2})},$$

$$v_{j+2}(x) = \frac{(x - x_{j+2})(x - x_{j+1})}{(x_j - x_{j+2})(x_j - x_{j+1})}. \qquad (13)$$

We construct the approximation of the function $f(x)$ in the form:

$$G^R(x) = f(x_j)v_j(x) + f(x_{j+1})v_{j+1}(x) +$$

$$f(x_{j+2})v_{j+2}(x),\ x \in [x_j, x_{j+1}].$$

When $x = x_j + th$, we obtain the formulas from (13) for $x \in [x_j, x_{j+1}]$:
$v_j(x_j + th) = 1 - (3/2)t + t^2/2$,
$v_{j+1}(x_j + th) = 2t - t^2$,
$v_{j+2}(x_j + th) = t^2/2 - t/2$.

4 The Theorem of the Approximation

In this section, we formulate a theorem of the approximation with the polynomial and the trigonometric splines. Let $\|f^{(3)}\|_{[\alpha,\beta]} = \max_{x\in[\alpha,\beta]} |f^{(3)}(x)|_{[\alpha,\beta]}$.

Theorem 1 *Let function $f(x)$ be such that $f \in C^3[\alpha, \beta]$, $[\alpha, \beta] \subset [a, b]$. Suppose the set of nodes is: $x_{j+2} - x_{j+1} = x_{j+1} - x_j = x_j - x_{j-1} = h$. Then for $x \in [x_j, x_{j+1}]$ we have*

$$\|f - G^L\|_{[x_j,x_{j+1}]} \le K_1^L h^3 \|f^{(3)}\|_{[x_{j-1},x_{j+1}]}, \qquad (14)$$

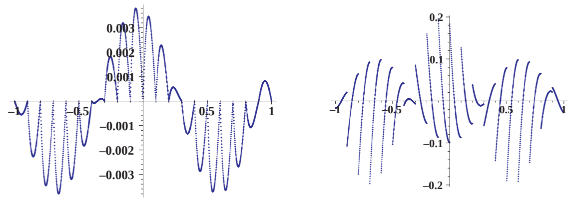

Figure 9: The plots of the error of approximation of the function $f(x) = \sin(3x)\cos(2x)$ and its first derivative with the right trigonometric splines.

$$\|f - G^R\|_{[x_j,x_{j+1}]} \le K_1^R h^3 \|f^{(3)}\|_{[x_j,x_{j+2}]}, \qquad (15)$$

$$\|f - F^L\|_{[x_j,x_{j+1}]} \le K_2^L h^3 \|f^{(3)} + f'\|_{[x_{j-1},x_{j+1}]}, \qquad (16)$$

$$\|f - F^R\|_{[x_j,x_{j+1}]} \le K_2^R h^3 \|f^{(3)} + f'\|_{[x_j,x_{j+2}]}, \qquad (17)$$

where $K_1^L = K_1^R = 0.385/3! \approx 0.0642$, $K_2^L = 0.0713$, $K_2^R = 0.0795$.

Proof: Using the properties of the basis splines it can easily be shown that for the left splines we receive $F^L(x_j) = f(x_j)$, $F^L(x_{j-1}) = f(x_{j-1})$, $F^L(x_{j+1}) = f(x_{j+1})$. So in the polynomial case $(F^L)'(x) = f(x_{j-1})\omega'_{j-1}(x) + f(x_j)\omega'_j(x) + f(x_{j+1})\omega'_{j+1}(x)$ is an interpolation polynomial. That is why we can apply the classic theory,

$$K_1^L = \max_{x\in[x_j,x_{j+1}]} |(x - x_j)(x - x_{j+1})(x - x_{j-1})| =$$

$$= h^3 \max_{t\in[0,1]} |t(t-1)(t+1)| \approx 0.385h^3,$$

and receive formula (14). Formula (15) can be obtained in the same way:

$$K_1^R = \max_{x\in[x_j,x_{j+1}]} |(x - x_j)(x - x_{j+1})(x - x_{j+2})| =$$

$$= h^3 \max_{t\in[0,1]} |t(t-1)(t-2)| \approx 0.385h^3.$$

The algorithm for obtaining the error estimation of non-polynomial splines is described in paper [14]. In short, for our trigonometric splines it is as follows. First let us represent the function $f(x)$ in the form convenient for obtaining the error estimation. We construct a homogeneous linear equation $Lf = 0$, which has the fundamental system of solutions $\varphi_0 = 1$, $\varphi_1 = \sin(x)$, $\varphi_2 = \cos(x)$. Let us construct Lf for $x \in [x_j, x_{j+1}] \in [a, b]$:

$$Lf = det \begin{pmatrix} 1 & \sin(x) & \cos(x) & f(x) \\ 0 & \cos(x) & -\sin(x) & f'(x) \\ 0 & -\sin(x) & -\cos(x) & f''(x) \\ 0 & -\cos(x) & \sin(x) & f'''(x) \end{pmatrix}$$

We can easily obtain: $Lf = -f'(t) - f^{(3)}(t)$. Here the Wronskian

$$W(x) = det \begin{pmatrix} 1 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \\ 0 & -\sin(x) & -\cos(x) \end{pmatrix}$$

does not equal zero.

Now we can construct a general solution of the nonhomogeneous equation $Lf = Q$ by the method of variation of the constants.

Suppose $f(x) = C_1(x)\sin(x) + C_2(x)\cos(x) + C_3(x)$, where C_i are some constants. Solving the system

$$\sum_{i=1}^{3} C_i'(x)\varphi_i(x) = 0,$$

$$\sum_{i=1}^{3} C_i'(x)\varphi_i'(x) = 0,$$

$$\sum_{i=1}^{3} C_i'(x)\varphi_i''(x) = Q(x).$$

We get:

$$C_i(x) = \int_{x_j}^{x} \frac{W_{3i}(t)Q(t)dt}{W(t)} + c_i,$$

where c_i are arbitrary constants, $W_{3i}(t)$ are algebraic complements (signed minor) of the element of i-th column of 3-th row of determinant $W(t)$. Thus

$$f(x) = \sum_{i=1}^{3} \varphi_i(x) \int_{x_j}^{x} \frac{W_{3i}(t)Q(t)dt}{W(t)} + \sum_{i=1}^{3} c_i\varphi_i(x).$$

But $Q = Lf$, thus we obtain:

$$f(x) = 2\int_{x_j}^{x} (f^{(3)}(t) + f'(t))\sin^2\frac{(t-x)}{2}dt+$$

$$c_1\sin(x) + c_2\cos(x) + c_3. \quad (18)$$

Now using expressions for F^L and F^R (see (1), (5), (8), (11)) and (18) we can obtain the errors of approximation of function f with the constants $K_2^L = 0.0713$, $K_2^R = 0.0795$

Thus, the formulae (16) and (17) are valid.

Remark 2 *There are examples that show that the constants K_1 and K_2 can't be reduced. They are the following: $f(x) = f_1(x) = x^3/6$ for the polynomial splines and $f(x) = f_2(x) = \sin(x) - \cos(x) + x$ for the trigonometrical splines. Let us consider the left splines. For the right splines the result will be the same. Let us take $h = 1$, $x_j = 0$, $x_{j+1} = 1$, $x_{j-1} = -1$ and construct the trigonometrical approximation $F^L(x)$ and polynomial approximation $G^L(x)$ using (1).*

We have for these functions:

$$f_1{}^{(3)}(x) = 1, \;\; f_2{}^{(3)}(x) + f_2{}'(x) = 1,$$

so at point $x = 0.5708$ we receive
$F^L(x) - f_2(x) = 0.0713,$
and at point $x = 0.57735$ we receive
$G^L(x) - f_1(x) = 0.0642.$

Remark 3 *Using the Taylor expansions in a vicinity of point x_j and the first derivatives of the formulae (7), (13), we obtain the following statements: For the approximation of the first derivative of the function the following relations are valid (everywhere except nodes)*

$$\|f' - (G^L)'\|_{[x_j,x_{j+1}]} \le K_3h^2\|f'''\|_{[x_{j-1},x_{j+1}]},$$

$$\|f' - (G^R)'\|_{[x_j,x_{j+1}]} \le K_4h^2\|f'''\|_{[x_j,x_{j+2}]},$$

$K_3 = 0.5$, $K_4 = 1$. *Using (9), (12) and (3), (6), we obtain the following statements:*

$$\|f' - (F^L)'\|_{[x_j,x_{j+1}]} \le K_5h^2\|f^{(3)}\|_{[x_{j-1},x_{j+1}]},$$

$$\|f' - (F^R)'\|_{[x_j,x_{j+1}]} \le K_6h^2\|f^{(3)}\|_{[x_j,x_{j+2}]},$$

where $K_5 = 0.472$, $K_6 = 1.1$.

Let us calculate the actual errors of the approximation with the left polynomial and trigonometric splines using the formulae:

$$R_F^L = \max_{x\in[-1,1]} |F^L(x) - f(x)|,$$

$$R_G^L = \max_{x\in[-1,1]} |G^L(x) - f(x)|.$$

The maximums of the errors of the approximation in absolute values (Act.Err.Func.) with the left polynomial splines (Pol.Left) and the left trigonometric splines (Trig.left) of the function $f(x) = \sin(3x)\cos(2x)$ and its first derivative (Act.Err.Deriv.) are shown in Table 1. For comparison, Table 1 shows the theoretical approximation errors of the function (Theor.Err.Func.) obtained using the theorem. Here $h = 0.1$, $[a, b] = [-1, 1]$.

The results of the application of the left trigonometrical splines for the approximation of functions are given in Table 2. The results of the application of the left polynomial splines for the approximation of the same functions are given in Table 3. In the second column of Table 2 and Table 3 the maximums

Table 1: The maximums of the error of the approximation in absolute values with the left polynomial and trigonometric splines of the function $f(x) = \sin(3x)\cos(2x)$

$f(x) = \sin(3x)\cos(2x)$	Pol.Left	Trig.left
Act.Err.Func.	0.00397	0.00378
Act.Err. Deriv.	0.197	0.189
Theor.Err.Func.	0.00427	0.00449

of the actual errors of approximations in absolute values are done. In the third column of Table 2 and Table 3 the maximums of the theoretical errors of approximations in absolute values are done. Calculations for both tables were made in Maple with Digits=15, $[a, b] = [-1, 1]$, $h = 0.1$.

Table 2: The errors of approximation with the left trigonometrical splines

$f(x)$	Actual err.	Theor.err.
$\sin(3x)$	0.153e-2	0.171e-2
$1/(1 + 25x^2)$	0.294e-1	0.413e-1
x^3	0.573e-3	0.642e-3
$\sin(x) - \cos(x) + x$	0.642e-4	0.713e-4

Table 3: The errors of approximation with the left polynomial splines

$f(x)$	Actual.err.	Theor.err.
$\sin(3x)$	0.1721e-2	0.1732e-2
$1/(1 + 25x^2)$	0.2957e-1	0.3754e-1
x^3	0.3849e-3	0.3850e-3
$\sin(x) - \cos(x) + x$	0.9061e-4	0.9074e-4

The results of the application of the right trigonometrical splines for the approximation of functions are given in Table 4. The results of the application of the right polynomial splines for the approximation of the same functions are given in Table 5. In the second column of Table 4 and Table 5 the maximums of the actual errors of approximations in absolute values are done. In the third column of Table 4 and Table 5 the maximums of the theoretical errors of approximations in absolute values are done. Calculations for both tables were made in Maple with Digits=15, $[a, b] = [-1, 1]$, $h = 0.1$.

The results of the application of the right and left trigonometric splines for the approximation of the first derivative of functions are given in Table 6. The results of the application of the right and left polynomial splines for the approximation of the first derivative of functions are given in Table 7.

Table 4: The errors of approximation with the right trigonometrical splines

$f(x)$	Actual err.	Theor.err.
$\sin(3x)$	0.153e-2	0.191e-2
$1/(1 + 25x^2)$	0.294e-1	0.461e-1
x^3	0.573e-3	0.715e-3
$\sin(x) - \cos(x) + x$	0.642e-4	0.795e-4

Table 5: The errors of approximation with the right polynomial splines

$f(x)$	Actual.err.	Theor.err.
$\sin(3x)$	0.1721e-2	0.1732e-2
$1/(1 + 25x^2)$	0.2957e-1	0.3754e-1
x^3	0.3849e-3	0.3850e-3
$\sin(x) - \cos(x) + x$	0.9061e-4	0.9074e-4

Calculations for both tables were made in Maple with Digits=15, $[a, b] = [-1, 1]$, $h = 0.1$.

Table 6: The actual errors of approximation of the first derivative of functioms with the right and left trigonometric splines

$f(x)$	Trig.left	Trig.right
$\sin(3x)$	0.754e-1	0.777e-1
$1/(1 + 25x^2)$	1.413	1.452
x^3	0.278e-1	0.287e-1
$\sin(x) - \cos(x) + x$	0.324e-2	0.334e-2

Table 7: The actual errors of approximation of the first derivative of functions with the right and left polynomial splines

$f(x)$	Pol.left	Pol.right
$\sin(3x)$	0.840e-1	0.873e-1
$1/(1 + 25x^2)$	1.42	1.46
x^3	0.194e-1	0.020e-1
$\sin(x) - \cos(x) + x$	0.456e-2	0.469e-2

The theoretical results of the application of the right and left trigonometric splines for the approximation of the first derivative of functions are given in Table 8. The theoretical results of the application the right and left polynomial splines for the approximation of the first derivative of functions are given in Table 9.

Calculations for both tables were made in Maple with Digits=15, $[a, b] = [-1, 1]$, $h = 0.1$.

The last four tables show that the constants in the inequalities of the approximation of the derivatives of

Table 8: The theoretical errors of approximation of the first derivative of functions with the right and left trigonometric splines

$f(x)$	Trig.left	Trig.right
$\sin(3x)$	0.127	0.296
$1/(1+25x^2)$	2.75	6.40
x^3	0.029	0.066
$\sin(x)-\cos(x)+x$	0.00667	0.0155

Table 9: The theoretical errors of approximation of the first derivative of functions with the right and left polynomial splines

$f(x)$	Pol.left	Pol.right
$\sin(3x)$	0.135	0.270
$1/(1+25x^2)$	2.918	5.836
x^3	0.030	0.06
$\sin(x)-\cos(x)+x$	0.707e-2	0.141e-1

the function are somewhat overestimated and can be slightly reduced.

5 Interval extention

As is known, the task of interval estimation is to find the narrowest estimation interval as possible.

For interval estimation of approximation with splines, we will use operations on intervals set forth, for example, in book [10]. Interval result over real intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ can be obtained using the formulas:

1. $A + B = [a_1 + b_1, a_2 + b_2]$,
2. $A - B = [a_1 - b_2, a_2 - b_1] = A + [-1, -1] \cdot B$,
3. $A \cdot B = [\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}]$,
4. $A : B = [a_1, a_2] \cdot [1/b_2, 1/b_1], 0 \notin B$.

For a unary operation we use the rule

5. $r(A) = [\min_{x\in A}(r(x)), \max_{x\in A}(r(x))]$,

where $r(A)$ is the unary operation.

Theorem 1 helps us to choose the correct length $h = x_{j+1} - x_j$ of the interval $[x_j, x_{j+1}]$.

Suppose we know the values of function $f(x)$ at nodes $\{x_k\}$. Using formulas (1), (3), (5), (6), (8), (9), (11), (12) with trigonometrical splines and the technique of interval analysis [10] we can construct the upper and lower boundaries for every interval $Y_j = [x_j, x_{j+1}]$. Thus we avoid the calculations of approximation $f(x)$ in many points of every interval $[x_j, x_{j+1}]$ if we need to know the boundaries of the interval, where the function f varieties. In order to obtain the boundaries of variety $f(x)$ we construct the approximation $F(x)$, $x \in Y_j$ in form (1) and consider $F(Y_j)$.

In order to get the narrowest estimation interval we transform formulas (5). First, we consider the estimate of the lower bound of the estimating interval of the basis spline $w_{j-1}(x)$.

Let X^{max}_{j-1} be the maximum

$$X^{max}_{j-1} = \max_{x\in[x_j,x_{j+1}]}(\cos(x_j/2 - x + x_{j+1}/2).$$

Then the upper boundary of w_{j-1} will be the following

$$w^{MA}_{j-1} = 2\sin(x_j/2 - x_{j+1}/2)/(\sin(x_j - x_{j-1}) -$$

$$\sin(x_{j+1} - x_{j-1}) - \sin(-x_{j+1} + x_j)) \cdot X^{max}_{j-1} + w^A_{j-1},$$

where

$$w^A_{j-1} = \sin(x_{j+1} - x_j)/(\sin(x_j - x_{j-1}) -$$

$$\sin(x_{j+1} - x_{j-1}) - \sin(-x_{j+1} + x_j)).$$

After calculating the upper boundaries of w_{j-1}, w_j and w_{j+1} we can calculate the upper boundary of $F(x)$. Now the upper boundary of $F(x)$ will be the following:

$$F^{MAX} = f(x_{j-1})w^{MA}_{j-1} + f(x_j)w^{MA}_j + f(x_{j+1})w^{MA}_{j+1}.$$

A program was developed in the MAPLE environment to visualize the interval estimation of the variation of a function and its first derivative. To obtain an interval estimate of the function or its first derivative, values of the function in grid nodes are required. The program uses trigonometric basis splines. Directional machine rounding is not used in this version of the program.

The results of the interval extension of function $f(x)$ are given in Figures 10-12, 14, 16, 18.The results of the interval extension of the first derivative of the function $f(x)$ are given in Figures 13, 15, 17. Here $[a, b] = [0, \pi]$, $h = \pi/15$. The plots of function $f(x)$ and its upper and lower boundaries are given in Fig. 10 when $f(x) = \sin(x)$ and in Fig. 11 when $f(x) = \cos(x)$.

The plots of function $1/(1 + 25x^2)$ and its upper and lower boundaries are given in Fig. 12. The plots of the first derivative of this function its upper and lower boundaries are given in Fig. 13.

The plots of function $\sin(3x)$ and its upper and lower boundaries are given in Fig. 14. The plots of the first derivative of this function its upper and lower boundaries are given in Fig. 15.

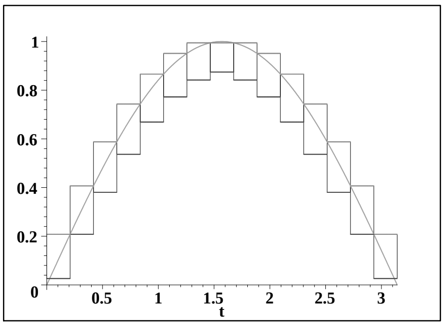

Figure 10: The plots of function $f(x) = \sin(x)$ and its upper and lower boundaries.

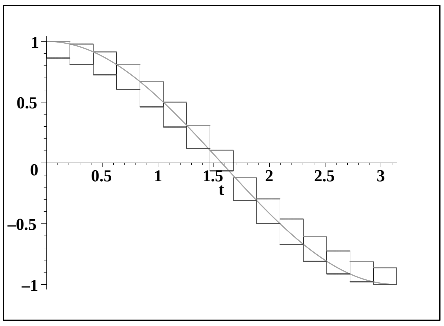

Figure 11: The plots of function $f(x) = \cos(x)$ and its upper and lower boundaries.

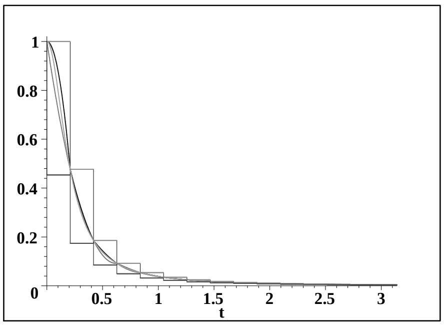

Figure 12: The plots of function $1/(1 + 25x^2)$ and its upper and lower boundaries.

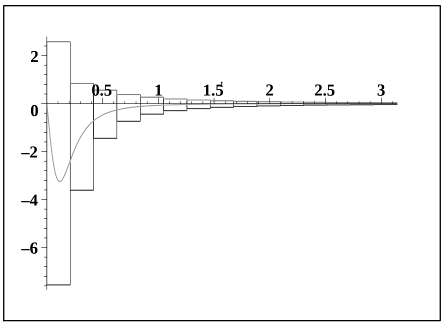

Figure 13: The plots of the first derivative of the function $1/(1 + 25x^2)$ and its upper and lower boundaries

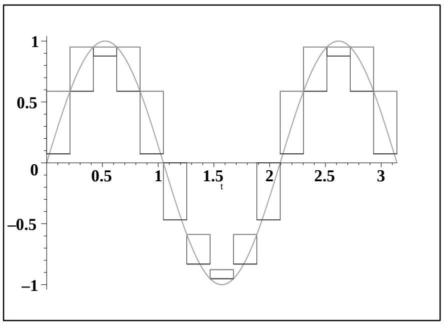

Figure 14: The plots of function $\sin(3x)$ and its upper and lower boundaries.

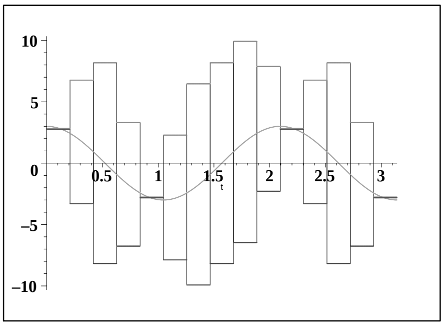

Figure 15: The plots of the first derivative of the function $\sin(3x)$ and its upper and lower boundaries.

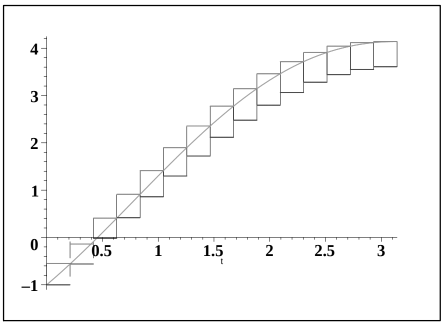

Figure 16: The plots of function $\sin(x) - \cos(x) + x$ and upper and lower boundaries.

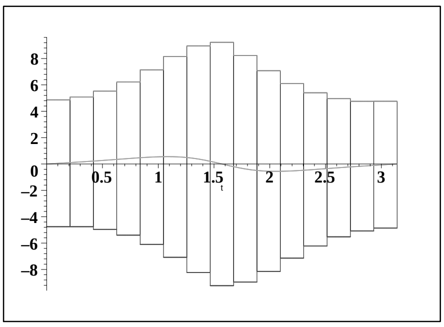

Figure 17: The plot of the first derivative of the function $1/(1+(cos(x))^2)$ and its upper and lower boundaries.

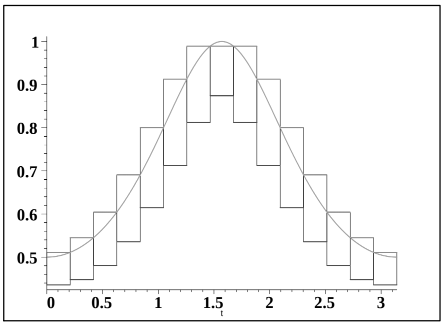

Figure 18: The plots of the function $1/(1+(cos(x))^2)$ and its upper and lower boundary.

6 Conclusion

In this paper we calculate the constants that cannot be reduced in the theorem of approximation with trigonometrical splines and present the results of working the program of constructing interval extension. The results of the numerical experiments show that trigonometrical approximation is preferred to polynomial approximation when we approximate a trigonometrical function. To avoid calculation in many points we can use interval extension if we need to know only the upper and the lower boundaries of variation of the function. But we have to hold in mind the theorem of approximation.

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