Stochastic Aspects of the Plastic Limit

IOANNIS DOLTSINIS
University of Stuttgart
Faculty of Aerospace Engineering and Geodesy
Pfaffenwaldring 27, 70569 Stuttgart
GERMANY
doltsinis@isd.uni-stuttgart.de

Abstract: This study addresses the significance of randomness for the plastic limit of perfectly plastic solids and structures. The response to random input is analyzed, the improvement of the load carrying capacity is considered along with the issue of robustness against randomness, and the probability of failure is explored in the context of reliability assessment. Application of the theoretical statements is demonstrated on a single as well as simple example. Eventually, system reliability is discussed for assemblies of structural components.

Key–Words: Plastic limit, stochastic analysis, robustness, reliability.

1 Introduction

The limit load in perfect plasticity may be considered as the first of the critical states encountered during the course of the elastic-plastic deformation process of a solid [1]. The load multiplier to the limit, the safety factor, can be determined exactly or approximately by utilizing the limit load theorems, the static and the kinematic ones [2], or other methods [3].

The present study deals with the significance of randomness for the safety factor. In this connection stochastic analysis is applied in order to obtain mean and variance as a function of stochastic input in general terms [4]. The next step concerns the issue of optimization in the presence of randomness and comprises the task of robustness. This is followed by reliability considerations regarding the safety of the solid with respect to the plastic limit. The analysis is demonstrated throughout by means of a single example in order to maintain coherence. Thereby various input quantities are considered as stochastic variables. These are, the loading, the geometry and the yield stress of the material. The discussion of the impact on the randomness of the safety factor and on the reliability of the solid is complemented by a consideration of the yield stress of the material as a random field within the solid. An assessment of the reliability, the probability of success or rather of the probability of failure by the plastic limit relies on the probability content of the input variables in the unsafe region; demonstration is provided. A last step deals with the safety of structures assembled of components. Given the characteristics of the components the reliability of the system is asked for. Series and parallel assemblies are standard configurations as are also combinations thereof. In this connection a scheme is presented based on the safety index of the system expressed in terms of the individual constituents. For clarity of the exposition, the theoretical argument is referred to truss structures.

The remainder of the account is organized as follows. Section 2 recalls the definition of the plastic limit and of the safety factor, the safe load multiplier. Section 3 deals with randomness presenting both a probabilistic approach and an approximation of mean and variance of the safety factor independently of the specific source of the randomness. Section 4 is concerned with the improvement of the load carrying capacity of the solid by optimization and takes care of the robustness with respect to the randomness of the input. Section 5 addresses the probability of failure. Section 6 discusses the theoretical arguments by means of an example. The considerations refer to the significance of the random input, demonstrate various occupations of the input covariance matrix and deal with the yield stress as a random field. Optimization of the limit load is attempted by an adjustment of the geometry of the solid. Section 7 touches the issue of the reliability of structural assemblies. Section 8 summarizes the essentials and concludes.

2 The Plastic Limit

The plastic limit state of a perfectly plastic solid or structure is characterized by the existence of a yield mechanism, that is a kinematically admissible velocity field $\dot{\mathbf{u}}(\mathbf{x})$ which induces exclusively plastic strain at a rate $\dot{\boldsymbol{\eta}}$. The yield mechanism, free of elastic constituents, enables deformation without change in stress $\boldsymbol{\sigma}$. The load carrying capacity of the solid is then exhausted, plastic yielding may occur at constant load. Volume forces are denoted $\mathbf{f}$, surface tractions $\mathbf{t}$. The symbols $\mathbf{f}$, $\mathbf{t}$, $\dot{\mathbf{u}}$ and $\dot{\boldsymbol{\eta}}$, $\boldsymbol{\sigma}$ refer to vector arrangements of work conjugate entities, the vector $\mathbf{x}$ specifies the location within the solid.

The rate of work computed with the yield mechanism is

$$L = \int_V \mathbf{f}^{\mathrm{t}}\dot{\mathbf{u}}\mathrm{d}V + \int_A \mathbf{t}^{\mathrm{t}}\dot{\mathbf{u}}\mathrm{d}A. \tag{1}$$

The integration extends over the volume V of the solid and the surface area A. The dissipation rate at yield is

$$D = \int_V \boldsymbol{\sigma}^{\mathrm{t}}\dot{\boldsymbol{\eta}}\mathrm{d}V, \tag{2}$$

the stress $\boldsymbol{\sigma}$ been associated to the plastic strain $\dot{\boldsymbol{\eta}}$ by the applicable flow rule.

The safety factor n as a load multiplier establishes the limit level in equilibrium with the stress $\boldsymbol{\sigma}$ involved in dissipation. This implies the work equality,

$$\begin{aligned} nL &= \int_V (n\mathbf{f})^{\mathrm{t}}\dot{\mathbf{u}}\mathrm{d}V + \int_A (n\mathbf{t})^{\mathrm{t}}\dot{\mathbf{u}}\mathrm{d}A \\ &= \int_V \boldsymbol{\sigma}^{\mathrm{t}}\dot{\boldsymbol{\eta}}\mathrm{d}V = D. \end{aligned} \tag{3}$$

Therefrom the safety factor

$$n = \frac{D}{L} = \frac{\int_V \boldsymbol{\sigma}^{\mathrm{t}}\dot{\boldsymbol{\eta}}\mathrm{d}V}{\int_V \mathbf{f}^{\mathrm{t}}\dot{\mathbf{u}}\mathrm{d}V + \int_A \mathbf{t}^{\mathrm{t}}\dot{\mathbf{u}}\mathrm{d}A}. \tag{4}$$

Safety with respect to the plastic limit requires that

$$L < D \Rightarrow n > 1. \tag{5}$$

For $n = 1$ the applied load is at the limit level, $n < 1$ refers to loading beyond the plastic limit.

3 Randomness

3.1 Probabilistic approach

Randomness in the geometry, the material data and in the magnitude of the imposed loading reflects on the dissipation rate and the rate of work. The safety factor n then is a random quantity with the probability characteristics to explore.

In the following randomness of variables will be indicated by a tilde such that common plastic limit terminology is maintained as far as possible. The probability distribution function of the safety factor $\tilde{n}$ is

$$F_{\mathrm{n}}(n) = P(\tilde{n} < n) = P\left(\frac{\widetilde{D}}{\widetilde{L}} < n\right). \tag{6}$$

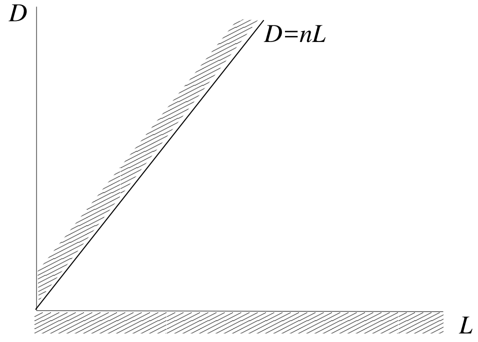


Figure 1: Domain of integration $D/L < n$.

If a joined probability density function $f_{\mathrm{L,D}}(L, D)$ is available,

$$\begin{aligned} F_{\mathrm{n}}(n) &= \int\int_{D/L<n} f_{\mathrm{L,D}}(L, D)\mathrm{d}D\mathrm{d}L \\ &= \int_0^{\infty}\int_0^{nL} f_{\mathrm{L,D}}(L, D)\mathrm{d}D\mathrm{d}L, \end{aligned} \tag{7}$$

with the domain of integration $D/L < n$ as in Fig. 1. The probability density function of the safety factor follows to

$$\begin{aligned} f_{\mathrm{n}}(n) = \frac{\mathrm{d}F_{\mathrm{n}}(n)}{\mathrm{d}n} &= \int_0^{\infty} L f_{\mathrm{L,D}}(L, nL)\mathrm{d}L \\ &= \int_0^{\infty} L f_{\mathrm{L}}(L) f_{\mathrm{D}}(nL)\mathrm{d}L. \end{aligned} \tag{8}$$

The second integral in eqn (8) presumes statistical independence of $\widetilde{L}$ and $\widetilde{D}$ with individual density functions $f_{\mathrm{L}}(L)$, $f_{\mathrm{D}}(D)$. In this case the expectation operations for mean value and variance of the safety factor give for the mean value

$$\mu_{\mathrm{n}} = \mathrm{E}\left[\frac{\widetilde{D}}{\widetilde{L}}\right] = \mathrm{E}\left[\frac{1}{\widetilde{L}}\right]\mathrm{E}[\widetilde{D}] = \mu_{1/\mathrm{L}}\mu_{\mathrm{D}}, \tag{9}$$

for the variance

$$\begin{aligned} \sigma_{\mathrm{n}}^2 &= \mathrm{E}\left[\left(\frac{\widetilde{D}}{\widetilde{L}} - \mu_{\frac{\mathrm{D}}{\mathrm{L}}}\right)^2\right] \\ &= \sigma_{1/\mathrm{L}}^2\sigma_{\mathrm{D}}^2 + \mu_{\mathrm{D}}^2\sigma_{1/\mathrm{L}}^2 + \mu_{1/\mathrm{L}}^2\sigma_{\mathrm{D}}^2. \end{aligned} \tag{10}$$

A Taylor-series expansion about the mean of the variable $\widetilde{L}$ helps approximating mean value and variance of the inverse.

3.2 Approximate mean and variance

Despite a desire for exact probabilistic expressions, it appears advisable to seek appropriate explicit relationships for the mean and the variance of the safety factor. Symbolic matrix notation collects the variables $\widetilde{D}$ and $\widetilde{L}$ in the random vector

$$\tilde{\mathbf{h}} = \{\widetilde{D}\ \ \widetilde{L}\} \quad \text{with mean} \quad \boldsymbol{\mu}_{\rm h} = \{\mu_{\rm D}\ \ \mu_{\rm L}\}. \quad (11)$$

The Taylor-series expansion of the safety factor $\tilde{n} = \widetilde{D}/\widetilde{L} = \tilde{n}(\tilde{\mathbf{h}})$ to the second order about the mean of the vector argument reads

$$\tilde{n}_2(\tilde{\mathbf{h}}) = n(\boldsymbol{\mu}_{\rm h}) + \left(\frac{{\rm d}\tilde{n}}{{\rm d}\tilde{\mathbf{h}}}\right)_\mu (\tilde{\mathbf{h}} - \boldsymbol{\mu}_{\rm h}) + \frac{1}{2}(\tilde{\mathbf{h}} - \boldsymbol{\mu}_{\rm h})^{\rm t} \left(\frac{{\rm d}^2\tilde{n}}{{\rm d}\tilde{\mathbf{h}}{\rm d}\tilde{\mathbf{h}}^{\rm t}}\right)_\mu (\tilde{\mathbf{h}} - \boldsymbol{\mu}_{\rm h}). \quad (12)$$

The derivatives entering eqn (12) are

$$\frac{{\rm d}\tilde{\rm n}}{{\rm d}\tilde{\mathbf{h}}} = \left[\frac{\partial\tilde{n}}{\partial\widetilde{D}}\ \frac{\partial\tilde{n}}{\partial\widetilde{L}}\right] = \frac{1}{\widetilde{L}}[1\ \ -\tilde{n}],$$
$$\frac{{\rm d}^2\tilde{n}}{{\rm d}\tilde{\mathbf{h}}{\rm d}\tilde{\mathbf{h}}^{\rm t}} = \frac{\rm d}{{\rm d}\tilde{\mathbf{h}}}\left(\frac{{\rm d}\tilde{n}}{{\rm d}\tilde{\mathbf{h}}}\right)^{\rm t} = \frac{1}{\widetilde{L}^2}\begin{bmatrix} 0 & -1 \\ -1 & 2\tilde{n} \end{bmatrix}. \quad (13)$$

Evaluation is for the mean values $\mu_{\rm D}$, $\mu_{\rm L}$.

The expectation of the expression in eqn (12) furnishes the mean value of the safety factor to the second order

$$(\mu_{\rm n})_2 = \frac{\mu_{\rm D}}{\mu_{\rm L}}\left(1 + \frac{\sigma_{\rm L}^2}{\mu_{\rm L}^2} - \frac{\sigma_{\rm LD}}{\mu_{\rm L}\mu_{\rm D}}\right). \quad (14)$$

The quotient $\mu_{\rm D}/\mu_{\rm L}$ refers to the zeroth term of the Taylor expansion, and $\sigma_{\rm LD} = \sigma_{\rm DL}$ stands for the covariance of the rate of work and the dissipation rate.

The first-order approximation of the variance of the safety factor from eqn (12) involves the covariance matrix $\boldsymbol{\Sigma}_{\rm h}$ of the argument variables

$$(\sigma_{\rm n}^2)_1 = \left(\frac{{\rm d}\tilde{n}}{{\rm d}\tilde{\mathbf{h}}}\right)_\mu \boldsymbol{\Sigma}_{\rm h} \left(\frac{{\rm d}\tilde{n}}{{\rm d}\tilde{\mathbf{h}}}\right)^{\rm t}_\mu,$$
$$\boldsymbol{\Sigma}_{\rm h} = \begin{bmatrix} \sigma_{\rm D}^2 & \sigma_{\rm DL} \\ \text{sym} & \sigma_{\rm L}^2 \end{bmatrix}. \quad (15)$$

Executing the matrix operations and rearranging

$$(\sigma_{\rm n}^2)_1 = \frac{\mu_{\rm D}^2}{\mu_{\rm L}^2}\left(\frac{\sigma_{\rm L}^2}{\mu_{\rm L}^2} - 2\frac{\sigma_{\rm LD}}{\mu_{\rm L}\mu_{\rm D}} + \frac{\sigma_{\rm D}^2}{\mu_{\rm D}^2}\right). \quad (16)$$

3.3 Sources of randomness

The simple expression of the safety factor n as the quotient of dissipation rate D and rate of work L suggested a first stochastic approach in this set. Eventually the randomness of the system arises from random input variables like loading, geometry and material parameters arranged in the vector array

$$\tilde{\boldsymbol{\alpha}} = \{\tilde{\alpha}_1\ \tilde{\alpha}_2 \cdots \tilde{\alpha}_q\}. \quad (17)$$

The dependence

$$\tilde{n}(\tilde{\boldsymbol{\alpha}}) = \frac{\widetilde{D}(\tilde{\boldsymbol{\alpha}})}{\widetilde{L}(\tilde{\boldsymbol{\alpha}})} = \tilde{n}[\tilde{\mathbf{h}}(\tilde{\boldsymbol{\alpha}})] \quad (18)$$

is case sensitive. The implicit appearance of the input variables $\tilde{\boldsymbol{\alpha}}$ last in eqn (18) should point on the relationship to the previous considerations in terms of $\tilde{\mathbf{h}} = \{\widetilde{D}\ \ \widetilde{L}\}$.

The Taylor-series expansion of $\tilde{n}(\tilde{\boldsymbol{\alpha}})$ about the mean $\boldsymbol{\mu}_\alpha = \{\mu_{\alpha_1}\ \mu_{\alpha_2} \cdots \mu_{\alpha_q}\}$ is

$$\tilde{n}(\tilde{\boldsymbol{\alpha}}) = \tilde{n}(\boldsymbol{\mu}_\alpha) + \left(\frac{{\rm d}\tilde{n}}{{\rm d}\tilde{\boldsymbol{\alpha}}}\right)_\mu (\tilde{\boldsymbol{\alpha}} - \boldsymbol{\mu}_\alpha) + \frac{1}{2}(\tilde{\boldsymbol{\alpha}} - \boldsymbol{\mu}_\alpha)^{\rm t}\left(\frac{{\rm d}^2\tilde{n}}{{\rm d}\tilde{\boldsymbol{\alpha}}{\rm d}\tilde{\boldsymbol{\alpha}}^{\rm t}}\right)_\mu (\tilde{\boldsymbol{\alpha}} - \boldsymbol{\mu}_\alpha). \quad (19)$$

With eqn (19) the second-order mean of the safety factor in terms of the input variables $\tilde{\boldsymbol{\alpha}}$ is substantiated to

$$(\mu_{\rm n})_2 = \tilde{n}(\boldsymbol{\mu}_\alpha) + \frac{1}{2}\sum_{i,j=1}^{q}\left(\frac{{\rm d}^2\tilde{n}}{{\rm d}\tilde{\alpha}_i{\rm d}\tilde{\alpha}_j}\right)_\mu \sigma_{\alpha_i\alpha_j}. \quad (20)$$

The first-order variance of $\tilde{n}$ is

$$(\sigma_{\rm n}^2)_1 = \left(\frac{{\rm d}\tilde{n}}{{\rm d}\tilde{\boldsymbol{\alpha}}}\right)_\mu \boldsymbol{\Sigma}_\alpha \left(\frac{{\rm d}\tilde{n}}{{\rm d}\tilde{\boldsymbol{\alpha}}}\right)^{\rm t}_\mu, \quad (21)$$

where $\boldsymbol{\Sigma}_\alpha = \{\sigma_{\alpha_i\alpha_j}\}$ denotes the covariance matrix of the random input variables.

Equivalent to the above straightforward approach in terms of the input variables, a two-step procedure first determines mean and variance/covariance of dissipation- and work rate because of $\tilde{\boldsymbol{\alpha}}$; then mean value and variance of $\tilde{n}$ are obtained via eqns (14) and (16).

4 Improving the Load Carrying Capacity – Robustness

The task concerns an adjustment of design variables to an improved load carrying capacity. The design variables $\tilde{\mathbf{z}} = \{\tilde{z}_1\,\tilde{z}_2 \cdots \tilde{z}_p\}$ and the non-adjustable input

$\tilde{\mathbf{y}} = \{\tilde{y}_1\,\tilde{y}_2 \cdots \tilde{y}_{q-p}\}$ are distinguished in the input vector

$$\tilde{\boldsymbol{\alpha}} = \{\tilde{\mathbf{y}}\ \tilde{\mathbf{z}}\}. \tag{22}$$

With loading and material fixed the design variables are geometric in nature; the task objective is the safety factor $\tilde{n}(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})$. It is requested that the mean values of the design variables are determined such that the mean value of the safety factor is maximized and the variance becomes a minimum for robustness against input scatter. In mathematical terms:

$$\begin{array}{rl} \text{find} & \boldsymbol{\mu}_{\mathrm{L}} \leq \boldsymbol{\mu}_{\mathrm{z}} \leq \boldsymbol{\mu}_{\mathrm{U}} \\ \text{minimizing} & \left[\begin{array}{c} -\mu_{\mathrm{n}}(\boldsymbol{\mu}_{\mathrm{z}}) \\ \sigma_{\mathrm{n}}(\boldsymbol{\mu}_{\mathrm{z}}) \end{array}\right] \\ \text{subject to} & \boldsymbol{\mu}_{\mathrm{c}}(\boldsymbol{\mu}_{\mathrm{z}}) + \boldsymbol{\zeta}\boldsymbol{\sigma}_{\mathrm{c}}(\boldsymbol{\mu}_{\mathrm{z}}) \leq 0. \end{array} \tag{23}$$

The design variables may be bounded in the mean by a lower limit $\boldsymbol{\mu}_{\mathrm{L}}$ and an upper limit $\boldsymbol{\mu}_{\mathrm{U}}$ due to technological restrictions. The constraint functions enter the minimization problem by the mean values in $\boldsymbol{\mu}_{\mathrm{c}}$ and standard deviations in $\boldsymbol{\sigma}_{\mathrm{c}}$. The coefficients in the diagonal matrix $\boldsymbol{\zeta}$ control the strictness of the constraint. The values specify the degree to which scatter is tolerated: the larger the value the closer the constraint is met under fluctuating conditions.

The definition of the optimum for the vector-valued objective in eqn (23) is not unique. Among various possibilities, a scalar substitute of the problem relies on the desirability function

$$G(\boldsymbol{\mu}_{\mathrm{Z}}, \xi) = -(1-\xi)\mu_{\mathrm{n}}(\boldsymbol{\mu}_{\mathrm{Z}}) + \xi\sigma_{\mathrm{n}}(\boldsymbol{\mu}_{\mathrm{Z}}), \quad 0 \leq \xi \leq 1. \tag{24}$$

This compromises the two requirements by the weighting factor ξ, and allows the utilization of standard optimization algorithms. The decision for a design may be based on a number of optimum solutions for a variety of ξ-values between the limits $\xi = 0$ and $\xi = 1$ appertaining to the maximum mean and to the minimum standard deviation of the safety factor.

The following is worth notice in connection with a first-order expansion of the safety factor about the mean input. This gives for the mean

$$\mu_{\mathrm{n1}} = n(\boldsymbol{\mu}_{\mathrm{y}}, \boldsymbol{\mu}_{\mathrm{z}}), \tag{25}$$

and for the variance

$$\sigma^2_{\mathrm{n1}} = \left[\frac{\partial \tilde{n}}{\partial \tilde{\mathbf{y}}}\,\frac{\partial \tilde{n}}{\partial \tilde{\mathbf{z}}}\right]_{\mu} \left[\begin{array}{cc} \boldsymbol{\Sigma}_{\mathrm{y}} & \boldsymbol{\Sigma}_{\mathrm{yz}} \\ \boldsymbol{\Sigma}_{\mathrm{zy}} & \boldsymbol{\Sigma}_{\mathrm{z}} \end{array}\right] \left[\frac{\partial \tilde{n}}{\partial \tilde{\mathbf{y}}}\,\frac{\partial \tilde{n}}{\partial \tilde{\mathbf{z}}}\right]^{\mathrm{t}}_{\mu}. \tag{26}$$

In order to be subject of improvement the output variance must depend on the design variables, a requirement that transfers to the derivatives in eqn (26).

The randomness of the safety factor can be caused by each one of the arguments. For instance if the design variables are deterministic, the other, random input gives rise to the variance

$$\sigma^2_{\mathrm{n1}} = \left(\frac{\partial \tilde{n}}{\partial \tilde{\mathbf{y}}}\right)_{\mu} \boldsymbol{\Sigma}_{\mathrm{y}} \left(\frac{\partial \tilde{n}}{\partial \tilde{\mathbf{y}}}\right)^{\mathrm{t}}_{\mu}, \tag{27}$$

As a rule random design variables contribute to the variance. However, an unconstrained extremum of the mean value of the safety factor in eqn (25) in the space of the design variables satisfies the condition

$$\left(\frac{\partial \tilde{n}}{\partial \tilde{\mathbf{z}}}\right)_{\mu} = \mathbf{0}, \tag{28}$$

which reduces eqn (26) for the variance back to eqn (27). At the extremum of the first-order mean the covariance matrix $\boldsymbol{\Sigma}_{\mathrm{z}}$ of the design variables is not effective, the contribution vanishes. If there is no other random input the first-order variance of the safety factor becomes zero:

$$(\sigma^2_{\mathrm{n1}}) = 0. \tag{29}$$

5 Probability of Failure

Reliability with respect to the plastic limit of the loaded structure equals the probability that the safe load multiplier is not less than unity: $n \geq 1$. The complementary measure is the probability of failure

$$\begin{aligned} P_{\mathrm{f}} &= P(\tilde{n} < 1) = P\left(\frac{\tilde{n} - \mu_{\mathrm{n}}}{\sigma_{\mathrm{n}}} < \frac{1 - \mu_{\mathrm{n}}}{\sigma_{\mathrm{n}}}\right) \\ &= P(\underline{\tilde{n}} < -\beta). \end{aligned} \tag{30}$$

The last expression implies testing of the standardized variate

$$\underline{\tilde{n}} = \frac{\tilde{n} - \mu_{\mathrm{n}}}{\sigma_{\mathrm{n}}}, \tag{31}$$

against the reliability index

$$\beta = \frac{\mu_{\mathrm{n}} - 1}{\sigma_{\mathrm{n}}}. \tag{32}$$

If a distribution function is available for the standardized variate $\underline{\tilde{n}}$, the failure probability can be obtained therefrom using $-\beta$ as argument. A log-normal distribution of $\tilde{n}$ with parameters μ and σ suggests the following reasoning

$$\begin{aligned} P_{\mathrm{f}} &= P(\tilde{n} < 1) = P(\ln \tilde{n} < 0) \\ &= P\left(\frac{\ln \tilde{n} - \mu}{\sigma} < -\frac{\mu}{\sigma}\right). \end{aligned} \tag{33}$$

In the above, the standardized variable $\underline{\tilde{n}} = (\ln \tilde{n} - \mu)/\sigma$ is normally distributed such that the failure

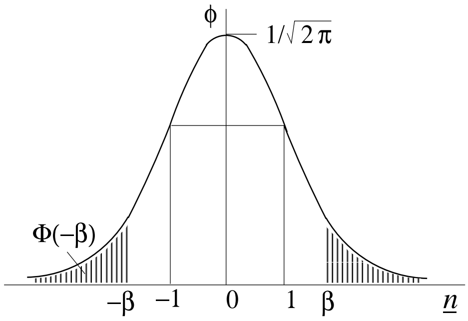


Figure 2: Failure and survival probability for standard normal distribution.

probability can be accessed from the standard normal distribution $\Phi(\underline{\tilde{n}})$ with the negative value of the quantity $\beta = \mu/\sigma$ as argument, (Fig 2).

In connection with the log-normal distribution mean and variance of $\tilde{n}$ are sufficient as resulting from the foregoing stochastic analysis. They enter the computation of the distribution parameters

$$\sigma^2 = \ln\left(\frac{\sigma_n^2}{\mu_n^2} + 1\right), \quad \mu = \ln\mu_n - \frac{\sigma^2}{2}. \tag{34}$$

More general, the failure probability is obtained with the probability density function for $\tilde{n}$, if available, as

$$P_f = \int_{n\le 1} f_n \mathrm{d}n = \int_{n(\boldsymbol{\alpha})\le 1} f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})\mathrm{d}\alpha_1\mathrm{d}\alpha_2\cdots\mathrm{d}\alpha_q. \tag{35}$$

Evaluation of the first integral in eqn (35) presumes knowledge of the probability density function $f_n(n)$ of the safety factor. Eventually, the failure probability is equated to the probability content of the basic random input $\tilde{\boldsymbol{\alpha}}$ in the unsafe region $n(\boldsymbol{\alpha}) \le 1$. The evaluation of the second integral in eqn (35) relies on the joined probability density function $f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$.

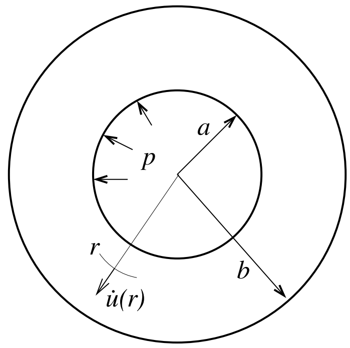
6 Application of Stochastic Analysis

6.1 Definition of problem

The thick-walled circular cylinder in Fig. 3 is subjected to internal pressure and deforms under the condition of plane strain [2]. The yield mechanism for the axial symmetric case in the context of perfect plasticity is specified by the radial velocity

$$\dot{u}(r) = \frac{a}{r}\dot{u}_a \tag{36}$$

where $\dot{u}_a = \dot{u}(r = a)$ denotes the velocity at the inner radius. Therefrom the associated plastic flow with



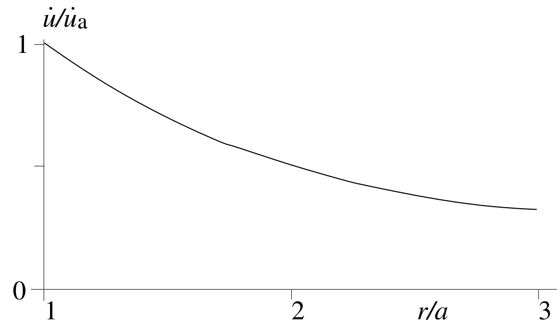


Figure 3: Thick-walled cylinder. Problem definition and distribution of yield velocity.

radial strain rate component $\dot{\eta}_r$ and tangential component $\dot{\eta}_t$, the sum vanishing due to the isochoric condition. The equivalent strain rate is the scalar quantity

$$\dot{\bar{\eta}} = \sqrt{\frac{2}{3}(\dot{\eta}_r^2 + \dot{\eta}_t^2)} = \frac{2}{\sqrt{3}}\frac{a}{r}\frac{\dot{u}_a}{r}. \tag{37}$$

In deviatoric plasticity the work equivalence

$$\boldsymbol{\sigma}^t\dot{\boldsymbol{\eta}} = \bar{\sigma}\dot{\bar{\eta}} = \sqrt{3}\tau\dot{\bar{\eta}}, \tag{38}$$

is used in order to introduce the yield stress τ of the material in shear in the expression for the dissipation rate. The first equality is formal while the second one involves the material parameter. Assuming a unique yield stress throughout the material, the dissipation rate for the unit length of the cylinder follows to

$$D = \int_V \sqrt{3}\tau\dot{\bar{\eta}}\mathrm{d}V = \sqrt{3}\tau\int_a^b \dot{\bar{\eta}}(2\pi r\mathrm{d}r) = 2\pi a\dot{u}_a 2\tau\ln\frac{b}{a}. \tag{39}$$

The rate of work of the pressure on the yield velocity is

$$L = 2\pi a p\dot{u}_a. \tag{40}$$

The safety factor becomes

$$n = \frac{D}{L} = 2\frac{\tau}{p}\ln\frac{b}{a}. \tag{41}$$

6.2 Significance of random input

Apart from yield stress and pressure, inner radius a and outer radius b of the cylinder may add to the randomness. The random input is collected in the vector array

$$\tilde{\boldsymbol{\alpha}} = \{\tilde{\tau}\ \tilde{p}\ \tilde{a}\ \ \tilde{b}\}, \tag{42}$$

with mean

$$\boldsymbol{\mu}_\alpha = \{\mu_\tau\ \mu_{\rm p}\ \mu_{\rm a}\ \mu_{\rm b}\}, \tag{43}$$

and covariance matrix

$$\boldsymbol{\Sigma}_\alpha = \begin{bmatrix} \sigma_\tau^2 & 0 & 0 & 0 \\ & \sigma_{\rm p}^2 & \sigma_{\rm pa} & \sigma_{\rm pb} \\ \text{sym} & & \sigma_{\rm a}^2 & \sigma_{\rm ab} \\ & & & \sigma_{\rm b}^2 \end{bmatrix}. \tag{44}$$

The covariance matrix reflects an assumed statistical independence between the material yield stress τ and the other input as from loading and geometry.

The derivatives entering the Taylor series expansion in eqn (19) are,

$$\left(\frac{{\rm d}\tilde{n}}{{\rm d}\boldsymbol{\alpha}}\right)_\mu = \left[\frac{\tilde{n}}{\tilde{\tau}}\ -\frac{\tilde{n}}{\tilde{p}}\ -\frac{2\tilde{\tau}}{\tilde{p}\tilde{a}}\ \ \frac{2\tilde{\tau}}{\tilde{p}\tilde{b}}\right]_\mu, \tag{45}$$

and

$$\left(\frac{{\rm d}^2\tilde{n}}{{\rm d}\tilde{\boldsymbol{\alpha}}{\rm d}\tilde{\boldsymbol{\alpha}}^{\rm t}}\right)_\mu = \begin{bmatrix} 0 & -\frac{\tilde{n}}{\tilde{\tau}\tilde{p}} & -\frac{2}{\tilde{p}\tilde{a}} & \frac{2}{\tilde{p}\tilde{b}} \\ & \frac{2\tilde{n}}{\tilde{p}^2} & \frac{2\tilde{\tau}}{\tilde{p}^2\tilde{a}} & -\frac{2\tilde{\tau}}{\tilde{p}^2\tilde{b}} \\ \text{sym} & & \frac{2\tilde{\tau}}{\tilde{p}\tilde{a}^2} & 0 \\ & & & -\frac{2\tilde{\tau}}{\tilde{p}\tilde{b}^2} \end{bmatrix}_\mu, \tag{46}$$

evaluated at the mean values of the input variables. The second-order mean of the safety factor is deduced as

$$\begin{aligned} (\mu_n)_2 \ = \ & \\ n_\mu \left[1+\left(\frac{\sigma_{\rm p}}{\mu_{\rm p}}\right)^2\right] \ &+ \ \frac{\mu_\tau}{\mu_{\rm p}}\left[\left(\frac{\sigma_{\rm b}}{\mu_{\rm b}}\right)^2-\left(\frac{\sigma_{\rm a}}{\mu_{\rm a}}\right)^2\right] \\ &+ \ 2\frac{\mu_\tau}{\mu_{\rm p}}\left(\frac{\sigma_{\rm pb}}{\mu_{\rm p}\mu_{\rm b}}-\frac{\sigma_{\rm pa}}{\mu_{\rm p}\mu_{\rm a}}\right), \end{aligned} \tag{47}$$

where $n_\mu = \tilde{n}(\boldsymbol{\mu}_\alpha)$, the material yield stress still is considered an independent variable, and with the third term in eqn (47) vanishing in the case where all variables are uncorrelated. The first-order approximate of the variance of the safety factor, eqn (21), becomes

$$\begin{aligned} (\sigma_{\rm n})_1^2 \ = \ & n_\mu^2\left[\left(\frac{\sigma_\tau}{\mu_\tau}\right)^2+\left(\frac{\sigma_{\rm p}}{\mu_{\rm p}}\right)^2\right] \\ &+ \ \left(\frac{2\mu_\tau}{\mu_{\rm p}}\right)^2\left[\left(\frac{\sigma_{\rm a}}{\mu_{\rm a}}\right)^2+\left(\frac{\sigma_{\rm b}}{\mu_{\rm b}}\right)^2\right], \end{aligned} \tag{48}$$

which refers to uncorrelated variables.

The previous procedure operating with eqn (14) and eqn (16) in terms of dissipation rate and rate of work implies a preliminary step computing the mean value and the variance of the functions $\widetilde{L}(\tilde{\boldsymbol{\alpha}})$, eqn (40), and $\widetilde{D}(\tilde{\boldsymbol{\alpha}})$, eqn (39), as a well as the covariance of the two quantities. The two-step procedure is completed by the evaluation of eqn (14) for the mean of $\tilde{n}$ and of eqn (16) for the variance. Reproduction of eqn (47) and eqn (48) confirms the equivalence to the direct approach when using first-order means and variances of $\widetilde{D}(\tilde{\boldsymbol{\alpha}})$ and $\widetilde{L}(\tilde{\boldsymbol{\alpha}})$.

6.3 Remarks on the input covariance matrix

Given the variances $\sigma_\tau^2, \sigma_{\rm p}^2, \sigma_{\rm a}^2, \sigma_{\rm b}^2$ of the individual variables the interest is in the covariances. In the absence of other information the following discusses the covariance of variables as from virtual functional dependencies. In a first approach fluctuations in the yield stress $\tilde{\tau}$ of the material are assumed independent of those of geometry and loading such that

$$\sigma_{\tau{\rm p}} = \sigma_{\tau{\rm a}} = \sigma_{\tau{\rm b}} = 0. \tag{49}$$

Inner radius $\tilde{a}$ and outer radius $\tilde{b}$ may be related by the condition of constant cross-section area:

$$\pi(\tilde{b}^2-\tilde{a}^2) = \text{constant} \Rightarrow \frac{{\rm d}\tilde{b}}{{\rm d}\tilde{a}} = \frac{\tilde{a}}{\tilde{b}}. \tag{50}$$

This implies a dependence of the variances and introduces covariance. To the first order for the variances,

$$\sigma_{\rm b}^2 = {\rm E}[(\tilde{b}-\mu_{\rm b})^2] = \left(\frac{\mu_{\rm a}}{\mu_{\rm b}}\right)^2\sigma_{\rm a}^2. \tag{51}$$

For the covariance,

$$\sigma_{\rm ab} = {\rm E}[(\tilde{a}-\mu_{\rm a})(\tilde{b}-\mu_{\rm b})] = \frac{\mu_{\rm a}}{\mu_{\rm b}}\sigma_{\rm a}^2. \tag{52}$$

Constancy of the applied force, if required, adjusts the pressure p to the inner radius a:

$$2\pi\tilde{a}\tilde{p} = \text{constant} \Rightarrow \frac{{\rm d}\tilde{p}}{{\rm d}\tilde{a}} = -\frac{\tilde{p}}{\tilde{a}}. \tag{53}$$

This relates the variances by

$$\sigma_{\rm p}^2 = {\rm E}[(p-\mu_{\rm p})^2] = \left(\frac{\mu_{\rm p}}{\mu_{\rm a}}\right)^2\sigma_{\rm a}^2, \tag{54}$$

and implies the covariance

$$\sigma_{\rm pa} = {\rm E}[(\tilde{p}-\mu_{\rm p})(\tilde{a}-\mu_{\rm a})] = -\frac{\mu_{\rm p}}{\mu_{\rm a}}\sigma_{\rm a}^2. \tag{55}$$

The covariance with the outer radius is

$$\sigma_{\rm pb} = {\rm E}[(\tilde{p} - \mu_{\rm p})(\tilde{b} - \mu_{\rm b})] \\ = -\frac{\mu_{\rm p}}{\mu_{\rm a}}\sigma_{\rm ab} = -\frac{\mu_{\rm p}}{\mu_{\rm b}}\sigma_{\rm a}^2. \quad (56)$$

For the aforementioned conditions the covariance matrix of the input variables, eqn (44), becomes

$$\mathbf{\Sigma}_\alpha = \sigma_{\rm a}^2 \begin{bmatrix} \frac{\sigma_\tau^2}{\sigma_{\rm a}^2} & 0 & 0 & 0 \\ & \left(\frac{\mu_{\rm p}}{\mu_{\rm a}}\right)^2 & -\frac{\mu_{\rm p}}{\mu_{\rm a}} & -\frac{\mu_{\rm p}}{\mu_{\rm b}} \\ & & 1 & \frac{\mu_{\rm a}}{\mu_{\rm b}} \\ & & & \left(\frac{\mu_{\rm a}}{\mu_{\rm b}}\right)^2 \end{bmatrix} \quad (57)$$

This accounts for a constant cross-section area in addition to the constancy of the force resulting from the applied pressure.

6.4 Yield stress as a random field

The yield stress may vary with the position within the material as a random field. The expression for the dissipation rate must account for this variation in space

$$\widetilde{D} = \int\limits_V \sqrt{3}\tilde{\tau}\dot{\overline{\eta}}{\rm d}V. \quad (58)$$

The expectation gives the mean value

$$\mu_{\rm D} = {\rm E}\left[\int\limits_V \sqrt{3}\tilde{\tau}\dot{\overline{\eta}}{\rm d}V\right] \\ = \int\limits_V \sqrt{3}\mu_\tau\dot{\overline{\eta}}{\rm d}V = \sqrt{3}\mu_\tau \int\limits_V \dot{\overline{\eta}}{\rm d}V. \quad (59)$$

The last, simplified form is applicable if the random field is homogeneous in the mean. In addition, the mean yield stress of an ergodic field equals the average within a single sample.

Regarding the variance of the dissipation rate the difference $(\widetilde{D} - \mu_{\rm D})$ is squared

$$(\widetilde{D} - \mu_{\rm D})^2 = \\ \int\limits_V \sqrt{3}(\tilde{\tau} - \mu_\tau)(\mathbf{x})\dot{\overline{\eta}}(\mathbf{x}){\rm d}V \\ \times \int\limits_V \sqrt{3}(\tilde{\tau} - \mu_\tau)(\mathbf{x}')\dot{\overline{\eta}}(\mathbf{x}'){\rm d}V', \quad (60)$$

where $\mathbf{x}$, $\mathbf{x}'$ denote individual positions within the material volume. The expectation defines the variance of the dissipation rate

$$\sigma_{\rm D}^2 = \int\int 3\sigma_\tau(\mathbf{x}, \mathbf{x}')\dot{\overline{\eta}}(\mathbf{x})\dot{\overline{\eta}}(\mathbf{x}'){\rm d}V{\rm d}V'. \quad (61)$$

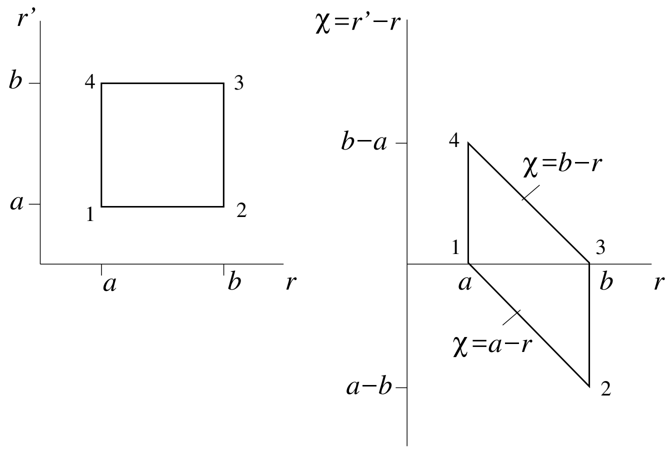


Figure 4: Change of variables.

The integrand involves the auto-covariance of the random field $\tilde{\tau}(\mathbf{x})$

$$\sigma_\tau(\mathbf{x}, \mathbf{x}') = {\rm E}[(\tilde{\tau} - \mu_\tau)(\mathbf{x})(\tilde{\tau} - \mu_\tau)(\mathbf{x}')]. \quad (62)$$

In case where the plastic flow field is also random the above formalism applies to the product $(\widetilde{\tau\dot{\overline{\eta}}})$.

Next the yield stress of the cylinder material is considered a random function of the radius, all other quantities deterministic. In this case the dissipation rate for the unit length is not expressed by eqn (39); it is stated as

$$\widetilde{D} = \int\limits_a^b \sqrt{3}\tilde{\tau}(r)\dot{\overline{\eta}}(r)2\pi r{\rm d}r = 2\pi a\dot{u}_a \int\limits_a^b 2\tilde{\tau}(r)\frac{{\rm d}r}{r}. \quad (63)$$

The mean value is

$$\mu_{\rm D} = 2\pi a\dot{u}_a \int\limits_a^b 2\mu_\tau(r)\frac{{\rm d}r}{r} = 2\pi a\dot{u}_a 2\mu_\tau \ln\frac{b}{a}. \quad (64)$$

The last expression, valid for a homogeneous field, is as for a random yield stress constant throughout the material. Also eqn (61) is interpreted for a homogeneous field where the auto-covariance depends on the distance between points, not on the distinct positions. Then the variance

$$\sigma_{\rm D}^2 = \int\int\limits_a^b 3\sigma_\tau(r, r')\dot{\overline{\eta}}(r)\dot{\overline{\eta}}(r')2\pi r{\rm d}r2\pi r'{\rm d}r' \\ = (4\pi a\dot{u}_a)^2 \int\int\limits_a^b \frac{\sigma_\tau(r - r')}{rr'}{\rm d}r{\rm d}r'. \quad (65)$$

This suggests employment of the variables [5]

$$r \quad \text{and} \quad \chi = r' - r, \quad (66)$$

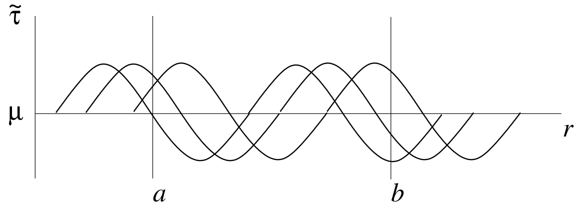


Figure 5: Harmonic variation of yield stress with random phase.

such that the variance of the dissipation rate for a homogeneous random field $\tilde{\tau}(r)$ becomes (see Fig. 4),

$$\sigma_{\rm D}^2 = (4\pi a\dot{u}_a)^2 \left\{ \int\limits_{a-b}^{0} \left[\int\limits_{a-\chi}^{b} \frac{\sigma_\tau(\chi)}{r^2+\chi r} {\rm d}r \right] {\rm d}\chi + \int\limits_{0}^{b-a} \left[\int\limits_{a}^{b-\chi} \frac{\sigma_\tau(\chi)}{r^2+\chi r} {\rm d}r \right] {\rm d}\chi \right\}. \quad (67)$$

Evaluation of the inner integral leaves,

$$\sigma_{\rm D}^2 = (4\pi a\dot{u}_a)^2 \left[\int\limits_{a-b}^{0} \frac{\sigma_\tau(\chi)}{\chi} \ln \frac{ab}{(a-\chi)(b+\chi)} {\rm d}\chi + \int\limits_{0}^{b-a} \frac{\sigma_\tau(\chi)}{\chi} \ln \frac{(a+\chi)(b-\chi)}{ab} {\rm d}\chi \right]. \quad (68)$$

The harmonic wave serves as an example

$$\tilde{\tau}(r) = \tau_0 + A\sin(\omega r + \tilde{\varphi}), \quad (69)$$

with τ_0, A, ω fixed while $\tilde{\varphi}$ is uniformly random in the interval $[-\pi,\, \pi]$, (Fig. 5). The mean value $\mu_\tau = \tau_0$ and the variance $\sigma_\tau^2 = A^2/2$ do not vary along the radius. The auto-covariance is a function of the distance between positions along the radius:

$$\begin{aligned} \sigma_\tau(r, r') &= {\rm E}[A\sin(\omega r + \tilde{\varphi})A\sin(\omega r' + \tilde{\varphi})] \\ &= \int\limits_{-\pi}^{\pi} A^2 \sin(\omega r + \varphi)\sin(\omega r' + \varphi) \frac{1}{2\pi} {\rm d}\varphi \\ &= \frac{A^2}{2}\cos\omega(r - r') = \sigma_\tau(r' - r). \quad (70) \end{aligned}$$

Use of the above result with $r' - r = \chi$ in eqn (68) returns the integral to be evaluated for the variance of the dissipation rate in this particular case.

6.5 Optimum inner radius

The optimization of the safety factor will be investigated for the thick-walled cylinder in Fig. 3; first- and second-order approximation are contrasted. The inner radius is the single random design variable all other input fixed. The applied pressure is assumed adjusted to the variation of the radius such that the product $(ap) = {\rm constant}$. The safety factor of eqn (41) then is replaced by the form

$$\tilde{n}(\tilde{a}) = \frac{2\tau}{(ap)} \tilde{a} \ln \frac{b}{\tilde{a}}. \quad (71)$$

This is a function of the inner radius $\tilde{a}$, the design variable. The first-order expansion of $\tilde{n}(\tilde{a})$ about the mean value of the argument gives

$$\mu_{\rm n1} = \tilde{n}(\mu_{\rm a}) = \frac{2\tau}{(ap)} \left(\ln \frac{b}{\mu_{\rm a}} \right) \mu_{\rm a} \quad (72)$$

for the mean, and the variance

$$\sigma_{\rm n1}^2 = \left(\frac{{\rm d}\tilde{n}}{{\rm d}\tilde{a}} \right)_\mu^2 \sigma_{\rm a}^2 = \left[\frac{2\tau}{(ap)} \left(\ln \frac{b}{\mu_{\rm a}} - 1 \right) \right]^2 \sigma_{\rm a}^2. \quad (73)$$

The mean assumes an extremum for

$$\ln \frac{b}{\mu_{\rm a}} = 1, \quad (74)$$

which in eqn (72) specifies the maximum safe load multiplier

$$\max(\mu_{\rm n1}) = \frac{2\tau}{(ap)} \mu_{\rm a}, \quad (75)$$

with $\mu_{\rm a}$ from eqn (74). At the same time the variance in eqn (73) is seen to vanish and so the standard deviation

$$(\sigma_{\rm n1})_{\max} = 0. \quad (76)$$

The second-order approximation of the mean is

$$\begin{aligned} \mu_{\rm n2} &= \mu_{\rm n1} + \frac{1}{2} \left(\frac{{\rm d}^2\tilde{n}}{{\rm d}\tilde{a}^2} \right)_\mu \sigma_{\rm a}^2 \\ &= \frac{2\tau}{(ap)} \left(\ln \frac{b}{\mu_{\rm a}} - \frac{1}{2} \frac{\sigma_{\rm a}^2}{\mu_{\rm a}^2} \right) \mu_{\rm a}. \quad (77) \end{aligned}$$

An extremum in conjunction with $\sigma_{\rm a}^2 = {\rm constant}$ is obtained for

$$\ln \frac{b}{\mu_{\rm a}} = 1 - \frac{1}{2} \frac{\sigma_{\rm a}^2}{\mu_{\rm a}^2}. \quad (78)$$

Accounting for in eqn (77) now gives the maximum mean of the safe load multiplier as

$$\max(\mu_{\rm n2}) = \frac{2\tau}{(ap)} \left(1 - \frac{\sigma_{\rm a}^2}{\mu_{\rm a}^2} \right) \mu_{\rm a}, \quad (79)$$

with the mean radius from eqn (78). The associated first-order variance from eqn (73) determines the standard deviation

$$\sigma_{\mathrm{n}1}|_{\max 2} = \frac{2\tau}{(ap)} \frac{1}{2} \frac{\sigma_{\mathrm{a}}^2}{\mu_{\mathrm{a}}^2} \sigma_{\mathrm{a}}. \tag{80}$$

Instead of the variance of the inner radius, the design variable, next the standard deviation is kept constant: $\sigma_{\mathrm{a}}/\mu_{\mathrm{a}} =$ constant. In this case the mean safety factor of eqn (77) becomes a maximum for the mean inner radius from

$$\ln \frac{b}{\mu_{\mathrm{a}}} = 1 + \frac{1}{2} \frac{\sigma_{\mathrm{a}}^2}{\mu_{\mathrm{a}}^2}. \tag{81}$$

The appertaining mean safety factor from eqn (77) formally is as in eqn (72) from the first-order study but the mean inner radii differ. The variance from eqn (73) is as for $\sigma_{\mathrm{a}}^2 =$ constant; eqn (80) for the standard deviation is still applicable. With this the coefficient of variation of the safety factor at maximum second-order mean

$$\left(\frac{\sigma_{\mathrm{n}1}}{\mu_{\mathrm{n}2}}\right)_{\max} = \frac{1}{2} \left(\frac{\sigma_{\mathrm{a}}}{\mu_{\mathrm{a}}}\right)^3, \tag{82}$$

is seen to assume a much smaller value than the input $\sigma_{\mathrm{a}}/\mu_{\mathrm{a}}$.

Summarizing, mean optimization with the first-order expansion of the safety factor makes the variance vanish; a result that previously has been stated in more general terms. The second-order expansion along with $\sigma_{\mathrm{a}}^2 =$ constant associates a diminished optimum to a higher inner radius. For $\sigma_{\mathrm{a}}/\mu_{\mathrm{a}} =$ constant the maximum mean safety factor is formally as for the first-order but the appertaining inner radius lower. However, the differences are small, second-order the coefficient of variation of the inner radius, the design variable. The variance at the second-order optimum is not zero: for either of the above constraints the standard deviation is by the squared coefficient of variation proportional to that of the inner radius.

6.6 Assessment of the failure probability

In eqn (35) the first and the last integral indicate two different approaches to the determination of the failure probability. They will be elucidated for the hollow cylinder problem whose safety factor is specified in eqn (41). For the purpose of demonstration the quotient of the outer to the inner radius is considered as a single random input variable: $\tilde{\alpha} = \widetilde{(b/a)}$. From eqn (41),

$$\alpha = \frac{b}{a} = \exp \frac{pn}{2\tau}, \tag{83}$$

which limits the region of unsafe input $0 \leq \tilde{n} \leq 1$ to

$$1 \leq \tilde{\alpha} \leq \exp \frac{p}{2\tau}. \tag{84}$$

If a probability density function $f_\alpha(\alpha)$ is available, the last integral in eqn (35) furnishes the failure probability as

$$P_{\mathrm{f}} = \int\limits_{n(\alpha)\leq 1} f_\alpha(\alpha)\mathrm{d}\alpha = \int\limits_{\alpha=1}^{\exp \frac{p}{2\tau}} f_\alpha(\alpha)\mathrm{d}\alpha. \tag{85}$$

Evaluation may be analytical or numerical by Monte Carlo integration techniques.

Alternatively, determination of the failure probability by the first integral in eqn (35) presumes knowledge of the probability density function of the safety factor $\tilde{n}$, an evenly increasing function of the input $\tilde{\alpha}$. It is inferred that F_{n} and $F_\alpha[\alpha(n)]$ of the two quantities are equal. This gives the probability density function of n as

$$\begin{aligned} f_{\mathrm{n}}(n) = \frac{\mathrm{d}F_{\mathrm{n}}(n)}{\mathrm{d}n} &= \frac{\mathrm{d}F_\alpha[\alpha(n)]}{\mathrm{d}\alpha} \frac{\mathrm{d}\alpha(n)}{\mathrm{d}n} \\ &= f_\alpha[\alpha(n)] \frac{\mathrm{d}\alpha(n)}{\mathrm{d}n}. \end{aligned} \tag{86}$$

Specifying on account of eqn (83),

$$f_{\mathrm{n}}(n) = f_\alpha \left(\exp \frac{pn}{2\tau}\right) \frac{p}{2\tau} \exp \frac{pn}{2\tau}. \tag{87}$$

The failure probability follows to

$$\begin{aligned} P_{\mathrm{f}} &= \int\limits_{n\leq 1} f_{\mathrm{n}}(n)\mathrm{d}n \qquad (88) \\ &= \frac{p}{2\tau} \int\limits_{n=0}^{1} \left(\exp \frac{pn}{2\tau}\right) f_\alpha \left(\exp \frac{pn}{2\tau}\right) \mathrm{d}n, \end{aligned}$$

which implies nothing but a change of variables with respect to the integral in eqn (85).

The determination of the failure probability is simple in the case of a log-normal input variable $\tilde{\alpha}$. From $P(\tilde{n} < 1) = P[\tilde{\alpha}(\tilde{n}) < \tilde{\alpha}(1)]$,

$$\begin{aligned} P_{\mathrm{f}} &= P(\tilde{n} < 1) \\ &= P\left[\frac{\ln \tilde{\alpha}(\tilde{n}) - \mu}{\sigma} < \frac{\ln \tilde{\alpha}(1) - \mu}{\sigma}\right]. \end{aligned} \tag{89}$$

The standardization is with the parameters μ and σ of the log-normal distribution. These are related to the mean and the variance of the original variable $\tilde{\alpha}$ analogously to eqn (34). The failure probability now

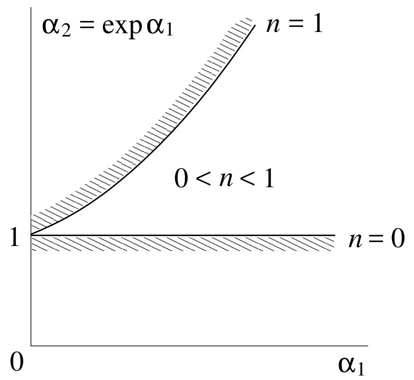


Figure 6: Two-dimensional unsafe region.

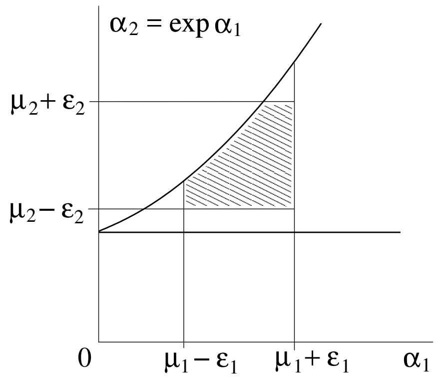


Figure 7: Truncated domain of integration.

can be accessed from the standard normal distribution as $P_\mathrm{f} = \Phi(-\beta)$ with the negative value of the quantity

$$\beta = \frac{\mu - \ln \tilde{\alpha}(1)}{\sigma} = \frac{1}{\sigma}\left(\mu - \frac{p}{2\tau}\right) \qquad (90)$$

as argument.

6.7 Two-dimensional input space

A two-dimensional input space for the hollow cylinder will be defined by the random variables

$$\tilde{\alpha}_1 = \widetilde{\left(\frac{p}{2\tau}\right)}, \quad \tilde{\alpha}_2 = \widetilde{\left(\frac{b}{a}\right)}. \qquad (91)$$

One refers to the magnitude of the loading, the other to the geometry of the cross-section. In terms of the above variables eqn (83) assumes the form

$$\tilde{\alpha}_2 = \exp \tilde{\alpha}_1 \tilde{n}. \qquad (92)$$

The input is unsafe for $0 \leq n \leq 1$ within the region

$$0 \leq \tilde{\alpha}_1 \leq \infty, \quad 1 \leq \tilde{\alpha}_2 \leq \exp \alpha_1, \qquad (93)$$

indicated in Fig. 6.

Implementation in eqn (35) gives the failure probability for the actual case

$$P_\mathrm{f} \int_{\alpha_1=0}^{\infty} \int_{\alpha_2=1}^{\exp \alpha_1} f_{\alpha_1,\alpha_2}(\alpha_1, \alpha_2) \mathrm{d}\alpha_2 \mathrm{d}\alpha_1 =$$
$$\int_{\alpha_1=0}^{\infty} f_{\alpha_1}(\alpha_1) \left(\int_{\alpha_2=1}^{\exp \alpha_1} f_{\alpha_2}(\alpha_2) \mathrm{d}\alpha_2 \right) \mathrm{d}\alpha_1. \qquad (94)$$

The last expression refers to statistically independent input variates $\tilde{\alpha}_1, \tilde{\alpha}_2$ such that the joined probability density function is composed of the individual ones by multiplication: $f_{\alpha_1,\alpha_2}(\alpha_1, \alpha_2) = f_{\alpha_1}(\alpha_1) f_{\alpha_2}(\alpha_2)$.

Evaluation of eqn (94) for independent uniform distributions in the interval

$$0 \leq \tilde{\alpha}_1 \leq 1, \quad 1 \leq \tilde{\alpha}_2 \leq e, \qquad (95)$$

with $f_{\alpha_1} = 1, f_{\alpha_2} = 1/(e-1)$ for the probability densities determines the failure probability

$$\begin{aligned} P_\mathrm{f} &= \int_0^1 \left(\int_1^{\exp \alpha_1} \frac{\mathrm{d}\alpha_2}{e-1} \right) \mathrm{d}\alpha_1 \\ &= \int_0^1 \frac{e^{\alpha_1} - 1}{e-1} \mathrm{d}\alpha_1 = \frac{e-2}{e-1} = 0.418. \end{aligned} \qquad (96)$$

More realistic constellations are defined by mean value μ and tolerance ε of the independent uniform variates:

$$\begin{aligned} \mu_1 - \varepsilon_1 \leq \tilde{\alpha}_1 \leq \mu_1 + \varepsilon_1, \\ \mu_2 - \varepsilon_2 \leq \tilde{\alpha}_2 \leq \mu_2 + \varepsilon_2. \end{aligned} \qquad (97)$$

For convenience the interval limits are abbreviated as

$$\begin{aligned} \underline{\varepsilon}_1 = \mu_1 - \varepsilon_1, \ \overline{\varepsilon}_1 = \mu_1 + \varepsilon_1 \\ \underline{\varepsilon}_2 = \mu_2 - \varepsilon_2, \ \overline{\varepsilon}_2 = \mu_2 + \varepsilon_2. \end{aligned} \qquad (98)$$

With reference to Fig. 7, if the domain of integration is entirely within the unsafe region $(n < 1)$ one confirms for the failure probability

$$P_\mathrm{f} = \int_{\underline{\varepsilon}_1}^{\overline{\varepsilon}_1} \frac{\mathrm{d}\alpha_1}{2\varepsilon_1} \int_{\underline{\varepsilon}_2}^{\overline{\varepsilon}_2} \frac{\mathrm{d}\alpha_2}{2\varepsilon_2} = 1. \qquad (99)$$

Analogously the probability of survival being $P_\mathrm{s} = 1$ in case that the variation of the input does not exceed the safe region $(n > 1)$.

In case that only part of the input is within the unsafe region the determination of the failure probability demands more attention. Defining the quantities

$$\begin{aligned} \underline{\Delta} &= \frac{\ln \underline{\varepsilon}_2 - \underline{\varepsilon}_1 + |\ln \underline{\varepsilon}_2 - \underline{\varepsilon}_1|}{2}, \\ \overline{\Delta} &= \frac{\overline{\varepsilon}_1 - \ln \overline{\varepsilon}_2 + |\overline{\varepsilon}_1 - \ln \overline{\varepsilon}_2|}{2}, \end{aligned} \qquad (100)$$

the integration is executed in accordance to

$$P_\mathrm{f} = \left[\int_{\underline{\varepsilon}_1 + \underline{\Delta}}^{\overline{\varepsilon}_1 - \overline{\Delta}} \left(\int_{\underline{\varepsilon}_2}^{\exp \alpha_1} \frac{\mathrm{d}\alpha_2}{2\varepsilon_2} \right) \frac{\mathrm{d}\alpha_1}{2\varepsilon_1} \right] + 2\varepsilon_2 \overline{\Delta}. \qquad (101)$$

Evaluation of the above expression with the numbers

$$\begin{aligned} &\tilde{\alpha}_1: \ \mu_1 = 0.5, \ \underline{\varepsilon}_1 = 0.45, \ \overline{\varepsilon}_1 = 0.55, \ 2\varepsilon_1 = 0.1, \\ &\tilde{\alpha}_2: \ \mu_2 = 1.5, \ \underline{\varepsilon}_2 = 1.35, \ \overline{\varepsilon}_2 = 1.65, \ 2\varepsilon_2 = 0.3, \end{aligned}$$

gives $\underline{\Delta} = 0, \ \overline{\Delta} = 0.55 - \ln 1.65$, and

$$\begin{aligned} P_\mathrm{f} &= \int_{0.45}^{\ln 1.65} \left(\int_{1.35}^{\exp \alpha_1} \frac{\mathrm{d}\alpha_2}{0.3} \right) \frac{\mathrm{d}\alpha_1}{0.1} + 0.3(0.55 - \ln 1.65) \\ &= \frac{1}{0.03} \int_{0.45}^{\ln 1.65} (e^{\alpha_1} - 1.35)\mathrm{d}\alpha_1 + 0.3(0.55 - \ln 1.65) \\ &= \frac{1.65 - e^{0.45} - 1.35(\ln 1.65 - 0.45)}{0.03} \\ &\quad + 0.3(0.55 - \ln 1.65) \\ &= 0.445. \end{aligned}$$

The probability of survival is $P_\mathrm{s} = 1 - P_\mathrm{f} = 0.555$.

7 Structural Assemblies

The following considers structures assembled of a number of individual components. Given the randomness of the components, the properties of the assembly are to be assessed. For the sake of transparency the issue is referred to truss structures assembled of bar members.

7.1 Randomness of the system

The dissipation rate at the plastic limit of a truss structure consisting of K bar members is

$$\widetilde{D} = \boldsymbol{\delta}^\mathrm{t} \widetilde{\mathbf{S}}. \qquad (102)$$

The vector array $\widetilde{\mathbf{S}} = \{\widetilde{S}_j\}$ comprises the random yield forces $\widetilde{S}_j > 0, \ j = 1 \cdots K$ of the individual bars. The vector array

$$\boldsymbol{\delta} = |\mathbf{a}\dot{\mathbf{u}}| \qquad (103)$$

defines the magnitude $\delta_j \geq 0$ of the elongation rate of the members as from the N nodal velocities $\dot{\mathbf{u}}$ at yield.

It is assumed that the randomness of the members is such that the deformation mechanism is not affected. Then the mean value of the dissipation rate in eqn (102) is

$$\mu_\mathrm{D} = \mathrm{E}[\boldsymbol{\delta}^\mathrm{t} \tilde{\mathbf{S}}] = \boldsymbol{\delta}^\mathrm{t} \boldsymbol{\mu}_\mathrm{S}, \qquad (104)$$

where the vector array $\boldsymbol{\mu}_\mathrm{S}$ comprises the mean values of the K yield forces. The variance of the dissipation rate requires knowledge of the covariance matrix $\boldsymbol{\Sigma}_\mathrm{S}$ of the yield forces

$$\begin{aligned} \sigma_\mathrm{D}^2 &= \mathrm{E}[(\widetilde{D} - \mu_\mathrm{D})^2] \qquad (105) \\ &= \boldsymbol{\delta}^\mathrm{t} \mathrm{E}[(\widetilde{\mathbf{S}} - \boldsymbol{\mu}_\mathrm{S})(\widetilde{\mathbf{S}} - \boldsymbol{\mu}_\mathrm{S})^\mathrm{t}] \boldsymbol{\delta} = \boldsymbol{\delta}^\mathrm{t} \boldsymbol{\Sigma}_\mathrm{S} \boldsymbol{\delta}. \end{aligned}$$

Particular cases are pointed out next. For bar yield forces equal in the mean, $\mu_1 \cdots \mu_K = \mu_\mathrm{S}$,

$$\boldsymbol{\mu}_\mathrm{S} = \mu_\mathrm{S} \mathbf{e}, \quad \mathbf{e} = \{1 \ 1 \ \cdots 1\}, \qquad (106)$$

the mean dissipation rate is

$$\mu_\mathrm{D} = \mu_\mathrm{S} \mathbf{e}^\mathrm{t} \boldsymbol{\delta} = \mu_\mathrm{S} \sum_{j=1}^{K} \delta_j. \qquad (107)$$

In the case of equal variances $\sigma_1^2 \cdots \sigma_K^2 = \sigma_S^2$ it is worth distinguishing two extremes. If the yield forces vary independently there are not off-diagonal entities in the covariance matrix

$$\boldsymbol{\Sigma}_\mathrm{S} = \sigma_\mathrm{S}^2 \lceil 1 \rfloor = \sigma_\mathrm{S}^2 \mathbf{I}. \qquad (108)$$

The variance of the dissipation rate becomes,

$$\sigma_\mathrm{D}^2 = \sigma_\mathrm{S}^2 (\boldsymbol{\delta}^\mathrm{t} \boldsymbol{\delta}) = \sigma_\mathrm{S}^2 \sum_{j=1}^{K} \delta_j^2. \qquad (109)$$

If, on the other hand, the yield force varies simultaneously in all bars,

$$\boldsymbol{\Sigma}_\mathrm{S} = \sigma_\mathrm{S}^2 \mathbf{E}, \ \mathbf{E} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = (\mathbf{e}\mathbf{e}^\mathrm{t}), \qquad (110)$$

and the variance of the dissipation rate is

$$\sigma_\mathrm{D}^2 = \sigma_\mathrm{S}^2 (\boldsymbol{\delta}^\mathrm{t} \mathbf{E} \boldsymbol{\delta}) = \sigma_\mathrm{S}^2 (\sum_{j=1}^{K} \delta_j)^2. \qquad (111)$$

The result of eqn (111) for equal and simultaneous variation of the yield force in the bars is as for a single variate $\widetilde{S}$ applying uniquely to all bar members:

$$\widetilde{\mathbf{S}} = \widetilde{S} \mathbf{e}. \qquad (112)$$

Considering in eqn (105),

$$\sigma_{\mathrm{D}}^2 = \mathrm{E}[(\widetilde{S}-\mu_{\mathrm{S}})^2](\mathbf{e}^{\mathrm{t}}\boldsymbol{\delta})^2 = \sigma_{\mathrm{S}}^2(\sum_{j=1}^{K}\delta_j)^2, \quad (113)$$

which reproduces eqn (111).

The rate of work of random nodal forces $\widetilde{\mathbf{Q}}$ on the yield velocities $\dot{\mathbf{u}}$ is

$$\widetilde{L} = \dot{\mathbf{u}}^{\mathrm{t}}\widetilde{\mathbf{Q}} \quad (114)$$

with mean and variance,

$$\mu_{\mathrm{L}} = \dot{\mathbf{u}}^{\mathrm{t}}\boldsymbol{\mu}_{\mathrm{Q}}, \quad \sigma_{\mathrm{L}}^2 = \dot{\mathbf{u}}^{\mathrm{t}}\boldsymbol{\Sigma}_{\mathrm{Q}}\dot{\mathbf{u}}. \quad (115)$$

The vector array $\boldsymbol{\mu}_{\mathrm{Q}}$ comprises the mean values of the applied forces, and $\boldsymbol{\Sigma}_{\mathrm{Q}}$ is the covariance matrix. Particular cases can be distinguished in analogy to the yield forces of the bar members.

Assuming independence, mean value and variance of the rate of work and the dissipation rate enable an estimation of the reliability index. Using in eqn (32) the approximations of eqn (14) and eqn (16) for the mean and the variance of the safe load multiplier,

$$\beta = \frac{\mu_{\mathrm{n}}-1}{\sigma_{\mathrm{n}}} = \frac{1-\frac{\mu_{\mathrm{L}}}{\mu_{\mathrm{D}}}+\frac{\sigma_{\mathrm{L}}^2}{\mu_{\mathrm{L}}^2}}{\sqrt{\frac{\sigma_{\mathrm{L}}^2}{\mu_{\mathrm{L}}^2}+\frac{\sigma_{\mathrm{D}}^2}{\mu_{\mathrm{D}}^2}}}. \quad (116)$$

The quantities entering eqn (116) are

$$\frac{\mu_{\mathrm{D}}}{\mu_{\mathrm{L}}} = \frac{\boldsymbol{\delta}^{\mathrm{t}}\boldsymbol{\mu}_{\mathrm{S}}}{\dot{\mathbf{u}}^{\mathrm{t}}\boldsymbol{\mu}_{\mathrm{Q}}},$$

$$\frac{\sigma_{\mathrm{L}}^2}{\mu_{\mathrm{L}}^2} = \frac{\dot{\mathbf{u}}^{\mathrm{t}}\boldsymbol{\Sigma}_{\mathrm{Q}}\dot{\mathbf{u}}}{\dot{\mathbf{u}}^{\mathrm{t}}\boldsymbol{\mu}_{\mathrm{Q}}\boldsymbol{\mu}_{\mathrm{Q}}^{\mathrm{t}}\dot{\mathbf{u}}}, \quad \frac{\sigma_{\mathrm{D}}^2}{\mu_{\mathrm{D}}^2} = \frac{\boldsymbol{\delta}^{\mathrm{t}}\boldsymbol{\Sigma}_{\mathrm{S}}\boldsymbol{\delta}}{\boldsymbol{\delta}^{\mathrm{t}}\boldsymbol{\mu}_{\mathrm{S}}\boldsymbol{\mu}_{\mathrm{S}}^{\mathrm{t}}\boldsymbol{\delta}}. \quad (117)$$

If the system is homogeneous in the means,

$$\frac{\mu_{\mathrm{D}}}{\mu_{\mathrm{L}}} = \frac{\mu_{\mathrm{S}}}{\mu_{\mathrm{Q}}}\frac{\sum^{K}\delta_j}{\sum_{N}\dot{u}_i}, \quad (118)$$

and for independent variations,

$$\frac{\sigma_{\mathrm{L}}^2}{\mu_{\mathrm{L}}^2} = \frac{\sigma_{\mathrm{Q}}^2}{\mu_{\mathrm{Q}}^2}\frac{\sum^{N}\dot{u}_i^2}{(\sum_{N}\dot{u}_i)^2}, \quad \frac{\sigma_{\mathrm{D}}^2}{\mu_{\mathrm{D}}^2} = \frac{\sigma_{\mathrm{S}}^2}{\mu_{\mathrm{S}}^2}\frac{\sum^{K}{\delta_j}^2}{(\sum_{K}\delta_j)^2}. \quad (119)$$

Alternatively, accounting for simultaneous variation of members and of actions,

$$\frac{\sigma_{\mathrm{L}}^2}{\mu_{\mathrm{L}}^2} = \frac{\sigma_{\mathrm{Q}}^2}{\mu_{\mathrm{Q}}^2}, \quad \frac{\sigma_{\mathrm{D}}^2}{\mu_{\mathrm{D}}^2} = \frac{\sigma_{\mathrm{S}}^2}{\mu_{\mathrm{S}}^2}. \quad (120)$$

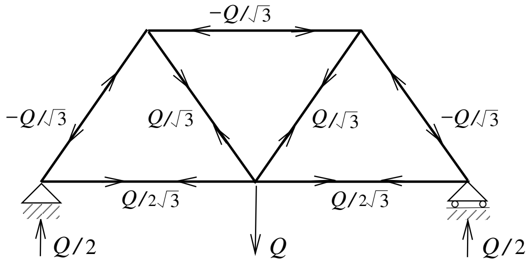


Figure 8: Assembly in series. Statically determinate system.

7.2 Components in series

Components are assembled in series if failure of a single one implies failure of the structure. Conversely, the structure sustains the loading as long as all components are below the plastic limit. Given the reliability of the K components $P_{\mathrm{s}j},\ j=1,\cdots,K$, and assuming statistical independence, the reliability of the structure with respect to the plastic limit is

$$P_{\mathrm{s}} = P_{\mathrm{s}1}P_{\mathrm{s}2}\cdots P_{\mathrm{s}K} = \prod_{j=1}^{K}P_{\mathrm{s}j}. \quad (121)$$

The probability of failure for the system in series is

$$P_{\mathrm{f}} = 1-P_{\mathrm{s}} = 1-\prod_{j=1}^{K}P_{\mathrm{s}j}. \quad (122)$$

From the reliability point of view structural assemblies are in series if successful operation requires all components to be stressed below the plastic limit. As an example the plane truss in Fig. 8 will fail by plastic collapse when anyone of the bar members reaches the yield limit of the perfectly plastic material [4]. The resistance capacity of the bar members, the yield force, is assumed normally distributed according to

$$\widetilde{S} \sim \mathrm{N}(115.5\mathrm{kN}, 12\mathrm{kN}^2), \quad (123)$$

and the applied force according to

$$\widetilde{Q} \sim \mathrm{N}(175\mathrm{kN}, 64\mathrm{kN}^2). \quad (124)$$

The seven bar members carry forces $\lambda_j\tilde{Q}$, with a distance to the yield limit

$$\widetilde{Z}_j = \widetilde{S}_j - |\lambda_j|\widetilde{Q},\ j=1,\cdots,7, \quad (125)$$

The probability of failure by the plastic limit is transferred according to

$$P_{\mathrm{f}j} = P(\tilde{n}_j<1) = P(\widetilde{Z}_j<0) = P(\underline{\widetilde{Z}}_j<-\beta_j), \quad (126)$$

where $\underline{\widetilde{Z}}_j$ refers to the standardized distance which determines the individual reliability index

$$\beta_j = \frac{\mu_{Zj}}{\sigma_{Zj}} = \frac{\mu_S - |\lambda_j|\mu_Q}{\sqrt{\sigma_S^2 + \lambda_j^2\sigma_Q^2}}, \quad j = 1, \cdots, 7. \quad (127)$$

The evaluation with the coefficients λ_j from Fig. 8 gives

$$\beta_1 = \beta_2 = 15.608, \quad \beta_3 \to \beta_7 = 2.505.$$

The failure probability of the bar members follows from the standard normal distribution as

$$\begin{aligned} P_{f1} = P_{f2} &= \Phi^{-1}(-15.608) = 0, \\ P_{f3} \to P_{f7} &= \Phi^{-1}(-2.505) = 0.006122, \end{aligned}$$

with reliability

$$\begin{aligned} P_{s1} = P_{s2} &= \Phi^{-1}(15.608) = 1 \\ P_{s3} \to P_{s7} &= \Phi^{-1}(2.505) = 0.993878. \end{aligned}$$

The probability for the assembly in series to stay below the plastic limit then is computed to

$$\begin{aligned} P_s = \prod_{j=1}^{7} P_{sj} &= (P_{s1}P_{s2})(P_{s3}P_{s4}P_{s5}P_{s6}P_{s7}) \\ &= (1^2)(0.993878^5) = 0.969763, \end{aligned}$$

with a probability of failure for the truss by plastic collapse

$$P_f = 1 - P_s = 0.030237,$$

which is five times higher than for the bar members with the high risk.

A decreasing number of constituents in series enhances the reliability of the structural assembly. For instance, if the force is instead carried by a two-bar truss, (Fig. 9), all other data unchanged, the reliability of the system with respect to plastic collapse is

$$P_s = 0.993878^2 = 0.98779, \quad P_f = 0.012206.$$

It is noted that the two-bar truss in Fig. 9 is nothing but the middle part of the seven-bar structure in Fig. 8.

8 Conclusion

The study deals with the significance of randomness with regard to the limit state of perfectly plastic solids. In this connection the formal background has been presented for the main issues which concern stochastic analysis, robustness against randomness and reliability with respect to the plastic limit; essential tasks have been treated with reference to a unique example.

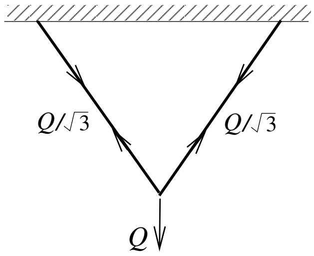


Figure 9: Two-bar assembly.

The subject of structural assemblies has been approached from a restricted point of view with the intention to expound in a separate account. Following throughout an elementary analytic approach aims at elucidating the influence of participating variables. Complex problems are accessible to numerical solution by computational techniques [4], employing also statistical simulation [6], which has not been the purpose of the present study.

References:

[1] I. Doltsinis, Critical States in Deformation Processes of Inelastic Solids, *B.H.V. Topping and C.A. Mota Soares (Eds), Progress in Computational Structures Technology*, Saxe-Coburg Publications, Stirling, Scotland, 2004.

[2] I. Doltsinis, *Elements of Plasticity*, WIT Press, Southampton 2010.

[3] I. Doltsinis, Plastic Limits of Structures and Energy Principles, *Arch Computat Methods Eng* 24, 2017, pp. 165–187.

[4] I. Doltsinis, *Stochastic Methods in Engineering*, WIT Press, Southampton 2012.

[5] A.M. Breipohl, *Probabilistic Systems Analysis*, John Wiley & Sons, Inc., New York 1970.

[6] M. Papadrakakis, N.D. Lagaros, Reliability-based Structural Optimization Using Neural Networks and Monte Carlo Simulation, *Comput Methods Appl Mech Eng (CMAME)* 191, 2002, pp. 3491–3507.