Radial Compression Stability of a Nonlinearly Elastic Cylinder with Internal Stresses

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Abstract: The effect of internal stresses caused by a linear defect such as a wedge disclination on the process of flat buckling of a hollow cylinder loaded on an external lateral surface with uniform pressure is investigated. For the analysis, the well-known models for describing compressible materials, the one-parameter model of Blatz and Ko and the five-constant Murnaghan model, were used. The stability analysis was carried out on the basis of a combination of the semi-inverse method of the nonlinear theory of elasticity and bifurcation approach. The value of a particular stress/strain characteristic for which a homogeneous boundary-value problem obtained by linearizing the original nonlinear equilibrium equations has a non-trivial solution was identified with the critical value of the loading parameter, i.e. value at which the system loses stability. The strength of disclination and the value of the applied pressure were used as such parameters. The stability regions are presented as areas on the plane of loading parameters. The possibility of the bifurcation curves condensing was demonstrated.

Key–Words: stability, buckling, semi-inverse method, bifurcation, disclination, finite strains

1 Introduction

The concept of disclinations, which arose in the study of certain properties of liquid crystals , later found application in the description of various biological objects, such as protein polymers, wood, nematoid structures of human skin et al. A new wave of interest to the nonlinear theory of elastic disclinations in recent years is associated with the active use of disclination models in the description of nanostructures of various kinds [1].

The study of disclinations in the framework of the nonlinear theory of elasticity was initiated in [2], a number of propositions and ideas of which were later refined and developed in the works of L.M. Zubov and his followers [3]-[5].

The use of models and methods of the nonlinear theory of elasticity allows not only to analyze the stress-strain state of a body with a defect but also to investigate questions of the stability of the constructed solutions. The traditional scheme of stability studies [6, 7] is based on the linearization of three-dimensional non-linear boundary problems in the neighborhood of the constructed solution and the study of the possibility of the existence of a nontrivial solution of these linear problems depending upon parameters, i.e. certain characteristics of the deformed state or external influences. This work can be viewed as a continuation of the studies begun in [4] and related to the study of the equilibrium and stability of a cylinder containing a wedge disclination, and a comparative analysis of the use in this connection more general models of nonlinear-elastic behavior of compressible media. The simplified version of the Blatz and Ko model and the five-constant Murnaghan model were used for the simulation. The main method to construct the equilibrium state was the semi-inverse method of the nonlinear theory of elasticity, the stability study was carried out within the framework of the static bifurcation approach.

2 Equilibrium and stability of a cylinder with disclination

Let's consider the hollow circular cylinder of the height h with inner stresses caused by the wedge disclination and with external pressure p uniformly distributed across its lateral surface.

Let r, φ, z and R, Φ, Z – cylindrical coordinates at the reference (non-deformed) and actual configurations, $\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_z$ and $\boldsymbol{e}_R, \boldsymbol{e}_\Phi, \boldsymbol{e}_Z$ – related Cartesian bases. The differential equations of equilibrium in the case when the body forces are absent and the boundary conditions at the lateral surfaces have the following form [8]

$$\nabla \cdot \mathbf{P} = 0, \tag{1}$$

$$\boldsymbol{e}_r \cdot \mathbf{P}|_{r=r_0} = 0, \boldsymbol{e}_r \cdot \mathbf{P}|_{r=r_1} = -pJ\mathbf{C}^{-1} \cdot \boldsymbol{e}_r, \tag{2}$$

$$\nabla = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\phi \frac{\partial}{r\partial \phi} + \boldsymbol{e}_z \frac{\partial}{\partial z},$$

$$\mathbf{C} = \nabla \boldsymbol{R}, J = \det \boldsymbol{C}, \mathbf{P} = \frac{\partial W}{\partial \mathbf{C}}.$$

Here r_0 and r_1 – inner and outer cylinder radii, respectively; $\boldsymbol{R}$ – radius-vector of the body point; $\mathbf{P}$ – nonsymmetric Piola stress tensor, W – strain energy function. At the ends of the cylinder, we assume the absence of tangential stresses

$$P_{zR} = 0, P_{z\Phi} = 0. \tag{3}$$

To construct of the solution of the boundary value problem (1)–(3) we use semi-inverse method of the static nonlinear elasticity by assuming the following form of the mapping from the reference configuration to the actual one [3]:

$$R = P\left(r\right), \ \Phi = \kappa\varphi, \ Z = z. \tag{4}$$

Here κ is the parameter characterizing the strength of the disclination. In the case $\kappa > 1$ the transformation (4) corresponds to the removal the sector $2\pi\kappa^{-1} \leq \varphi \leq 2\pi$ from the cylinder and rotation the section $\varphi = 2\pi\kappa^{-1}$ around the cylinder axis till the alignment with the plane $\varphi = 0$. The case $0 < \kappa < 1$ corresponds to the deformation when a wedge of the angle $2\pi\left(1-\kappa\right)$ is inserted into the cylinder (see Fig. 1).

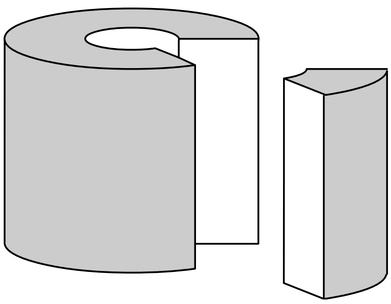

Fig. 1: Scheme of the disclination formation in the hollow cylinder

The analysis of the stability of the constructed solutions is carried out in the framework of the bifurcation approach based on the studies of the equations of neutral equilibrium, which are obtained in the framework of the theory of imposing a small deformation on the finite one [8]. Restricting analysis to the flat forms of buckling, let's add small disturbances to (4), putting

$$R = P(r) + \varepsilon u(r,\varphi), \Phi = \chi\varphi + \varepsilon v(r,\varphi). \tag{5}$$

After the calculation of the deformation gradient $\mathbf{C}$ and its linearization by the formula

$$\dot{\mathbf{C}} = \frac{\partial}{\partial \varepsilon}\mathbf{C}|_{\varepsilon=0},$$

the expression of the Piola stress tensor in (1), (2) should be also replaced by its linearized variant $\dot{\mathbf{P}}$. As a result we obtain the following linear homogenous system of differential equations

$$\nabla \cdot \dot{\mathbf{P}} = 0 \tag{6}$$

with boundary conditions:

$$\boldsymbol{e}_r \cdot \dot{\mathbf{P}} = 0, \boldsymbol{e}_r \cdot \dot{\mathbf{P}} = -p\left(\dot{J}\mathbf{C}^{-1} + J\dot{\mathbf{C}}^{-1}\right) \cdot \boldsymbol{e}_r. \tag{7}$$

The analysis of the system above is carried out by the method of the separation of variables:

$$\begin{aligned} u(r,\varphi) &= U(r)\cos(n\kappa\varphi), \\ v(r,\varphi) &= V(r)\sin(n\kappa\varphi), \end{aligned} \tag{8}$$

where natural number n is the mode of the stability loss. The separation (8) automatically satisfies the conditions (3) so the the subject of the analysis now is the liinear homogeneous system of ordinary differential equations for the functions $U(r), V(r)$. The case of $n = 0$ is excluded from consideration, the case of $n = 1$ corresponds to the motion of an absolutely rigid body, therefore it is also not considered. Thus, further we assume $n \geq 2$ [4]. The value of the disclination parameter κ or the value of external pressure p at which the constructed linear homogeneous boundary value problem has non-trivial solutions is treated as the critical value of the loading parameter, i.e. the value at which the system loses its stability.

3 Material models and boundary problems

To begin with we consider so called simplified version of Blatz and Ko material model [8]. The specific strain energy function for this model is

$$W = \frac{1}{2}\mu\left(\frac{I_2}{I_3} + 2\sqrt{I_3} - 5\right). \tag{9}$$

Here $I_k = I_k(\mathbf{G})$ – principal invariants of Cauchy strain measure $\mathbf{G} = \mathbf{C} \cdot \mathbf{C}^{\mathrm{T}}$, material parameter μ has the meaning of the shear modulus for the small strains.

In this case the equilibrium boundary value problem for the function $P(r)$ has the form

$$P'' = -\frac{1}{3}\frac{\left(r^3 P'^3 - \kappa^2 P^3\right) P'}{\kappa^2 r P^3}; \tag{10}$$

$$\begin{aligned} &\kappa P(r_0)'^3 P(r_0) = r_0, \\ &\kappa P(r_1)'^3 P(r_1)\,(q+1) = r_1, \end{aligned} \tag{11}$$

where prime denotes differentiation with respect to r, $q = p/\mu$ – dimensionless pressure.

Linearized by means (5) system of equilibrium equations (6) can be written as

$$\begin{aligned} U'' &= \left(\frac{1}{3r} - \frac{4P'^3 r^2}{3\kappa^2 P^3}\right) U' + \left(\frac{P'^4 r^2}{\kappa^2 P^4} + \frac{P'^2 n^2}{3P^2}\right) U - \\ &- \left(\frac{P'^3 r^2 n}{3\kappa^2 P^2} + \frac{P' n}{3}\right) V' + \frac{2P'^4 r^2 n}{3\kappa^2 P^3} V, \\ V'' &= \left(\frac{\kappa^2 n}{r^2 P'} + \frac{P' n}{P^2}\right) U' + \left(\frac{P'^2 n}{P^3} + \frac{P'^4 r^2 n}{3\kappa^2 P^5}\right) U - \\ &- \left(\frac{2r^2 P'^3}{3\kappa^2 P^3} + \frac{7}{3r}\right)\left(\frac{\kappa^2 n}{r^2 P'} + \frac{P' n}{P^2}\right) V' + \frac{3P'^2 n^2}{P^2} V \end{aligned} \tag{12}$$

Its boundary conditions are the following: at $r = r_0$

$$\frac{3}{P'^4} U' + \frac{\kappa}{r} U + \frac{\kappa P n}{r} V = 0,$$

$$\left(\frac{\kappa}{r} - \frac{r^2}{\kappa^2 P^3 P'} - \frac{1}{P P'^3}\right) nU + \frac{r^2}{\kappa^2 P^2 P'^2} V = 0,$$

at $r = r_1$

$$\frac{3}{P'^4} U' + \frac{\kappa(q+1)}{r} U + \frac{\kappa(q+1)n}{r} V = 0,$$

$$\left(\frac{\kappa(q+1)}{r} - \frac{r^2}{\kappa^2 P^3 P'} - \frac{1}{P P'^3}\right) nU + \frac{r^2}{\kappa^2 P P'^2} V' = 0$$

Despite the rather compact form, the boundary problem (10)–(11) is essentially nonlinear and it is not possible to construct its solution in an explicit analytical form. Therefore, the variable coefficients of the linearized boundary value problem also cannot be written analytically. At each step of the computational scheme (i.e., for each pair of values of the disclination parameter and external pressure) the boundary value problem (10)–(11) was solved numerically, and then this numerical solution was used to specify the coefficients and study the possibility of the existence of the nontrivial solution.

The Murnaghan model was used as the second model of the material. Specific strain energy function is a cubic form of the strain tensor $\mathbf{K} = (\mathbf{G} - \mathbf{I})/2$ (where $\mathbf{I}$ – identity tensor):

$$\begin{aligned} W = &\frac{1}{2}(\lambda + 2\mu) I_1^2(\mathbf{K}) - 2\mu I_2(\mathbf{K}) + \\ &+ \frac{1}{3}(l + 2m) I_1^3(\mathbf{K}) - 2m I_1(\mathbf{K}) I_2(\mathbf{K}) + n I_3(\mathbf{K}). \end{aligned}$$

The material parameters λ, μ have the meaning of the Lame coefficients, l, m, n are Murnaghan constants.

Equilibrium equations and linearized boundary value problems for the Murnaghan material will not be given because of their cumbersomeness.

4 Non-homogeneous cylinder

The semi-inverse approach based upon transformation (4) turns out to be applicable in some cases of an inhomogeneous cylinder. For brevity, in this section we restrict ourselves to the consideration of a model (9), in which we assume the parameter μ depending on the radial coordinate r. The most typical are two cases: a composite cylinder and a functional gradient material, in which the μ modulus is a continuous function (Fig. 2).

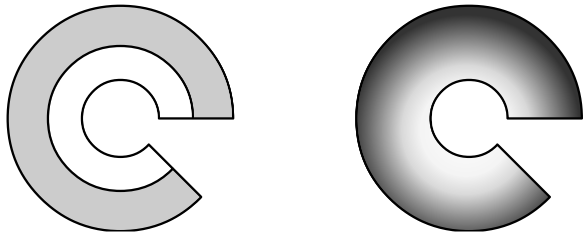

Fig. 2: Two types of heterogeneity: composite cylinder and continuous inhomogeneity

In the first case, to determine the stress-strain state of the cylinder, that is, to find the function $P(r)$, it is required to solve a set of two systems of the form (10), (11) supplemented by the conditions of continuity of displacements and strains on the material dividing line. In the second case, the equilibrium equation will be modified as follows:

$$\begin{aligned} P'' = &-\frac{1}{3}\frac{\left(r^3 P'^3 - \kappa^2 P^3\right) P'}{\kappa^2 r P^3} - \\ &- \frac{1}{3}\frac{P' \mu' \left(P'^3 P \kappa - r\right)}{r\mu}. \end{aligned} \tag{13}$$

Boundary conditions will not change.

Linearized systems also will either represent a combination of two systems of the form (12), or will be corrected taking into account the fact that the parameter μ is a function and, therefore, will contain its derivative. Due to the bulkiness, these systems are not given here.

We consider following types of continuous heterogeneity:

- linear heterogeneity

$$\mu(r) = \mu_0 \left(1 + (\delta - 1)\frac{r - r_0}{r_1 - r_0}\right),$$

- exponential heterogeneity

$$\mu(x) = \mu_0 \exp\left(\delta \frac{r - r_0}{r_1 - r_0}\right),$$

so in both cases $\mu(r_0) = \mu_0$ and $\mu(r_1) = \mu_0\delta$ and $\mu(r_1) = \mu_0 \exp(\delta)$ for linear and exponential functions, respectively.

5 Numerical results

First of all, it should be noted that the choice of the model did not show a qualitative effect on the stress-strain state. If the parameters of the models were chosen so that in the linear approximation they corresponded to the same material, the difference in the distribution of normal stresses was no more than 10 percent.

Fig. 3 and 4 represent the cylinder stability regions on the plane of two loading parameters – the external pressure and the disclination parameter for the Blatz and Ko model. Fig. 3 corresponds to a cylinder that is close to solid ($r_0/r_1 = 0.1$), Fig. 4 presents the results for the thick-walled tube ($r_0/r_1 = 0.9$).

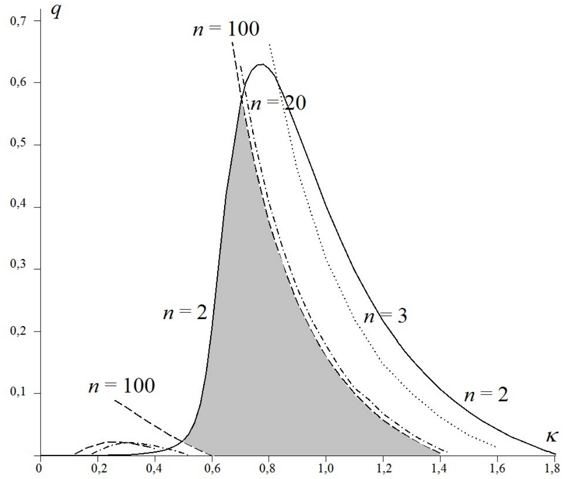

Fig. 3: Stability region for Blatz and Ko material, $r_0/r_1 = 0.1$.

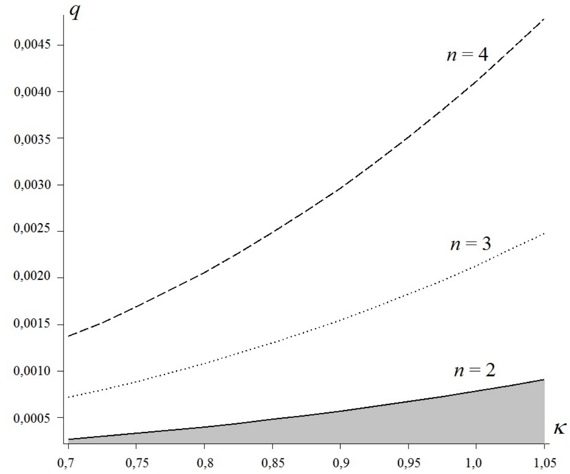

Fig. 4: Stability region for Blatz and Ko material, $r_0/r_1 = 0.9$.

An important difference of the first case is the fact that the stability region is described by curves with different mode numbers. In addition, one can see the condensing of bifurcation curves at one of the boundaries of the stability region. Increasing the mode number n, starting from a certain value (about fifty), practically does not change the critical pressure. This fact agrees with the results obtained earlier in [4] for the harmonic material model. This, in particular, means that for such body it is impossible to predict the form of the stability loss based on the analysis of linearized equations. For a thinner cylinder, the $n = 2$ mode is always preferred; the corresponding critical pressure is minimal. Another feature of thick cylinders is the existence of a zone where the loss of stability can occur only due to the presence of disclination without the application of external pressure. Such an area may be absent in the thinner cylinders.

Fig. 5 presents the stability region for a close-to-solid cylinder made of Murnaghan material, whose characteristics correspond to plexiglass from [8]: $\lambda = 3.9 \cdot 10^4$ MPa, $\mu = 1.86 \cdot 10^4$ MPa, $l = 1.09 \cdot 10^4$ MPa, $m = 2.4 \cdot 10^3$ MPa, $n = 1.88 \cdot 10^4$ MPa.

Here again, the stability region is delineated by curves corresponding to different mode numbers. In addition, in this case, also there is the possibility of stability loss due to the disclination of sufficiently large strength, without external pressure. It is clear that both of these features are related not to the model, but to the geometry of the body under consideration.

The last figure in this section shows the influence of continuous inhomogeneity upon the region of stability of the disclination strength when no external pressure acts for the close-to-solid cylinder. Comparing Fig. 6 and Fig. 3 one can see that both types of in-

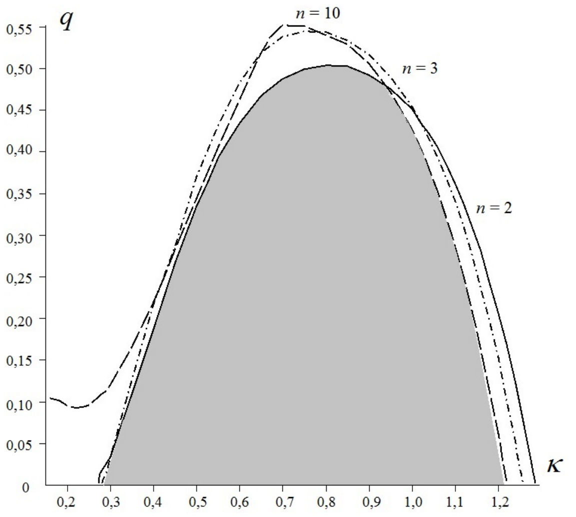

Fig. 5: Stability region for Murnaghan material, $r_0/r_1 = 0.1$.

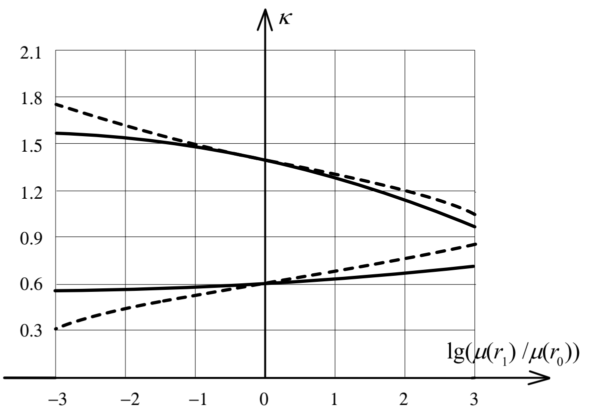

Fig. 6: Stability regions with respect to disclination strength; solid line – the linear inhomogeneity, dotted line – the exponential one.

homogeneities have the similar influence. In the case of the soft outer layer and the rigid inner layer (left half of the plane) the cylinder is slightly more stable. The inverse case – softer inner layer – has a more pronounced negative effect on the stability region.

6 Conclusion

The study of the equilibrium and stability of a hollow circular cylinder of compressible nonlinear elastic material were carried out. Internal stresses in the cylinder are caused by the presence of a wedge disclination; a uniformly distributed normal load acts on the lateral surface.

A comparative analysis of the models used at various values of the parameters showed, in particular, that thick-walled cylinders are characterized by condensing of bifurcation curves, which can also occur in cases where there is no disclination, but its presence makes the condensation areas much more extensive. The presence of disclination can be both a stabilizing factor and vice versa – significantly reduce the value of the critical pressure.

Different types of the inhomogeneity of the cylinder material can also have noticeable influence upon the stability. In particular they can significantly change the region of stability of the unloaded cylinder having wedge disclination of the given strength.

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References:

[1] A. E. Romanov, Mechanics and physics of disclinations in solids, *European Journal of Mechanics A/Solids*, 22, 2003, pp. 727–741

[2] R. de Wit, Linear theory of static disclinations, *Fundamental Aspects of Dislocation Theory*, Nat. Bur. Stand. (US), 1970, pp. 651–673.

[3] L. M. Zubov, *Nonlinear Theory of Dislocations and Disclinations in Elastic Bodies*, Springer –Verlag, Berlin–Heidelberg–New York–Tokyo, 1997.

[4] A. A. Zelenin, L. M. Zubov, Stability and postcritical behavior of an elastic cylinder with disclination, *Mechanics of solids*, 24, 1989, pp. 97–103.

[5] M. I. Karyakin, L. M. Zubov, Theory of Isolated and Continuously Distributed Disclinations and Dislocations in Micropolar Media, *Advanced Structured Materials, Vol 7. Mechanics of Generalized Continua*, Springer–Verlag, Berlin–Heidelberg, 2011, pp. 275–290

[6] L. M. Zubov, D. N. Sheidakov, Instability of a hollow elastic cylinder under tension, torsion, and inflation *Journal of Applied Mechanics, Transactions ASME*, 75, 2008, pp. 0110021–0110026.

[7] L. Obrezkov, Equilibrium and stability of nonlinearly elastic cylinder made of Blatz-Ko material, *Engineering Transactions*, 4, 2016, pp. 457–463.

[8] A. I. Lurie *Non-linear theory of elasticity*, North Holland, Amsterdam, 1990.