About self-similar solution of elastic contact problem

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Abstract: - Using equations of the static theory of elasticity, the problem of calculation stress distribution inside cylindrical solid at its point contact with semi-infinite elastic medium solved. For the displacement vector of the media $\mathbf{u} = \left(u_r\left(r,z\right), 0, u_z\left(r,z\right)\right)$, where $u_r -$ radial deformations, $u_z -$ longitudinal strain of the cylinder, the system of two linear equations in partial derivatives in cylindrical coordinate system was obtained. The problem was solved in the stationarity case when the times of δt exceed of the contact time τ. It is shown that the both functions $u_r\left(r,z\right)$ and $u_z\left(r,z\right)$ obey the equations of the third-order which accept an existence of physically correct self-similar solutions. The numerical solutions of the equations for the different boundary conditions found.

Key-Words: - static theory of elasticity, deformation, cylindrical coordinate, numerical solution.

1 Introduction

The problem of our research here belongs to the category of analytically solvable problems from the classical elasticity theory [1, 2] and for the case when the load diffusion is stationary. We are considering the contact interaction flying horizontally at the velocity $\mathbf{V}$ of the cylindrical rod with semi-infinite elastic medium. The kinetic energy is completely transforming into the strain potential energy one of the elastic medium, the rod itself and its heating. Here we are assume that the thermal conduction of elastic medium is very high, that means heating and cooling occur almost instantaneously. Essentially that the heating of the rod itself can be neglect, but its size is much smaller than the linear dimensions of the elastic medium. There is a certainly temperature increase in the contact area. As its known (see ref. [1]) the thermal deformations are not great and we can neglect them. Nevertheless, with the local heating, which lasts for some small but finite time Δt, exceeding the contact time τ, there is always take place the additional thermoelastic deformation. In some cases it can be important but in our problem it is not essential and will not be taken into account in.

2 The energy conservation equations and its consequences

The law of the conservation of the energy with the fundamental losses for the both bodies represented as

$$\frac{1}{2}\int_{V_c} \rho \mathbf{V}^2 dV = \\ = \int_V F_V' dV + \int_V \int_{t_0}^{t_1} \left[c_P \dot{T} + \frac{\kappa \left(\nabla T\right)^2}{2T_0} \right] dt dV + \Delta F_s \quad , \quad (1)$$

where the left part is the kinetic energy of the rod, going entirely to its internal deformations (we talk below about it), $\rho -$ is the density of the rod. $V_c -$

its volume, the first value on the right-hand side is the energy of semi-infinite elastic medium ("prime mark" of the free energy density shows on it). $V-$ is the volume of the media, the second value is the dissipation function and clean heating. c_P- is the isobaric heat capacity of volume unit of the body (we should note that often introduced specific thermal capacity that belongs to the volume unit of the body, differs from the heat capacity c_P of the mass unit introduced by us by an additional factor of ρ), T_0- is the thermostat temperature.

In this context we'd like should notice that the dissipation function is proportional to the square-law of the gradient of the temperature is nothing else than the linear approximation of the Fourier law. Indeed, considering the gradient of the temperature is small, we could use the expansion of the dissipative function in a series in powers of the temperature gradient. As it's shown in the ref. [3], the dissipative function is quadratic function of the small quantity $(\nabla T)^2$. It means that the inequality $\frac{|T-T_0|}{T_0}<<1$ should be satisfied. We shell talking about such temperatures when we use the Gaussian theorem transformation to reduce the integrals in (1) to the surface integrals.

Indeed, we have that

$$\int_V\int_{t_0}^{t_1}\left[c_P\dot{T}+\frac{\kappa(\nabla T)^2}{2T_0}\right]dtdV=$$
$$=\int_V\int_{t_0}^{t_1}c_P\dot{T}dtdV+\frac{1}{2}\int_{t_0}^{t_1}dt\int_V\nabla(\kappa\nabla T)dV-\frac{1}{2}\int_{t_0}^{t_1}dt\int_V(\kappa\Delta T)dV=$$
$$=\int_V\int_{t_0}^{t_1}c_P\dot{T}dtdV$$

Due to the heat conductivity equation of the Fourier we have that $\kappa\Delta T=c_P\dot{T}$. I.e. we could find from here

$$\int_V\int_{t_0}^{t_1}\left[c_P\dot{T}+\frac{\kappa(\nabla T)^2}{2T_0}\right]dtdV=\int_{t_0}^{t_1}\int_V c_P\dot{T}dtdV=$$
$$=\kappa\int_{t_0}^{t_1}\int_V\Delta TdtdV=\kappa\int_{t_0}^{t_1}dt\int_V\Delta TdV=\kappa\int_{t_0}^{t_1}dt\int_S\nabla Td\mathbf{S}=0$$
.

The equality to zero of this expression is connected with trivial fulfillment of the condition of the constancy of the temperature on the boundary. It means that the condition $T=T_0$ is satisfied. As for the last summand in eq. (1) present the sound losses and are not very significant. In this connection we are will not take them into account.

It is very convenient introduce the cylindrical coordinate system for the rod. Really then $dV=2\pi rdrdz$ - is the element of the rod volume. Therefore for the kinetic energy loss related to the deformation energy of interest to us, we can rewrite the equation (1) in the following form

$$\varepsilon(\lambda)\int_0^{h_1}\rho\mathbf{V}^2dz=\frac{2}{R^2}\int_0^{h_1}\int_0^R F_V(r,z)rdrdz\,, \qquad (2)$$

where $R-$ is the radius of the no deforming the cylinder yet, h_1- is the linear dimension of the contact area (see, for example the Hertz's problem in the ref. [1]), measured in the direction opposite to the direction of the lying inside the rod. The variable $\varepsilon(\lambda)$ characterizing the relative energy losses of the rod which would be strictly calculated below, $\lambda=\frac{h_1}{R}$ - is dimensionless parameter.

In the process of the contact rod + media the deformation decays exponentially fast with the coordinate z . But this fact we can account due to the integrating over a finite interval from 0 to h and write an improper integral due to its rapid convergence. Indeed, we should notice that due to the Hertz's theory, the linear dimension of the contact area can be calculate as $h=h_1+h_2$, where h_2- is the deforming region of semi-infinite elastic medium. We could find out from the simple evaluation that $h\approx|\mathbf{V}|\tau$, where $\tau-$ is the contact time. As to the free-energy density, due to ref. [1] we can represent it in the following form

$$F_V = \mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}\right)^2 + \frac{K}{2}u_{ll}^2 + \mathbf{f}\cdot\mathbf{u}, \qquad (3)$$

where $u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$ - is the symmetrical tensor of the deformations, $u_{ll} = div\mathbf{u} -$ is its spur, $\mathbf{u} = \mathbf{u}(\mathbf{r}) -$ is the required displacement vector of the tangent points of the body (it is consequence of the force effects leading to the deformation of the internal structure), $\mu -$ is the modulus of rigidity, $K -$ is the coefficient of contraction (expansion). It has a pressure dimension $\left(\frac{erg}{cm^3}\right)$. It connects with the Young modulus E and the Poisson's coefficient σ due to the following formulae

$$\mu = \frac{E}{2(1+\sigma)}, K = \frac{E}{3(1-2\sigma)}. \qquad (4)$$

The last summand, introduced by us in the expr. (3), is a potential energy which connects by manipulation of the applied forces. This is usually a gravitational force and centrifugal force for which of the volume force is determined by the simple dependence:

$$\mathbf{f} = \rho\mathbf{g} + \rho\frac{[\boldsymbol{\omega}\times\mathbf{r}]^2}{r}, \qquad (5)$$

where $\boldsymbol{\omega} -$is the angular velocity vector of rotation.

3 Basic equation and its solution

If write the functional $S\{\mathbf{u}\} = \int_V F_V dV$ and then the varying it on the variable deformations vector $\mathbf{u} = \mathbf{u}(\mathbf{r})$, taking into account the formulas (4), we are getting the equation of the static deformation theory

$$\Delta\mathbf{u} + \frac{1}{1-2\sigma}graddiv\mathbf{u} = -\mathbf{f}\frac{2(1+\sigma)}{E}. \qquad (6)$$

In the solvation of the equation (6), we would not take into account the gravitational force (we are assume that the rod flies horizontally) and its possible rotation around it axis. So, let us the force $\mathbf{f} = 0$. Then we obtain equation

$$\Delta\mathbf{u} + \frac{1}{1-2\sigma}graddiv\mathbf{u} = 0. \qquad (7)$$

To solve the equation (7) within the framework of the contact problem formulated above, using the cylindrical coordinate system, whose the axis Z is directed along the axis of the cylinder. It is clear that the strain vector has only two possible projections on the axis Z and r, i.e. $\mathbf{u} = \left(u_r(r,z), 0, u_z(r,z)\right)$.

Projecting the equation (7) on these axes we are coming to the following system of the equations

$$\begin{aligned} &\Delta u_r - \frac{u_r}{r^2} + \frac{1}{1-2\sigma}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right] = 0, \\ &\Delta u_z + \frac{1}{1-2\sigma}\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right] = 0. \end{aligned} \qquad (8)$$

Where the second summand in the upper equation of the system (8) is a result of the covariant differentiation after transition to a curvilinear coordinates (see, for example, ref. [3], p. 614).

The solution of the equations (8) as we have abovementioned, would be founded due to the auto modelling and should lead to a qualitatively correct result (which we shall see below).

I.e. introduce the follow dimensionless value

$$x = \frac{z}{r}. \qquad (9)$$

As a result we are getting the following system of equations

$$\begin{aligned} &\left[x^2 + \frac{1-2\sigma}{2(1-\sigma)}\right]u_r'' + xu_r' - u_r + \\ &+\frac{1}{2(1-\sigma)}\left(u_z'' - u_z'\right) = 0 \end{aligned}, \qquad (10)$$

$$\begin{aligned} &x^2u_r'' + \left[2(1-\sigma) + (1-2\sigma)x^2\right]u_z'' + \\ &+(1-2\sigma)xu_z' = 0 \end{aligned}. \qquad (11)$$

As we can see from the equations (10) and (11) the founding system of the equations is a linear algebraic system with reference to the derivatives u_z' and u_z''. Solving it for u_z' and u_z'' we are finding the following result after simple algebraic manipulations:

$$u_z' = \frac{b-c}{1+a}, \quad u_z'' = \frac{b+ac}{1+a}, \tag{12}$$

where the functions are

$$a = \frac{(1-2\sigma)x}{2(1-\sigma)+(1-2\sigma)x^2}, \quad b = -\frac{x^2 u_r''}{2(1-\sigma)+(1-2\sigma)x^2}, \quad c = 2(1-\sigma)\left\{\left[x^2 + \frac{1-2\sigma}{2(1-\sigma)}\right]u_r'' + xu_r' - u_r\right\}. \tag{13}$$

Differentiating the upper equation in the system (12) with respect to x and setting to the lower one, we can easily finding the following automodeling equation of the third order. Indeed, the differentiating shows that

$$b' - c' = \left(1 + \frac{a'}{1+a}\right)b + \left(1 - \frac{a'}{1+a}\right)c. \tag{14}$$

By substituting the explicit functions a, b, c determined by (13) to eq. (14) and grouping terms we are getting required equation after clumsy manipulations for $u_r(x)$

$$\varphi_1(x)u_r''' + \varphi_2(x)u_r'' + \varphi_3(x)u_r' - \varphi_4(x)u_r = 0, \tag{15}$$

where the new functions are the following

$$\begin{aligned} &\varphi_1(x) = 1-2\sigma+2(1-\sigma)x^2 + \frac{x^2}{s(x)}, \\ &\varphi_2(x) = 6(1-\sigma)x + \left[1-2\sigma+2(1-\sigma)x^2\right]\times \\ &\times\left(1-\frac{a'}{1+a}\right) - \frac{x^2}{s(x)}\left(1+\frac{a'}{1+a}\right), \\ &\varphi_3(x) = 2(1-\sigma)\left(1-\frac{a'}{1+a}\right)x, \\ &\varphi_4(x) = 2(1-\sigma)\left(1-\frac{a'}{1+a}\right), \\ &s(x) = 2(1-\sigma)+(1-2\sigma)x^2. \end{aligned} \tag{16}$$

Introducing one more function $\beta(x) = \frac{a'}{1+a}$, for the convenience and carrying out simple actions of an arithmetic nature, we are finding for it the following expression

$$\begin{aligned} &\beta(x) = (1-2\sigma)\left[2(1-\sigma)-(1-2\sigma)x^2\right]\times \\ &\times\left[2(1-\sigma)+(1-2\sigma)x^2\right]^{-1}\times \\ &\times\left[2(1-\sigma)+(1-2\sigma)(x+x^2)\right]^{-1} \end{aligned} \tag{17}$$

Thus, changing all no integrals $\frac{a'}{1+a}$ in the functions $\varphi_2, \varphi_3, \varphi_4$ on $\beta(x)$ from eq. (15), we are getting following third-order differential equation in which we introduce a more abbreviated dimensionless notation for the convenience, $y = \frac{u_r(x)}{R}$, where $R-$ is the radius of the no deform rod.

$$\begin{aligned} &\left[1-2\sigma+2(1-\sigma)x^2 + \frac{x^2}{s(x)}\right]y''' + \\ &+\left\{ 2x(1-\sigma)\left[3+\frac{2}{s^2(x)}\right] + \right. \\ &+\left[1-2\sigma+2(1-\sigma)x^2\right]\times \\ &\left.\times(1-\beta(x)) - \frac{x^2}{s(x)}(1+\beta(x)) \right\}y'' + \\ &+2(1-\sigma)(1-\beta(x))xy' - 2(1-\sigma)(1-\beta(x))y = 0 \end{aligned} \tag{18}$$

4 Numerical solution of the equation (18)

To solve the eq. (18) we are introduce the following boundary conditions

$$y(0) = 0,\;\; y(\infty) = 0,\;\; y'(0) = C\,.$$

In the accordance with the parameter C we could show the numerical solution of the equation (18) due to the Fig. 1-6 for different values of the Poisson's coefficient.

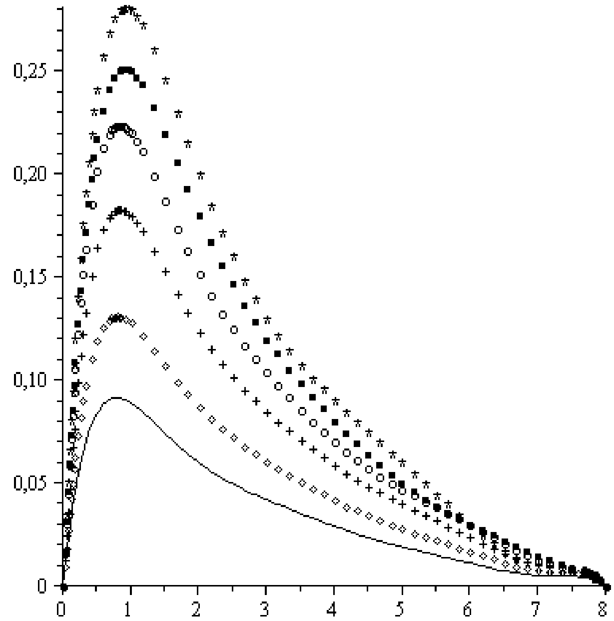

Fig.1. The function $y(x)$, obtained for the fixed initial condition $y'(0) = 1$ and at different coefficient values of the Poisson's ratio σ. All the curves are in accord with theoretical meanings of Poisson's coefficient (from now on). Solid curve conforms with $\sigma = 0.45$, quadrangles - $\sigma = 0.4$, crosses- $\sigma = 0.3$, circles - $\sigma = 0.2$, black boxes - $\sigma = 0.1$, asterisks- $\sigma = 0.01$.

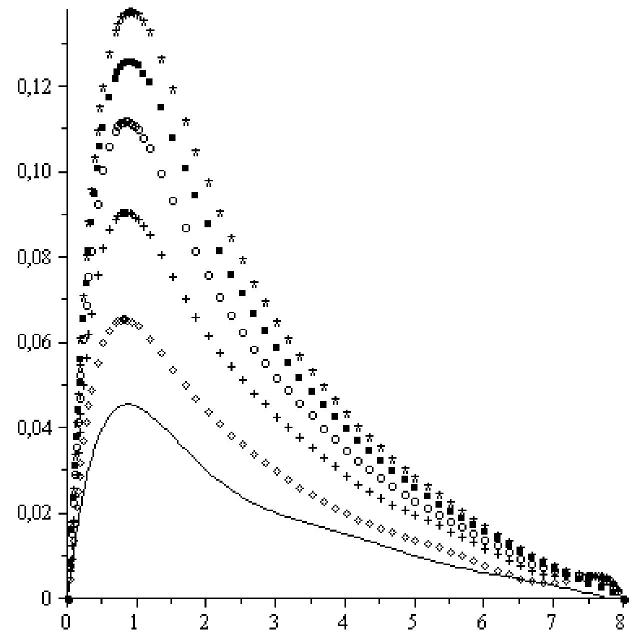

Fig.2. The function $y(x)$, obtained for an fixed initial condition $y'(0) = 0.5$ and at different coefficient values of the Poisson's ratio σ. Solid curve conforms with $\sigma = 0.45$, quadrangles - $\sigma = 0.4$, crosses - $\sigma = 0.3$, circles - $\sigma = 0.2$, black boxes - $\sigma = 0.1$, asterisks - $\sigma = 0.01$.

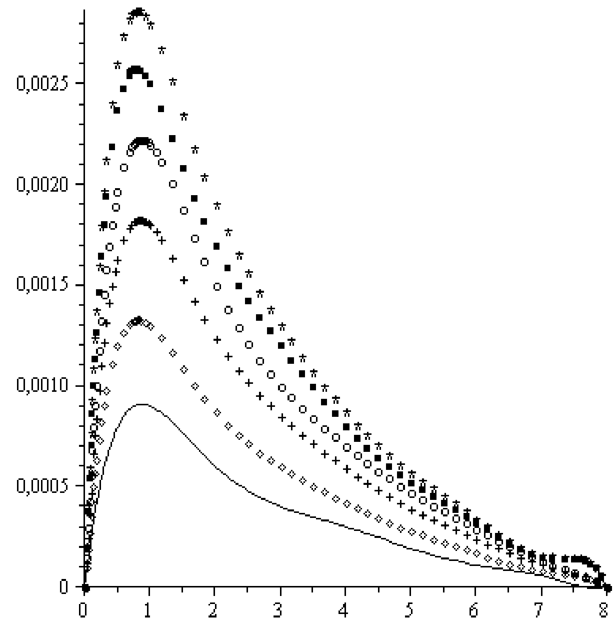

Fig.3. The function $y(x)$, obtained for an fixed initial condition $y'(0) = 0.01$ and at different coefficient values of the Poisson's ratio σ. Solid curve conforms with $\sigma = 0.45$, quadrangles- $\sigma = 0.4$

, crosses - $\sigma = 0.3$, circles - $\sigma = 0.2$, black boxes - $\sigma = 0.1$, asterisks - $\sigma = 0.01$.

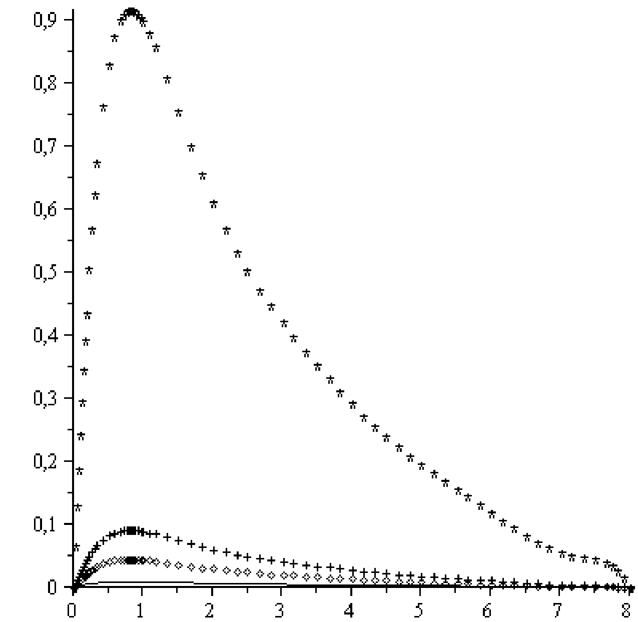

Fig.4. The function $y(x)$, obtained at a fixed value of the Poisson's ratio $\sigma = 0.45$ and at different initial conditions. The solid curve, enlarged tenfold, corresponds to $y'(0) = 0.01$, quadrangles- $y'(0) = 0.5$, crosses - $y'(0) = 1$, asterisks - $y'(0) = 10$.

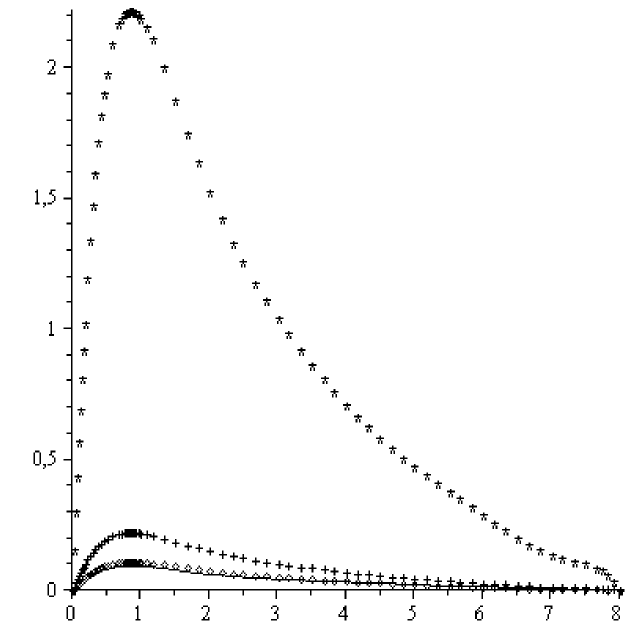

Fig.5. The function $y(x)$, obtained at a fixed value of the Poisson's ratio $\sigma = 0.2$ and at different initial conditions. Solid centuplicated curve corresponds to $y'(0) = 0.01$, quadrangles - $y'(0) = 0.5$, crosses - $y'(0) = 1$, asterisks - $y'(0) = 10$.

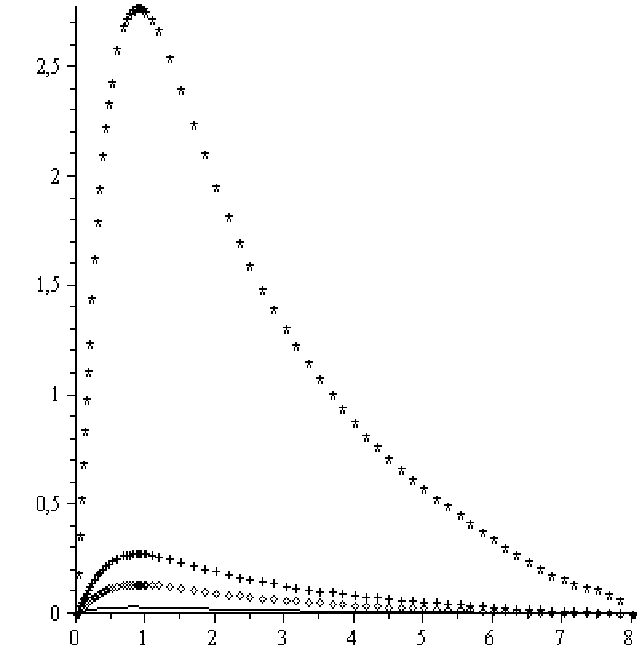

Fig.6. The function $y(x)$, obtained at a fixed value of the Poisson's ratio $\sigma = 0.01$ and at different initial conditions. The solid curve, enlarged tenfold corresponds to $y'(0) = 0.01$, quadrangles - $y'(0) = 0.5$, crosses - $y'(0) = 1$, asterisks - $y'(0) = 10$.

We should notice that such assemblage of curves of strain distribution is totally single-valued if we recall energy conservation principle (1). However before using of this equation we should first find of the dependence $u_z(x)$. Indeed, in the accordance with (12) we obtain

$$u_z = -\int_0^x dt \frac{b(t) - c(t)}{1 + a(t)}.$$

Substituting in here the expr. (13), as a result we are finding that

$$u_z(x) = \int_0^x \left[p(t) y''(t) + q(t)\left(ty'(t) - y(t)\right) \right] dt, \quad (19)$$

where the functions are

$$p(t)=\frac{t^2+2(1-\sigma)\left[2(1-\sigma)+(1-2\sigma)t^2\right]}{2(1-\sigma)+(1-2\sigma)(t+t^2)}\times$$
$$\times\left[t^2+\frac{1-2\sigma}{2(1-\sigma)}\right], \tag{20}$$
$$q(t)=\frac{2(1-\sigma)\left[2(1-\sigma)+(1-2\sigma)t^2\right]}{2(1-\sigma)+(1-2\sigma)(t+t^2)}.$$

Numerical analysis of the function (19) illustrates by the Figures 7 – 12 at different values of parameter C and the Poisson's coefficient σ.

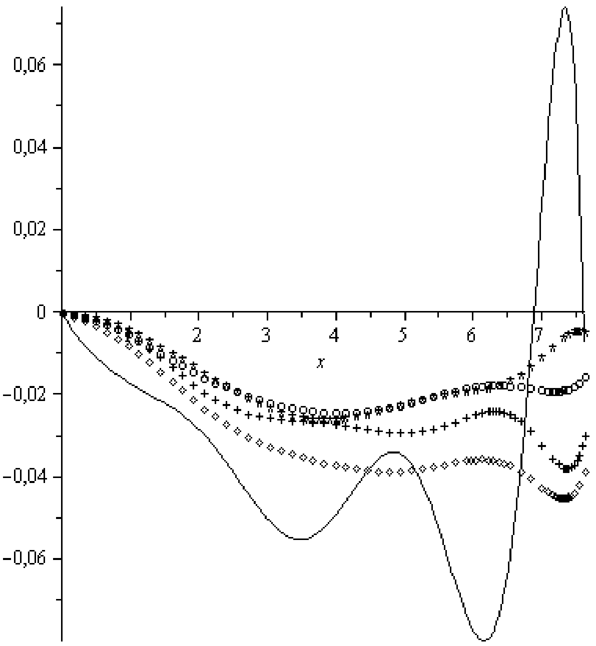

Fig.7. The dependence $u_z(x)$, constructed under a constant condition $y'(0)=0.01$ for different values of the Poisson's ratio σ. The solid curve corresponds to $\sigma=0.1$, quadrangles - $\sigma=0.2$, crosses - $\sigma=0.3$, circles - $\sigma=0.4$, asterisks - $\sigma=0.45$.

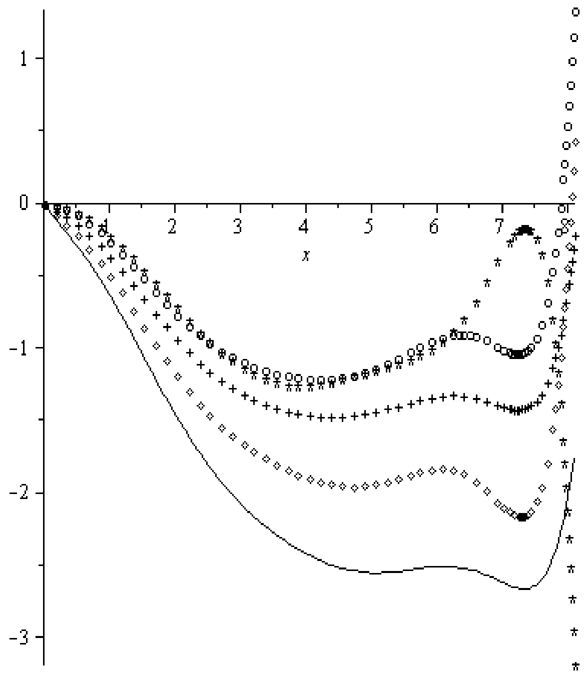

Fig.8. The dependence $u_z(x)$, constructed under a constant condition $y'(0)=0.5$ for the different values of the Poisson's ratio σ. The solid curve corresponds to $\sigma=0.1$, quadrangles - $\sigma=0.2$, crosses - $\sigma=0.3$, circles - $\sigma=0.4$, asterisks - $\sigma=0.45$.

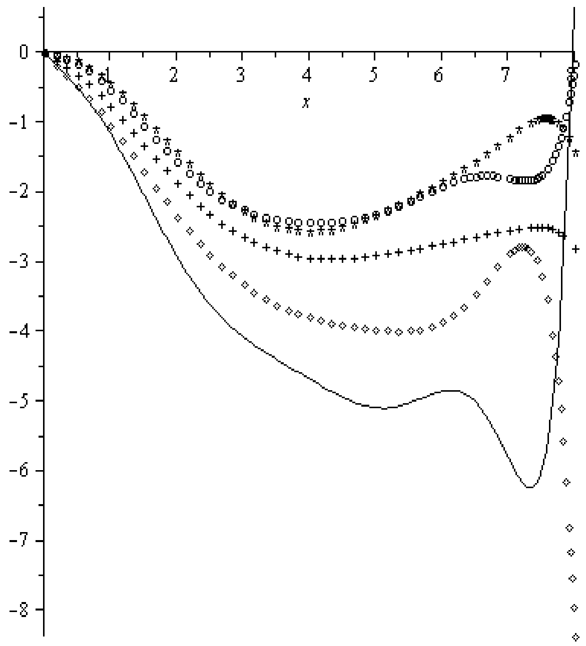

Fig.9. The dependence $u_z(x)$, constructed under a constant condition $y'(0)=1$ for different values of the Poisson's ratio σ. The solid curve corresponds to $\sigma=0.1$, quadrangles - $\sigma=0.2$, crosses - $\sigma=0.3$, circles - $\sigma=0.4$, asterisks - $\sigma=0.45$.

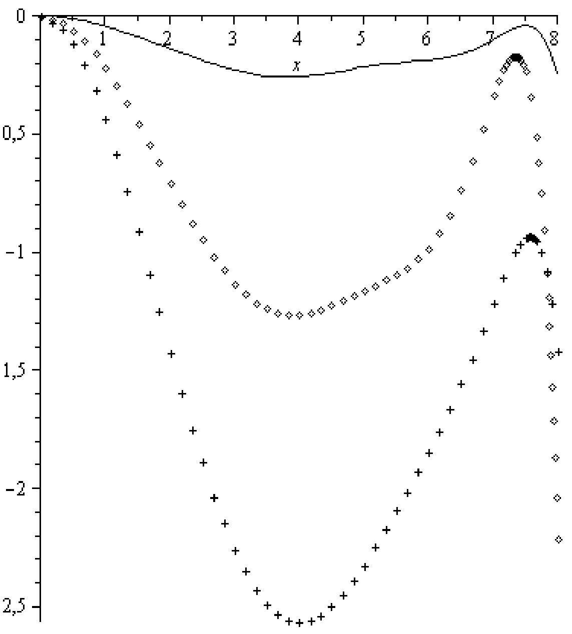

Fig.10. The dependence $u_z(x)$, constructed under constant value of the Poisson's ratio $\sigma = 0.45$ for different initial conditions. The solid curve, enlarged tenfold corresponds to $y'(0) = 0.01$, circles - $y'(0) = 0.5$, crosses - $y'(0) = 1$.

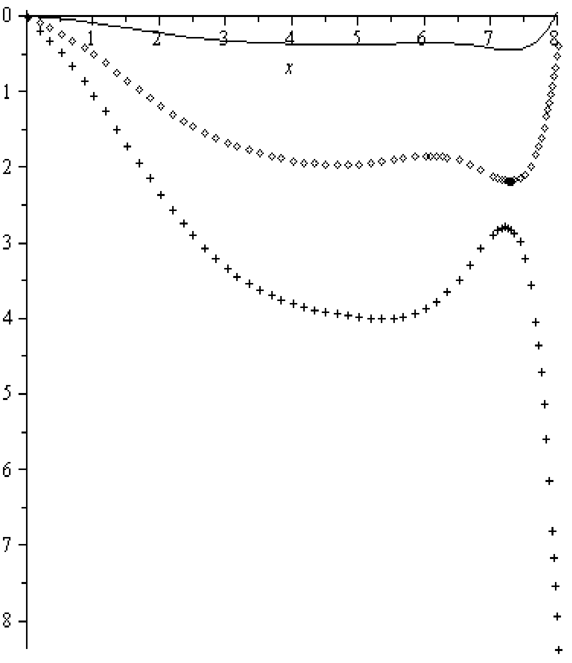

Fig.11. The dependence $u_z(x)$, constructed under constant value of the Poisson's ratio $\sigma = 0.2$ for different initial conditions. The solid curve, enlarged tenfold corresponds to $y'(0) = 0.01$, circles - $y'(0) = 0.5$, crosses - $y'(0) = 1$.

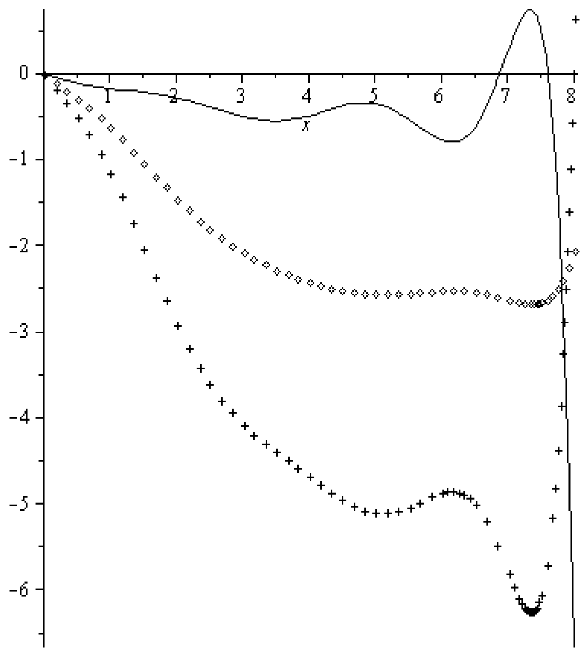

Fig.12. The dependence $u_z(x)$, constructed under constant value of the Poisson's ratio $\sigma = 0.1$ for different initial conditions. The solid curve, enlarged tenfold corresponds to $y'(0) = 0.01$, circles- $y'(0) = 0.5$, crosses - $y'(0) = 1$.

Therefore we could consider strain distribution is given and we could calculate free-energy density which in accordance with the expr. (3) is

$$F_V = \mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}\right)^2 + \frac{K}{2}u_{ll}^2.$$

Such simple manipulations leads us to the following expression

$$F_V = \frac{E}{4(1+\sigma)}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)^2 + \\ + \frac{(5-7\sigma)E}{6(1+\sigma)(1-2\sigma)}\left[\left(\frac{\partial u_z}{\partial z}\right)^2 + \left(\frac{\partial u_r}{\partial r}\right)^2\right],$$

Where we just proceed from differentiating over r and z to the differentiating on x As the result we are getting that the free energy is

$$F_V = \frac{E}{4(1+\sigma)r^2}\left(u_r' - xu_z'\right)^2 + \\ + \frac{(5-7\sigma)E}{6(1+\sigma)(1-2\sigma)r^2}\left(u_z'^2 + x^2u_r'^2\right). \quad (21)$$

In accordance with (19)

$$u_z'(x) = p(x)y'' + q(x)\left(xy' - y\right)$$

and from the expr. (21) we have

$$F_V = \frac{E}{4(1+\sigma)r^2}\left(y' - x\left[p(x)y'' + q(x)(xy' - y)\right]\right)^2 + \\ + \frac{(5-7\sigma)E}{6(1+\sigma)(1-2\sigma)r^2}\times \\ \times\left(x^2y'^2 + \left[p(x)y'' + q(x)(xy'-y)\right]^2\right) \qquad (22)$$

Taking into account (2), we obtain the following integral condition

$$\varepsilon\int_0^h \rho\mathbf{V}^2 dz = \frac{2}{R^2}\int_0^{h_1}\int_0^R rF_V drdz\,, \qquad (23)$$

where $h = h_1 + h_2 -$ is the total deformation which is divided between the rod where it is equal to h_1 and the elastic medium, where it is equal to h_2.

To calculate the double integral figuring in here it is very convenient to come to the new variables. One of the variables we have already know i.e. $x = \frac{z}{r}$. And the other one we choose not as in a parabolic coordinate system $u = rz$, but in a more convenient form, as

$$u = z^2 + r^2\,. \qquad (24)$$

Therefore, for the backward transformation, we are getting that

$$r = \sqrt{\frac{u}{1+x^2}}, z = x\sqrt{\frac{u}{1+x^2}}\,. \qquad (25)$$

Then the transition Jacobian is $J = \frac{1}{2\left(1+x^2\right)}$ and integration limits are piecewise smooth.

Domain of integration are

$$x \in \left[0, \frac{h}{R}\right],\ u \in \left[h^2, R^2\left(1+x^2\right)\right], \qquad (26)$$

$$x \in \left[\frac{h}{R}, \infty\right),\ u \in \left[h^2, \frac{h^2\left(1+x^2\right)}{x^2}\right]. \qquad (27)$$

As we can easily check with the use of the domain (26), (27) the area limited with piecewise smooth lines with the use of transition Jacobian is stable and equal to the hR.

In the result from the eq. (23) we obtain the following equation

$$\varepsilon\int_0^{\frac{h}{R}}\rho\mathbf{V}^2dx = \frac{1}{2R^3}\left[\int_0^{\frac{h}{R}}\frac{F_V(x)}{\left(1+x^2\right)^{\frac{1}{2}}}dx\int_{h^2}^{R^2\left(1+x^2\right)}\frac{du}{\sqrt{u}} + \right. \\ \left. +\int_{\frac{h}{R}}^{\infty}\frac{F_V(x)}{\left(1+x^2\right)^{\frac{1}{2}}}dx\int_{h^2}^{\frac{h^2\left(1+x^2\right)}{x^2}}\frac{du}{\sqrt{u}}\right].$$

Dimensionless variable $x = \frac{z}{R}$ is introduced for more convenient further purposes on the left – hand side. Inner integral on the right side is calculated elementary. As a result we obtain that

$$\varepsilon\int_0^{\frac{h}{R}}\rho\mathbf{V}^2dx = \frac{E}{R}\left[\int_0^{\frac{h}{R}}\frac{f_V(x)\left[R\left(1+x^2\right)^{\frac{1}{2}} - h\right]}{\left(1+x^2\right)^{\frac{1}{2}}}dx + \right. \\ \left. +h\int_{\frac{h}{R}}^{\infty}\frac{f_V(x)}{\left(1+x^2\right)^{\frac{1}{2}}}\left[\frac{\left(1+x^2\right)^{\frac{1}{2}}}{x} - 1\right]dx\right], \qquad (28)$$

where the dimensionless function is

$$f_V(x) = \frac{F_V(x)}{E} = \\ = \frac{1}{4(1+\sigma)}\left(y' - x\left[p(x)y'' + q(x)(xy'-y)\right]\right)^2 + \\ + \frac{(5-7\sigma)}{6(1+\sigma)(1-2\sigma)}\left(x^2y'^2 + \left[p(x)y'' + q(x)(xy'-y)\right]^2\right).$$

Reduced equation let us the possible calculate the dependence of the relative energy losses into the rod as a function of the parameter $\lambda = \frac{h_1}{R}$. Indeed, since according to the energy conservation principle, all the kinetic energy of the rod is expended on its own deformation and deformation of the semi-infinite elastic medium (remember: we ignore the heating of both bodies, as well as the sound losses). As a result we have get the simple relation

$$\pi\rho R^2\int_0^h \mathbf{V}^2 dz = 2\int_{V_c} F_V dV + 2\int_V F_V' dV \,,$$

where the first summand on the right belongs to the rod as it is shows the inferior index c near the volume. The elastic medium as we have already mentioned at the beginning of the paper is characterized simply by a prime mark.

The second summand is a total energy of deformation, which the rod gives to the elastic medium. For the rod integration is conveniently carried out in cylindrical coordinates, where the volume element is $dV_c = 2\pi r dr dz$, where coordinate z varies from 0 to h_1 and radius $r \in [0, R]$.

The second integral on the right is convenient to calculate in spherical coordinates the center of which must be taken in the point of contact of the rod with the medium. In this case

$$dV = r^2 dr d\text{O}\,,$$

where $d\text{O} = \sin\theta d\theta d\varphi$ - is the element of the solid angle and angle θ varies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, polar angle φ belongs to the range from 0 to 2π, $r \in [h_2, \infty)$ as always.

The first integral was calculated above in (28), but we are going to find the second one. To calculate it we use the classical solution of the point Thomson's problem [1]. In this connection the dependence of the deformation vector on the coordinates can be represented as

$$\mathbf{u} = \frac{1+\sigma'}{8\pi E'(1-\sigma')}\frac{(3-4\sigma')\mathbf{F} + \mathbf{n}(\mathbf{nF})}{r}, \quad (29)$$

where $\mathbf{F}-$ is the adding contact force, which is determined here as $\mathbf{F} = M\mathbf{a} = \pi R^2 H\rho\mathbf{V}\tau^{-1}$, $H-$ a length of the rod, $\mathbf{n} = \frac{\mathbf{r}}{r}-$ unit vector in the direction of the radius vector $\mathbf{r}$, $\tau-$ is the contact time. σ', E' is the Poisson's coefficient and the Young's modulus of the elastic medium accordance.

Calculating the deformation tensor $u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$ with the use of expr. (29), we obtain in the result

$$u_{ik} = \frac{1+\sigma'}{8\pi E'(1-\sigma')r^2}\times \\ \times\left[(3-4\sigma')(n_i F_k + n_k F_i) + (\delta_{ik} - 3n_i n_k)\mathbf{nF}\right]. \quad (30)$$

From where the divergence of the displacement vector is

$$u_{ll} = div\mathbf{u} = \frac{(1+\sigma')(3-4\sigma')}{4\pi E(1-\sigma')}\frac{(\mathbf{nF})}{r^2}.$$

Calculating the free – energy density of elastic medium

$$F_V' = \frac{E}{2(1+\sigma')}u_{ik}^2 + \frac{\sigma' E}{2(1+\sigma')(1-2\sigma')}u_{ll}^2,$$

we are finding after the simple mathematical manipulations:

$$F_V' = \frac{1+\sigma'}{E'\left[8\pi(1-\sigma')r^2\right]^2}\times \\ \times\left[(3-4\sigma')^2 F^2 + \left(3-(1+4\sigma')(3-4\sigma')\right)(\mathbf{nF})^2\right]. \quad (31)$$

Therefore, the strain energy of an semi – infinite elastic medium could be written as

$$F' = \int_V F_V' dV = \int_{h_2}^{\infty} dr \int_\Omega F_V' d\text{O}$$

Integration with respect to the angular variables is carried out elementary with the help of the formula $\bar{F}_V'(r) = \frac{1}{4\pi}\int_\Omega F_V' d\text{O}$, where the upper prime mark means averaging over the angular variables, value of Ω it is means domain of integration with respect to φ from 0 to 2π, but with respect to θ - from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. In the result we have

$$\bar{F}_V'(r) = \frac{F^2(1+\sigma')\left[3 + 8(3-4\sigma')\left(1 - \frac{5}{4}\sigma' - \sigma'^2\right)\right]}{3E'\left[8\pi(1-\sigma')r^2\right]^2},$$

After integrating with respect to r we obtain that

$$F' = 2\pi\int_{h_2}^{\infty} \overline{F}_V'(r)r^2dr = = \frac{2\pi F^2\left(1+\sigma'\right)\left[3+8\left(3-4\sigma'\right)\left(1-\frac{5}{4}\sigma'-\sigma'^2\right)\right]}{3E'h_2\left[8\pi\left(1-\sigma'\right)\right]^2}. \quad (32)$$

Hence, the total energy is the sum $W = F + F'$, from where according to (28):

$$F = 2\pi ER^3\left[\int_0^{\lambda}\frac{f_V(x)\left[\left(1+x^2\right)^{\frac{1}{2}}-\lambda\right]}{\left(1+x^2\right)^{\frac{1}{2}}}dx + +\lambda\int_{\lambda}^{\infty}\frac{f_V(x)}{\left(1+x^2\right)^{\frac{1}{2}}}\left[\frac{\left(1+x^2\right)^{\frac{1}{2}}}{x}-1\right]dx\right]. \quad (33)$$

It means that the relative energy losses in the rod that we are interested can be estimated using the formula

$$\varepsilon\left(\lambda\right) = \frac{F}{F+F'}.$$

Substituting in it (32) and (33) we get as a result

$$\varepsilon\left(\lambda\right) = \frac{S_{\sigma}\left(C,\lambda\right)}{S_{\sigma}\left(C,\lambda\right)+S'\left(\lambda,\nu\right)}, \quad (34)$$

where

$$S_{\sigma}\left(C,\lambda\right) = \int_0^{\lambda} f_V(x)\left(1-\frac{\lambda}{\sqrt{1+x^2}}\right)dx + +\lambda\int_{\lambda}^{\infty} f_V(x)\left(\frac{1}{x}-\frac{1}{\sqrt{1+x^2}}\right)dx, \quad (35)$$

and

$$S' = \frac{F^2\left(1+\sigma'\right)\left[1+8\left(1-\frac{4}{3}\sigma'\right)\left(1-\frac{5}{4}\sigma'-\sigma'^2\right)\right]}{EE'R^3h_2\left[8\pi\left(1-\sigma'\right)\right]^2}. \quad (36)$$

As $\mathbf{F} = \pi\rho R^2H\mathbf{V}\tau^{-1}$ for the force of the modulus with allowance for the solution of the Hertz's contact problem the contact time is determined as

$$\tau = 2.94\left(\frac{25m^2D^2}{16R\left|\mathbf{V}\right|}\right)^{\frac{1}{5}}, \quad (37)$$

where the stiffness factor

$$D = \frac{3}{4E}\left[1-\sigma^2+\frac{E}{E'}\left(1-\sigma'^2\right)\right]. \quad (38)$$

If we introduce instead of the mass of the rod m, its expression in terms of the density according to the formula $m = \pi\rho R^2 H$ and also introducing the average velocity of the sound as

$$\overline{c}_s = \sqrt{\frac{E}{\rho\left(1-\sigma^2+\frac{E}{E'}\left(1-\sigma'^2\right)\right)}} \quad (39)$$

We are getting from (37) that contact time

$$\tau = \frac{R}{\left|\mathbf{V}\right|}\left(\frac{\left|\mathbf{V}\right|}{\overline{c}_s}\right)^{\frac{4}{5}} \quad (40)$$

Finally, we obtain for free energy F

$$F = \xi\rho\overline{c}_s^2 R^{\frac{7}{5}} H^{\frac{3}{5}}\left(\frac{\left|\mathbf{V}\right|}{\overline{c}_s}\right)^{\frac{6}{5}}, \quad (41)$$

where the numerical parameter is $\xi = \frac{2\pi}{2.94}\left(\frac{8}{195\pi^2}\right)^{\frac{1}{5}} \approx 0.71$. If we substitute (41) into the expr. (36), we are finding the required expression

$$S'\left(\lambda,\nu\right) = \xi^2\frac{H^{\frac{6}{5}}}{h_2R^{\frac{1}{5}}}\left(\frac{\overline{c}_s}{u_s}\right)^4\left(\frac{\left|\mathbf{V}\right|}{\overline{c}_s}\right)^{\frac{12}{5}} \times \times\frac{\left(1+\sigma'\right)\left[1+8\left(1-\frac{4}{3}\sigma'\right)\left(1-\frac{5}{4}\sigma'-\sigma'^2\right)\right]}{\left[8\pi\left(1-\sigma'\right)\right]^2}, \quad (42)$$

where the velocity

$$u_s = \left(\frac{EE'}{\rho^2}\right)^{\frac{1}{4}}. \quad (43)$$

For the qualitative evaluation of the function $\varepsilon(\lambda)$, we need to assume an obvious correlation is done

$$\frac{h_1}{h_2} = \frac{E'}{E}, \qquad (44)$$

Hence,

$$\frac{h_2}{R} = \lambda \frac{E}{E'}. \qquad (45)$$

And from the expr. (42) we are getting that

$$S'(\lambda,\nu) = \xi^2 \lambda^{\frac{1}{5}} \nu^{\frac{6}{5}} \left(\frac{\overline{c}_s}{u_s}\right)^4 \left(\frac{|\mathbf{V}|}{\overline{c}_s}\right)^{\frac{12}{5}} \frac{E'}{E} \times$$
$$\times \frac{(1+\sigma')\left[1+8\left(1-\frac{4}{3}\sigma'\right)\left(1-\frac{5}{4}\sigma'-\sigma'^2\right)\right]}{\left[8\pi(1-\sigma')\right]^2}, \qquad (46)$$

where $\nu = \frac{H}{h_1}$.

Hence, the formulas (35) and (46) let us the possible estimate the fractional variation of strain energy of the rod upon impact with the use of the eq. (34).

Indeed, if for definiteness we choose the values of the Young's modulus and the Poisson's coefficient as

$$E = 2\cdot 10^{11}\frac{erg}{cm^3},\ E' = 2.2\cdot 10^{11}\frac{erg}{cm^3},\ \sigma' = 0.33, \qquad (47)$$

The density is $\rho = 7.8\frac{g}{cm^3}$. Then due to (39) for different values of the Poisson's ratio medium velocity of the sound σ is always equal to $\overline{c}_s \approx 10^5 \frac{cm}{s}$, and velocity which defined by the formula (43) is $u_0 = 1.64\cdot 10^5 \frac{cm}{s}$. As a result, for the theoretical and experimental values of Young's modulus and Poisson's ratio shown in the Table 1.

Table 1. Theoretical and experimental values of the Young's modulus and Poisson's ratio, adopted for the construction of the dependence $\varepsilon(\lambda)$.

	Young's modulus E, MPa	The Poisson's coefficient for calculations	The Poisson's coefficient (true value)
Concrete	1.5*10^3	0.1	0.16
Cast iron	1.15*10^5	0.2	0.23
Cuprum (aluminum)	1.1*10^5	0.3	0.31
Plumbum	1.7*10^4	0.4	0.42
Caoutchouc	8	0.45	0.47

The dependencies $\varepsilon(\lambda)$ illustrated by Figures 13 – 15.

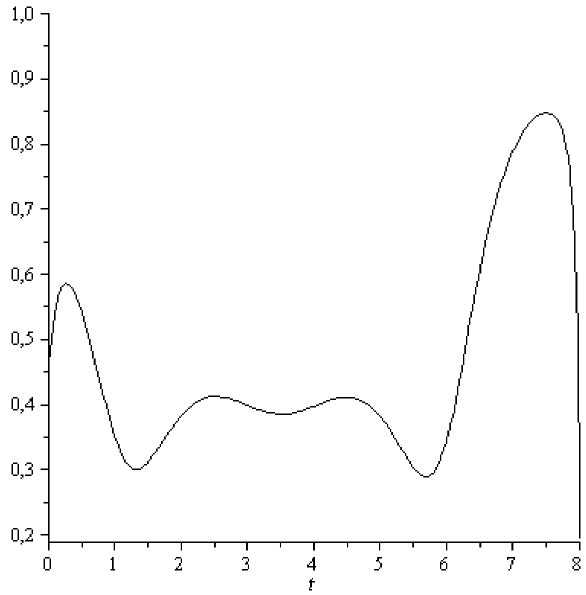

Fig.13. The dependence $\varepsilon(\lambda)$, obtained when a cylindrical rod hits a rubber moving at a speed of 30 m /s after contact with the steel medium.

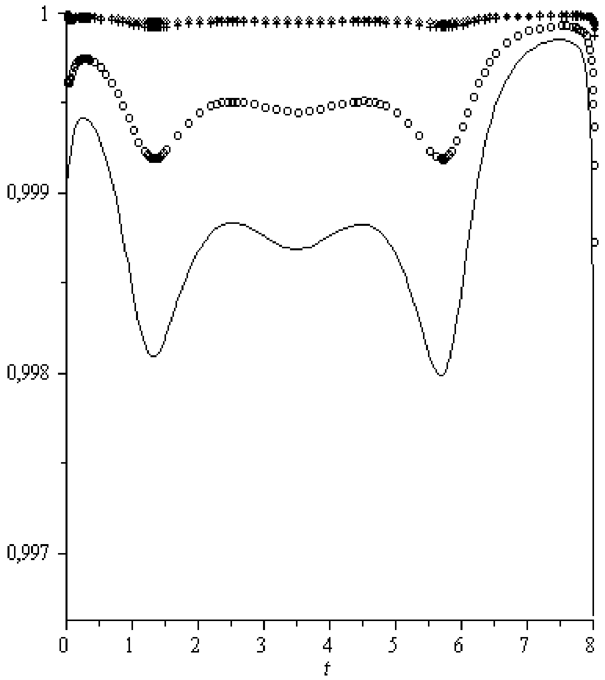

Fig.14. The dependencies $\varepsilon(\lambda)$, obtained when a cylindrical rod hits different materials moving at a speed of 30 m / s after contact with the steel medium. The solid curve corresponds to $\sigma = 0.1$ (concrete), quadrangles - $\sigma = 0.2$ (cast iron), crosses - $\sigma = 0.3$ (cuprum), circles - $\sigma = 0.4$ (plumbum).

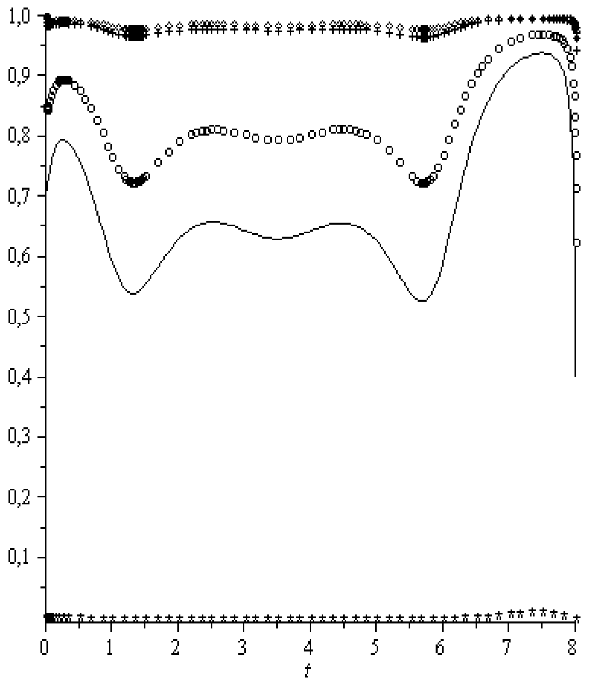

Fig. 15. The dependencies $\varepsilon(\lambda)$, obtained when a cylindrical rod hit different materials moving at a speed of 10 m / s after contact with the rubber medium. The solid curve corresponds to $\sigma = 0.1$ (concrete), quadrangles - $\sigma = 0.2$ (cast iron), crosses - $\sigma = 0.3$ (cuprum), circles - $\sigma = 0.4$ (plumbum), asterisks - $\sigma = 0.45$ (cautchouc).

As to the function $S_\sigma(C,\lambda)$, its dependence on the parameter λ could be represented using the graph shown in the Fig. 16.

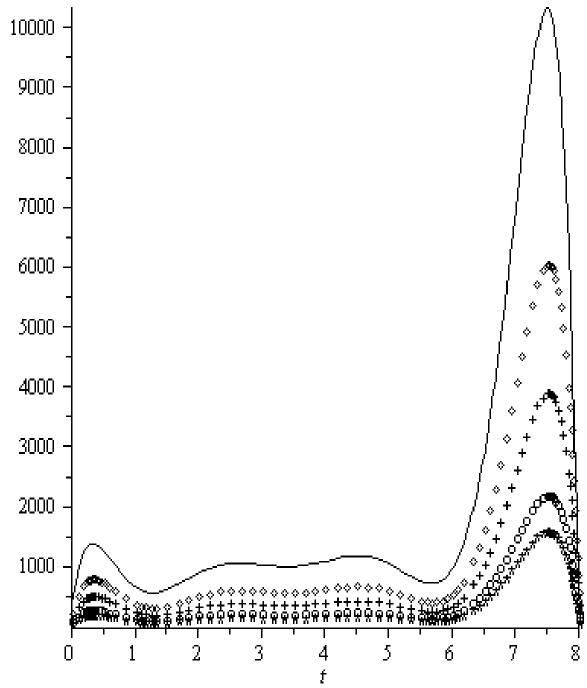

Fig. 16. The dependence $S_\sigma(C,\lambda)$, obtained for different varies of the Poisson's ratio. The solid curve corresponds to $\sigma = 0.1$ (concrete), quadrangles - $\sigma = 0.2$ (cast iron), crosses - $\sigma = 0.3$ (cuprum), circles - $\sigma = 0.4$ (plumbum), asterisks - $\sigma = 0.45$ (caoutchuc). The initial condition for function $y(x)$ here $y'(0) = 0.5$.

5 Classification analysis of the equation of the static elasticity

And the last thing we would like to pay attention is the classification of the equation (7). Indeed, if we formulate a standard equation for proper values k (see, for example, ref. [4]), then as a result we obtain the following algebraic equation of the third order:

$$k^3 - 2\frac{1-3\sigma}{1-2\sigma}k^2 + \frac{(1-\sigma)(1-3\sigma)}{(1-2\sigma)^2}k + \frac{8\sigma(1-\sigma)^2}{(1-2\sigma)^3} = 0 \cdot \quad (48)$$

Its graphical solution could be illustrated due to the Fig. 17.

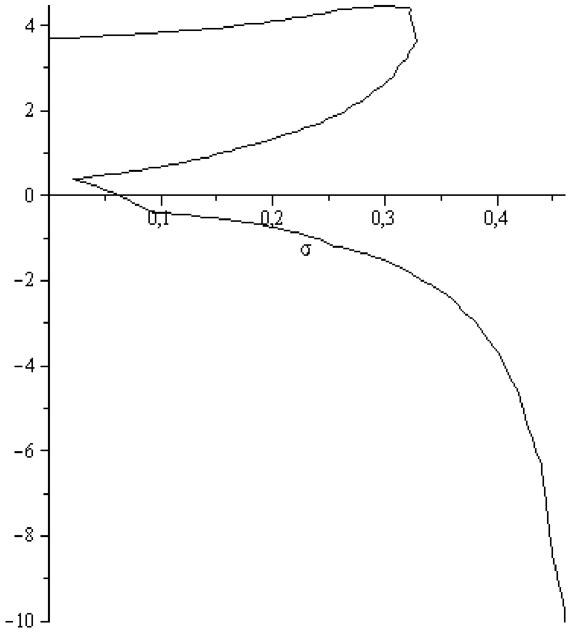

Fig. 17. The Poisson's ratio is plotted along the abscissa axis and the function along the ordinate axis $y = k(\sigma)$. We could see that with the value of the Poisson's ratio $\sigma \approx 0.067$, the equation could be attributed to an equation of parabolic type, which would mean the possibility of dissipation of energy in such a material. However, because of the specificity of the equation of the static theory of elasticity (7), this only means that it must describe a purely two-dimensional situation. That is why, at this value of the Poisson's ratio the material could be principal only two-dimensional.

As we could see from this Figure, for the values $0.067 < \sigma < 0.35$, equation (2) will be an equation of the hyperbolic type (as $k_1 > 0, k_2 > 0, k_3 < 0$), and for the values of $0 < \sigma < 0.067$, the equation (2) will be an equation of the elliptic type (here $k_1 > 0, k_2 > 0, k_3 > 0$).

It means that in the according to the general theory of partial differential equations [4, 5], in the case of an equation of hyperbolic type, the solutions in the two-dimensional case will be the imposition of proper periodic oscillations. In the case, if the equation belongs to an equation of the elliptic type, solutions could be found, for example, using the Green's function. We also note that the problems of a close orientation could be found, for example, in publications [6 – 10].

The case when $\sigma \approx 0.067$ should be searched more detail. The point is that at this value of the Poisson's ratio, as we could see from the graphical solution (18), the material can be only two – dimensional, as one of the values k vanishes! This interesting conclusion comes from an obvious way, although, practically the case should be somewhat different. In fact, since the basic equation of the static elasticity (7) was obtained in the quadratic approximation with respect to the strain tensor u_{ik}. When summands $A_{ijkl}^{nmpq} u_{ij} u_{kl} u_{nm} u_{pq}$ are taken into account in free-energy density F_V, where coefficients A_{ijkl}^{nmpq} are the tensor of the eight range, the nonzero values of which are found with the help of group theory and are dictated by the specific symmetry of the crystal, equation (7) becomes nonlinear, but already three – dimensional.

6 Conclusion

In the conclusion of this communication we would like to highlight of the main aspects of the abovementioned researches.

1. An analytical and numerical solution of the problem of calculating the distribution of the elastic stresses in the inner end zone of the rod contacting in a pulsed mode with a semi bounded elastic medium is given;
2. The solution turned out to be possible in the self – similar approximation, when the partial differential equation was reduced to an ordinary differential equation of the third order with variable coefficients;
3. Using the energy conservation equation, the relative change in the deformation energy in the rod is calculated:
4. Areas of the values of the Poisson's coefficient for which the basic equation of the static elasticity could be either elliptical or hyperbolic, which was not previously done;
5. It is noted that with the value of the Poisson's coefficient $\sigma \approx 0.067$ it is necessary to take into account the fourth – order corrections in the strain tensor in the free energy density, which naturally leads to nonlinear equations of the static theory of elasticity.

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