On the Numerical Solution of Some Two-Dimensional Non-Classical Elasticity Problems by Boundary Element Method

NATELA ZIRAKASHVILI
I. Vekua Institute of Applied Mathematics
Iv. Javakhishvili Tbilisi State University University
2 University St., Tbilisi, 0186
GEORGIA
natzira@yahoo.com http://www.viam.science.tsu.ge/geo/structure

Abstract: The paper sets non-classical problems, and formulates the problems of stress and displacement localization for a homogeneous isotropic elastic half-plane based on them. The problems are solved with a boundary element method. Test examples are given in the work showing the value of the normal stress to be applied to the part of the half-plane boundary to obtain the pre-given localized stress or displacement at the midpoint of the segment located inside the body. By using MATLAB software, the numerical results are obtained and corresponding graphs are constructed.

Key-Words: Non-classical problem; Boundary element method; Localization problem; Homogeneous isotropic half plane.

1 Introduction

In the theory of elasticity, there are a number of problems [1]-[13] that could be called non-classical due to the fact that the boundary conditions on a part of the surface boundary are either overdetermined or underdetermined, or the conditions on the boundary are associated with the conditions inside the body (so called non-local problems).

The author's (coauthor's) early work [14] deals with the solution of a non-classical three-dimensional thermoelasticity problem. The problem is to define the temperature on the upper and lower faces of the parallelepiped so that on some two planes inside the body that are parallel to the bases normal displacements or tangential ones would take a priori defined values.

Although there are a number of papers devoted to non-classical problems, we hope that our paper will of a certain interest. Indeed, it presents how a problem can be used in certain practical cases, in particular, for strain-localization problems.

The present paper sets and solves non-classical two-dimensional elasticity problems by boundary element method (BEM) [15]. The considered problems do not coincide with the above-mentioned non-classical problems and are of a great applicable importance.

Our goal is to set strain-localization problems (to be more precise, the problems of localization of stress or displacement) for a homogeneous isotropic half plane by selecting the boundary conditions of non-classical problems. The problem is to find the kind of distribution of normal stress along the section of the half plane boundary so that the normal stress, or normal displacement along the section parallel to the boundary of a given length inside the body parallel to the boundary of the half plane should equal to the value of a given function. If taking this function in a form describing the point force or point displacement applied to the middle point of the segment, the localization problem of the stress or the displacement is obtained. In a certain sense, the problem of localization of stresses in a body is a problem inverse to the delocalization problem [16].

In many structures subject to the action of extreme loading conditions, the initially smooth distribution of strain changes into a highly localized one. Typically, the strain increments are concentrated in a narrow zone, while a great part of the structure is free of load. Such kind of strain localization is caused by geometrical effects (e.g., necking of metallic bars) or by material instabilities (e.g., micro cracks, frictional slip, or when it will enable us to destroy a military structure, for example, an underground facilities). In this work, we will focus on the latter case.

In case of isothermal processes, the rate-independent solids and strain localization has been analyzed as a material instability within a theoretical framework according to Hadamard [17], Thomas [18], Hill [19], Mandel [20] and Rice [21]. In works [22]-[25], the main goal is to give a numerical analysis of strain-localization in multi-dimensional initial value problems. In work [22], the main goal is to give a numerical analysis of strain-localization in initial boundary value problems. In paper [23], localization is studied for an unconstrained, elasto-plastic, strain-softening Cosserat continuum. A finite element model for strain localization analysis of elasto-plastic solids corresponding to the discontinuous displacement fields is presented by Ronaldo I. Borja in [24]. Authors in paper [25] have presented a mathematical model for analyzing strain localization in frictional solids exhibiting displacement jumps.

The localization problems presented in this article are totally different from the above-mentioned problems both, in respect of the solution method (BEM) and mathematical formulation.

The current article examines two localization problems, which have the following physical meaning: on the middle point of the segment lying inside a body parallel to the border half plane in first case a point force is applied, and we must find such value of the normal stress along the section of the border half plane, which will cause this point force (stresses localization), while in the second case, there is given a vertical narrow deep trench outgoing of this point, and we must find such value of the normal stress along the section of the border half plane, which will result in such a pit (displacements localization). It is clear that these problems are non-classical because of the boundary conditions.

Finally, there are test examples given showing the value of normal stress supposed to apply to the section of the half-plane boundary to obtain the pre-given localized stress or displacement at the midpoint of the segment inside the body. The numerical results of these problems are presented together with appropriate graphs, and mechanical and physical interpretations of the problems.

2 Problems Statement

Let us set some non-classical static problems for homogeneous isotropic half plane S (See. Fig.1).

It is known that a homogeneous system of elastic static equilibrium in displacements in the Cartesian system of coordinates has the folowing form [26]:

$$\begin{cases}(\lambda+\mu)\theta_{,x}+\mu\Delta u=0\\(\lambda+\mu)\theta_{,y}+\mu\Delta \mathrm{v}=0\end{cases} \quad \text{in } S \qquad (1)$$

where λ, μ are Lamé constants and $\lambda=\dfrac{\nu E}{(1-2\nu)(1+\nu)}$, $\mu=\dfrac{E}{2(1+\nu)}$, where E is elasticity modulus, and ν is Poisson's ratio; $\Delta(\cdot)=(\cdot)_{,xx}+(\cdot)_{,yy}$ is Laplacian, $(\cdot)_{,x}=\dfrac{\partial(\cdot)}{\partial x}$, $(\cdot)_{,y}=\dfrac{\partial(\cdot)}{\partial y}$; $(\cdot)_{,xx}=\dfrac{\partial^2(\cdot)}{\partial x^2}$; $(\cdot)_{,yy}=\dfrac{\partial^2(\cdot)}{\partial y^2}$; $\vec{U}=(u,\mathrm{v})$ is the displacement vector; $\theta=\operatorname{div}\vec{U}=u_{,x}+\mathrm{v}_{,y}$.

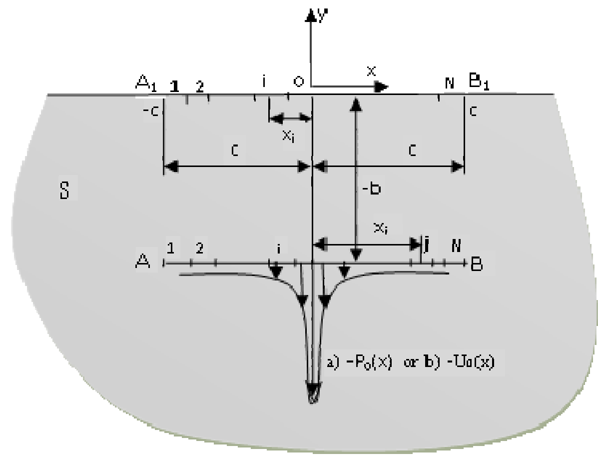

Fig. 1. Illustration of localization problems of stresses and displacements for elastic half plane.

2.1 Statement of a problem, when normal stress is applied to the segment inside a half plane

Let us consider a non-classical problem for half plane S (Fig. 1), when the tangent stress along the entire border and normal stress along boundary segment $|x|>c,\ y=0$ equal to zero. Along segment $|x|\le c,\ y=-b$ inside the body, the value of normal stress σ_{yy} is known. So, let us find the solutions to the system of equilibrium equations (1) satisfying the following boundary conditions:

$$\begin{aligned}&\text{when } |x|<\infty \text{ and } y=0: && \sigma_{yx}=0,\\&\text{when } |x|>c \text{ and } y=0: && \sigma_{yy}=0,\\&\text{when } |x|\le c \text{ and } y=-b: && \sigma_{yy}=-P_0(x),\end{aligned} \qquad (2)$$

where $P_0(x)$ is the sufficiently smooth function given along segment $[-c,c]$.

We can formulate the set problem as follows: let us find the kind of distribution of normal stress σ_{yy} along section $|x|\le c,\ y=0$ of the boundary of a half plane (see Fig. 1) so that the normal stress along segment $|x|\le c,\ y=-b$ inside the body equals to the values of given function $P_0(x)$.

If we consider function of kind $P_0(x) = P \cdot 10^{-|4x|}$ $(P = \text{constant})$, which describes a force similar to the concentrated one, then we will have the following localization problem: we must find the kind of distribution of normal stress σ_{yy} along segment A_1B_1 to obtain the concentrated force of the given value (localization of stresses) along section AB (see Fig. 1).

2.2 Statement problem when normal displacement is given on the segment lying inside a half plane

Let us consider a non-classical problem, when along the entire border of half plane S (see Fig. 1) the tangent stress is equal to zero, and normal displacement u_y on segment $|x| \le c, \quad y = -b$ lying inside the body is known. Also, normal stress along part $|x| \ge c, \quad y = 0$ of boundary is equal to zero. Thus, we have the following boundary conditions:

$$\begin{aligned} &\text{when } |x| < \infty \text{ and } y = 0: \quad \sigma_{yx} = 0, \\ &\text{when } |x| > c \text{ and } y = 0: \quad \sigma_{yy} = 0, \\ &\text{when } |x| \le c \text{ and } y = -b: \quad u_y = -U_0(x), \end{aligned} \tag{3}$$

where $U_0(x)$ is the sufficiently smooth function given along segment $[-c, c]$.

We can formulate this problem as follows: let us find the distribution of normal stress σ_{yy} along part $|x| \le c, \; y = 0$ of the boundary of the half plane when normal displacement along segment $|x| \le c, \; y = -b$ lying inside half plane S equals to $-U_0(x)$.

Let us consider this function of the following kind $U_0(x) = P \cdot 10^{-|4x|}$, $(P = \text{constant})$, which describes clearly expressed non-uniform normal displacement. Thus, we will have the following localization problem: let us find the distribution of normal stress σ_{yy} along segment A_1B_1 to obtain the pit of a given value along segment AB (displacements localization) (see Fig. 1).

3 Solving the set problems

Let us solve the set problems by BEM. When solving the boundary value problems for half planes by BEM, we use a singular solution of the Flaman problem (see Appendix A).

3.1 Solving problem (1), (2)

Let us divide segments $|x| \le c, \; y = 0$ and $|x| \le c, \; y = -b$ (see Fig. 1) into N segments (elements) of the same size $2a$ and smaller sizes (i.e. $a = c/N$). We mean that constant normal stresses P_y^j act on each j^{th} element of length $2a$ with centre $(x^j, 0)$ of segment $|x| \le c, \; y = 0$. We need to find such values of these stresses, for which the values of the normal stresses in middle points $(x^i, -b)$ of each i^{th} segment with a length of $2a$ along segment $|x| \le c, \; y = -b$ inside body will equal to the given value of $-P_0(x^i)$.

Normal stress in the centre of the i^{th} element lying on segment $|x| \le c, \; y = -b$ caused by the action of constant normal load P_y^j on the j^{th} element of segment $|x| \le c, y = 0$ will be found by inserting $y = -b$, $\quad x = x^i - x^j$ in equation (A.1) of the Appendix.

$$\sigma_{yy}(x^i, -b) = \frac{1}{\pi}\left[\left(\arctan\frac{b}{x^i - x^j - a} - \arctan\frac{b}{x^i - x^j + a}\right) + \frac{b(x^i - x^j + a)}{(x^i - x^j + a)^2 + b^2} - \frac{b(x^i - x^j - a)}{(x^i - x^j - a)^2 + b^2}\right]P_y^j .$$

Normal stress in the centre of the i^{th} element lying on segment $|x| \le c, y = -b$ will equal to following sum:

$$\sigma_{yy}(x^i, -b) = \sum_{j=1}^{N} A^{ij} P_y^j, \qquad i = 1, 2, \dots, N,$$

where we use the following formula for influence coefficients A^{ij}:

$$A^{ij} = \frac{1}{\pi}\left[\left(\arctan\frac{b}{x^i - x^j - a} - \arctan\frac{b}{x^i - x^j + a}\right) + \frac{b(x^i - x^j + a)}{(x^i - x^j + a)^2 + b^2} - \frac{b(x^i - x^j - a)}{(x^i - x^j - a)^2 + b^2}\right].$$

Thus, we obtain the following system of N linear algebraic equations with N unknown quantities P_y^j, $j = 1, 2, \dots, N$.

$$\sum_{j=1}^{N} A^{ij} P_y^j = P_0(x^i), \qquad i = 1, 2, \dots, N. \tag{4}$$

If solving (4) system in relation to unknown quantities P_y^j by means of any standard method of numerical analysis (by method of Gauss in our case), then we can assume that the set problem is solved and $\sigma_{yy}^j = P_y^j, \quad j = 1, \dots, N$ (see Appendix).

After solving these equations, we can express the displacements and stresses at any point $\left(x^i, y^k\right)$ of the body by means of other linear combination of load P_y^j. For example, the stresses and displacements have the following form:

$$\sigma_{xx}(x^i, y^k) = \frac{1}{\pi}\sum_{j=1}^{N}\left[\left(\arctan\frac{y^k}{x^i - x^j + a^j} - \arctan\frac{y^k}{x^i - x^j - a^j}\right) - \frac{y^k\left(x^i - x^j + a^j\right)}{(x^i - x^j + a^j)^2 + \left(y^k\right)^2} + \frac{y^k\left(x^i - x^j - a^j\right)}{(x^i - x^j - a^j)^2 + \left(y^k\right)^2}\right]P_y^j,$$

$$\sigma_{yy}(x^i, y^k) = \frac{1}{\pi}\sum_{j=1}^{N}\left[\left(\arctan\frac{y^k}{x^i - x^j - a^j} - \arctan\frac{y^k}{x^i - x^j + a^j}\right) - \frac{y^k\left(x^i - x^j + a^j\right)}{(x^i - x^j + a^j)^2 + \left(y^k\right)^2} + \frac{y^k\left(x^i - x^j - a^j\right)}{(x^i - x^j - a^j)^2 + \left(y^k\right)^2}\right]P_y^j, \quad (5)$$

$$\sigma_{xy}(x^i, y^k) = \frac{1}{\pi}\sum_{j=1}^{N}\left(y^k\right)^2\left[\frac{1}{(x^i - x^j + a^j)^2 + \left(y^k\right)^2} - \frac{1}{(x^i - x^j - a^j)^2 + \left(y^k\right)^2}\right]P_y^j,$$

$$i = 1, 2, ..., M_1, \quad k = 1, 2, ..., M_2.$$

$$u_x^j(x^i, y^k) = -\frac{1}{2\pi\mu}\sum_{j=1}^{N}\left\{(1-2\nu)\left[\left(x^i - x^j - a^j\right)\arctan\frac{y^k}{x^i - x^j - a^j} - \left(x^i - x^j + a\right)\arctan\frac{y^k}{x^i - x^j + a^j} - \pi a\right] + (1-\nu)y^k \ln\frac{(x^i - x^j - a^j)^2 + \left(y^k\right)^2}{(x^i - x^j + a^j)^2 + \left(y^k\right)^2}\right\}P_y^j,$$

$$u_y^j(x^i, y^k) = \frac{1}{2\pi\mu}\sum_{j=1}^{N}\left\{-y^k(1-2\nu)\left(\arctan\frac{y^k}{x^i - x^j - a^j} - \arctan\frac{y^k}{x^i - x^j + a^j}\right) + (1-\nu)\left[\left(x^i - x^j - a^j\right)\ln\left((x^i - x^j - a^j)^2 + \left(y^k\right)^2\right) - \left(x^i - x^j + a^j\right)\ln\left((x^i - x^j + a^j)^2 + \left(y^k\right)^2\right) + \left(L - x^j + a^j\right)\ln\left(L - x^j + a^j\right)^2 - \left(L - x^j - a^j\right)\ln\left(L - x^j - a^j\right)^2\right]\right\}P_y^j.$$

3.2 Solving problem (1), (3)

Let us divide segments $|x| \le c,\ y = 0$ and $|x| \le c,\ y = -b$ into N segments (elements) with equal $2a$ and smaller lengths. We mean that constant normal stresses P_y^j act on each j^{th} segment of segment $|x| \le c,\ y = 0$, each with the length of $2a$ and with centre $(x^j, 0)$. We must find such values of these stresses, for which the values of normal displacement in middle point $(x^i, -b)$ of each i^{th} element with length $2a$ of $|x| \le c,\ y = -b$ segment inside the body should equal to the given value of $-U_0(x^i)$.

Normal displacement $u_y^j(x^i, -b)$ in the centre $(x^i, -b)$ of the i^{th} element lying on the segment $|x| \le c,\ y = -b$ caused by the action of constant load P_y^j on the j^{th} element lying of segment $|x| \le c,\ y = 0$ will be found by inserting $y = -b$, $x = x^i - x^j$ and $L = L - x^j$ in equation (A.1) of the Appendix.

$$u_y^j(x^i, -b) = \frac{P_y^j}{2\pi\mu}\left\{-b(1-2\nu)\left(\arctan\frac{b}{x^i - x^j - a} - \arctan\frac{b}{x^i - x^j + a}\right) + (1-\nu)\left[\left(x^i - x^j - a\right)\ln\left((x^i - x^j - a)^2 + b^2\right) - \left(x^i - x^j + a\right)\ln\left((x^i - x^j + a)^2 + b^2\right) + \left(L - x^j + a\right)\ln\left(L - x^j + a\right)^2 - \left(L - x^j - a\right)\ln\left(L - x^j - a\right)^2\right]\right\}$$

Further, normal displacement in the centre of the i^{th} element lying on segment $|x| \le c,\ y = b$ will be computed with the following formula:

$$u_y(x^i, -b) = \sum_{j=1}^{N} B^{ij} P_y^i, \qquad i = 1, 2, ..., N.,$$

where we have the following formula for influence coefficients B^{ij}:

$$B^{ij} = \frac{1}{2\pi\mu}\left\{-b(1-2\nu)\left(\arctan\frac{b}{x^i - x^j - a} - \arctan\frac{b}{x^i - x^j + a}\right) + (1-\nu)\left[\left(x^i - x^j - a\right)\ln\left((x^i - x^j - a)^2 + b^2\right) - \left(x^i - x^j + a\right)\ln\left((x^i - x^j + a)^2 + b^2\right) + \left(L - x^j + a\right)\ln\left(L - x^j + a\right) - \left(L - x^l - a\right)\ln\left(L - x^j - a\right)^2\right]\right\}$$

Thus, the set problem is reduced to solving the following system of linear algebraic equations (N equations with N unknown values):

$$\sum_{j=1}^{N} B^{ij} P_y^i = -U_0(x^i), \quad i = 1, 2, ..., N. \tag{6}$$

If we solve system (6) in relation to unknown values P_y^j, then the set problem can be considered as solved, like the problem set in 3.1.

4 Numerical results and discussion

4.1 Numerical simulations of problems in stresses

By using MATLAB software, we obtained the numerical values of the normal stresses along segment AB (the given normal load) and distribution of normal stresses along segment A_1B_1 (the obtained normal stress) shown in Fig. 1 for the following data: $c = 1m,\ 2m,\ 3m,\ 4m$, $15m, 18m, 20m, 30m$ and $b = 5m,\ 6{,}5m,\ 8m,\ 10m$, $15m,\ 18m,\ 20m,\ 30m$; $N = 120$; $P = 10\, kg/cm^2$. Below are the tables and relevant graphs of some of the obtained results. Namely, Fig. 2 shows load $P_0(x)$ along AB segment and distribution of obtained normal stress P_y along A_1B_1 segment, when $c = 1m$ and $b = 5m,\ 6{,}5m,\ 8m,\ 10m$. In addition, a table of numerical values of normal stresses $\sigma_{yy} = P_y$ in the middle points of the boundary elements lying on segment A_1O is presented, when $c = 1m$ and $b = 5m$ (see Table 1). Table 2 gives minimum and maximum numerical values of normal stresses $\sigma_{yy} = P_y$ in the middle points of the boundary elements lying on segment A_1B_1 for different values of c and b.

Besides, represented 3D graphs of the distribution of stresses and displacements in the body section relevant to domain $-c < x < c,\ -30 < y < -10$, when $c = 1m$, $b = 30m$; for steel $E = 2 \times 10^6\ kg/cm^2$, $\nu = 0.3$ (see Fig. 3 and Fig. 4) and technical rubber $E = 2 \times 10^2\ kg/cm^2$, $\nu = 0.42$ (see Fig. 5). Formula (5) evidences that the stresses in the stress problems do not depend on Young's modulus and Poison's ratio. As for the displacements, the normal displacement less and tangential displacement is bigger in steel than in technical rubber.

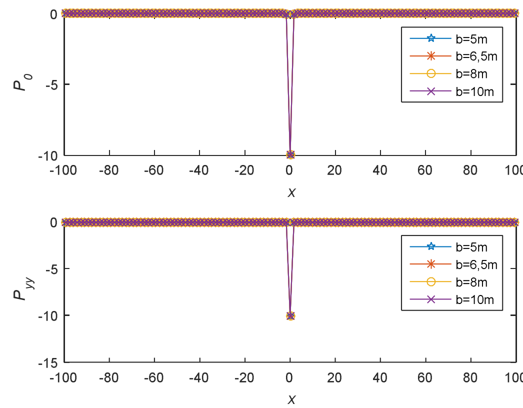

Fig. 2. The load $P_0(x)$ along segment AB and distribution of obtained normal stress P_y along segment A_1B_1, when $c = 1m$.

Table 1. Numerical values of stress $\sigma_{yy} = P_y$ in the middle points of the some boundary elements lying on segment A_1O, when $c = 1m$ and $b = 5m$

x	P_y
-99.17355	-2.576561×10^{-2}
-66.11570	-2.686224×10^{-2}
-33.05785	-2.755382×10^{-2}
0.000000	-10.02779

The numerical values of stress σ_{yy} in the middle points of the boundary elements lying on segment OB_1 exactly coincide with the numerical values given in Table 1.

Similar numerical results are obtained, when $c = 2m,\ 3m,\ 4m$ and $b = 5m,\ 6{,}5m,\ 8m,\ 10m$, and minimum and maximum numerical values of normal stresses P_y in the middle points of the boundary elements lying on segment A_1B_1 are given in Table 2.

Table 2. Numerical values of stresses $\sigma_{yy}=P_y$ in the middle points of the boundary elements lying on segment A_1B_1 for different values of c and b

b	$c=1$		$c=4$	
	Minimum	Maximum	Minimum	Maximum
5	-2.5766×10^{-2}	-10.0278	-1.1415×10^{-1}	-10.2249
6.5	-1.9113×10^{-2}	-10.0200	-1.0225×10^{-1}	-10.1452
8	-1.5132×10^{-2}	-10.0156	-7.1632×10^{-2}	-10.1037
10	-1.1814×10^{-2}	-10.0120	-5.6407×10^{-2}	-10.0733
15	-7.6026×10^{-3}	-10.0077	-3.5777×10^{-2}	-10.0407
18	-6.2571×10^{-3}	-10.0063	-2.9040×10^{-2}	-10.0318
20	-5.5958×10^{-3}	-10.0056	-2.5740×10^{-2}	-10.0278
30	-5.3147×10^{-3}	-10.0053	-1.6250×10^{-2}	-10.0168

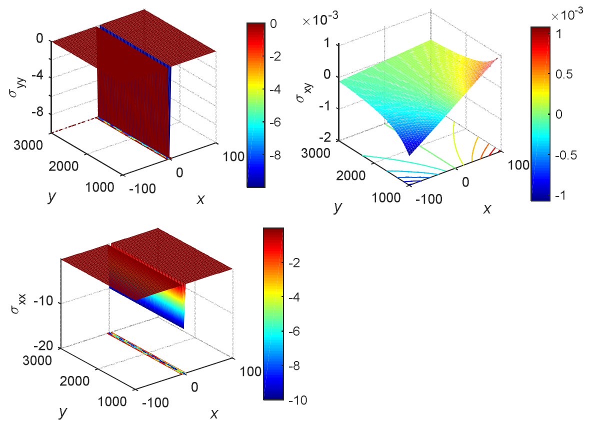

Fig. 3. Distribution of stresses in domain $-c<x<c$, $-30<y<-10$, when $c=1m$, $b=30m$, $\nu=0.3$ (in stresses for the problem, when $P_0(x)=P\cdot10^{-|4x|}$)

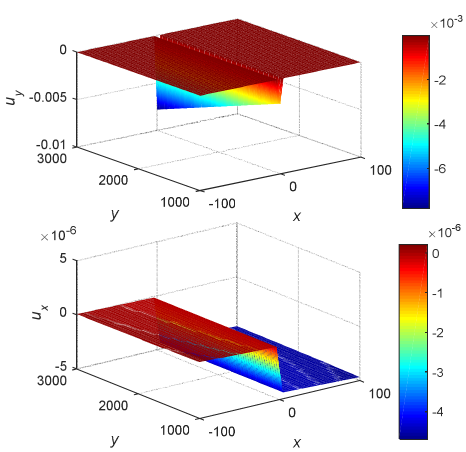

Fig. 4. Distribution of displacements for steel in domain $-c<x<c$, $-30<y<-10$, when $c=1m$, $b=30m$, $E=2\times10^6\ kg/cm^2$, $\nu=0.3$ (in stresses for the problem, when $P_0(x)=P\cdot10^{-|4x|}$)

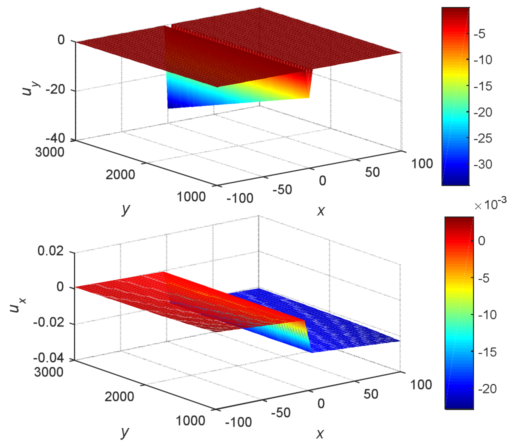

Fig. 5. Distribution of displacements for technical rubber in domain $-c<x<c$, $-30<y<-10$, when $c=1m$, $b=30m$, $E=2\times10^2\ kg/cm^2$, $\nu=0.42$ (in stresses for the problem, when $P_0(x)=P\cdot10^{-|4x|}$)

4.2 Numerical simulations of problems in displacements

For the following data: $E=2\times10^2\ kg/cm^2$, $\nu=0.42$ (technical rubber) or $E=2\times10^6\ kg/cm^2$, $\nu=0.3$ (steel); $c=1m,\ 2m,\ 3m,\ 4m$ and $b=5m,\ 6{,}5m,\ 8m,\ 10m$, $15m, 18m, 20m, 30m$; $N=120$, $P=10m$ numerical values of normal displacements at segment AB (the given normal displacement) and distribution normal stresses at segment A_1B_1 (the obtained normal stress) are obtained (see Fig. 1). Below are the tables and relevant graphs of some of the obtained results. Namely, they show normal displacement $U_0(x)$ given along segment AB, and the distribution of the obtained normal stress P_y along segment A_1B_1 is presented in Fig. 3, when $c=1m$ and $E=2\times10^2\ kg/cm^2$, $\nu=0.42$ (technical rubber), and in Fig. 4, when $c=1m$ and $E=2\times10^6\ kg/cm^2$, $\nu=0.3$ (steel). Moreover, there are tables of numerical values of normal stresses $\sigma_{yy}=P_y$ in the middle points of the boundary elements lying on segment A_1O , when $c=1m$ და $b=5m$, $E=2\times10^2\ kg/cm^2$ for technical rubber (see Table 3), and $c=1m$ და $b=5m$, $E=2\times10^6\ kg/cm^2$ for steel (see Table 5). There are also numerical values of normal stresses $\sigma_{yy}=P_y$ in the middle points of the boundary elements lying on segment A_1B_1 for different values of c and b when $E = 2\times10^2\ kg/cm^2$, $\nu = 0.42$ (rubber) (see Table 4), and when $E = 2\times10^6\ kg/cm^2$, $\nu = 0.3$ (steel) (see Table 6).

Furthermore, represented 3D graphs of distribution of stresses and displacements in the part of the body bordered by domain $-c < x < c,\ -30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2\times10^2\ kg/cm^2$, $\nu = 0.42$ for technical rubber (see Fig. 8 and Fig. 9) and $E = 2\times10^6\ kg/cm^2$, $\nu = 0.3$ for steel (see Fig. 10 and Fig. 11). In this case, stresses σ_{yy} and σ_{xx} in the steel are too big, while tangent stress τ_{xy} is less as compared to the technical rubber. In addition, the normal displacement in both materials are almost equal, while tangent displacement is bigger in steel than in technical rubber.

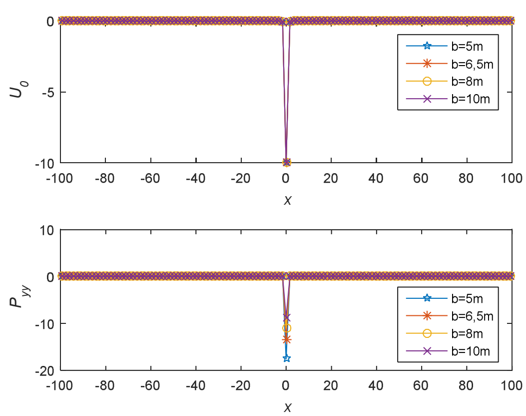

Fig. 6. Displacement $U_0(x)$ along segment AB and distribution of obtained normal stress P_y along segment A_1B_1, when $c = 1m$ and $E = 2\times10^2\ kg/cm^2$, $\nu = 0.42$ (technical rubber)

Table 3. Numerical values of normal stresses $\sigma_{yy} = P_y$ in the middle points of the some boundary elements lying on segment A_1O, when $c = 1m$ and $b = 5m$, $E = 2\times10^2\ kg/cm^2$, $\nu = 0.42$ (technical rubber)

x	P_y
-99.17355	6.191295×10^{-2}
-66.11570	64.13679×10^{-2}
-33.05785	6.551168×10^{-2}
0.000000	-17.53966

Table 4. Numerical values of normal stresses $\sigma_{yy} = P_y$ in the middle points of the boundary elements lying on segment A_1B_1 for different values of c and b, when $E = 2\times10^2\ kg/cm^2$, $\nu = 0.42$ (technical rubber)

b	$c = 1$		$c = 4$	
	Minimum	Maximum	Minimum	Maximum
5	6.1682×10^{-2}	-17.5399	1.6875×10^{-2}	-17.5204
6.5	4.1378×10^{-2}	-13.4999	2.2983×10^{-2}	-13.4909
8	2.9671×10^{-2}	-10.9732	2.1985×10^{-2}	-10.9676
10	2.0471×10^{-2}	-8.7820	1.8538×10^{-2}	-8.7780
15	1.0119×10^{-2}	-5.8584	1.1567×10^{-2}	-5.8555
18	7.2940×10^{-3}	-4.8832	9.0013×10^{-3}	-4.8807
20	6.0216×10^{-3}	-4.3954	7.7289×10^{-3}	-4.3932
30	2.8381×10^{-3}	-2.9314	4.1300×10^{-3}	-2.9300

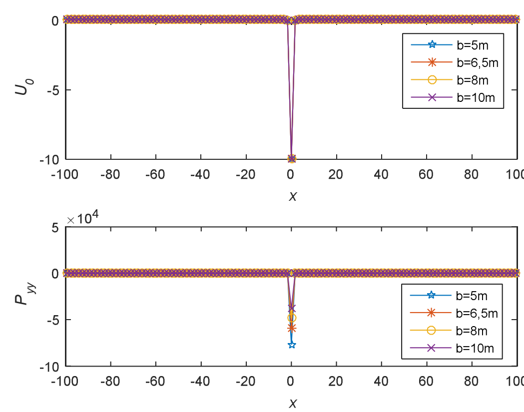

Fig. 7. Displacement $U_0(x)$ along segment AB and distribution of obtained normal stresses P_y along segment A_1B_1, when $c = 1m$ and, $E = 2\times10^6\ kg/cm^2$, $\nu = 0.3$ (steel)

Table 5. Numerical values of normal stresses $\sigma_{yy} = P_y$ in the middle points of the some boundary elements lying on segment A_1O, when $c = 1m$ and $b = 5m$, $E = 2\times10^6\ kg/cm^2$, $\nu = 0.3$ (steel)

x	P_y
-99.17355	1.455292×10^2
-66.11570	1.511012×10^2
-33.05785	1.545391×10^2
0.000000	7.676738×10^4

Table 6. Numerical values of normal stresses $\sigma_{yy} = P_y$ in the middle points of the boundary elements lying on segment A_1B_1 for different values of c and b, when $E = 2\times 10^6\ kg/cm^2$, $\nu = 0.3$ (steel)

b	$c = 1$		$c = 4$	
	Minimum	Maximum	Minimum	Maximum
5	145.181	-7.6768×10^4	53.0648	-7.6684×10^4
6.5	93.0356	-5.9075×10^4	67.4164	-5.9027×10^4
8	64.4930	-4.8011×10^4	61.4388	-4.7978×10^4
10	43.0377	-3.8418×10^4	49.5399	-3.8395×10^4
15	20.2100	-2.5621×10^4	28.6766	-2.5608×10^4
18	14.2920	-2.1353×10^4	21.5879	-2.1344×10^4
20	11.6818	-1.9219×10^4	18.1883	-1.9211×10^4
30	53.3406	-1.2815×10^4	9.06856	-1.2811×10^4

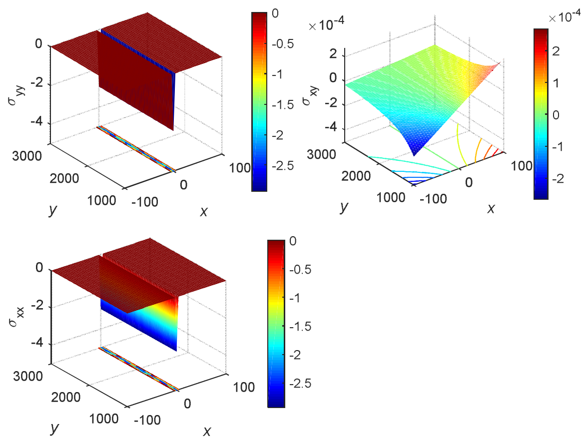

Fig. 8. Distribution of stresses in the part of the body of technical rubber bordered by domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2\times10^2\ kg/cm^2$, $\nu = 0.42$ (in displacements for the problem when $P_0(x) = P\cdot 10^{-|4x|}$)

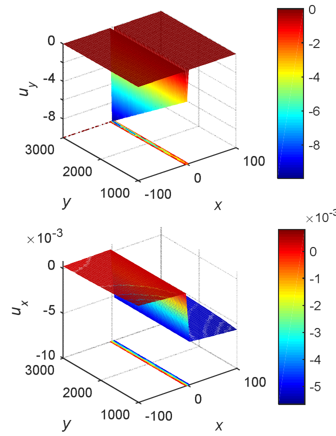

Fig. 9. Distribution of displacements in the part of the body of technical rubber bordered by domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2\times10^2\ kg/cm^2$, $\nu = 0.42$ (in displacements for the problem when $P_0(x) = P\cdot 10^{-|4x|}$)

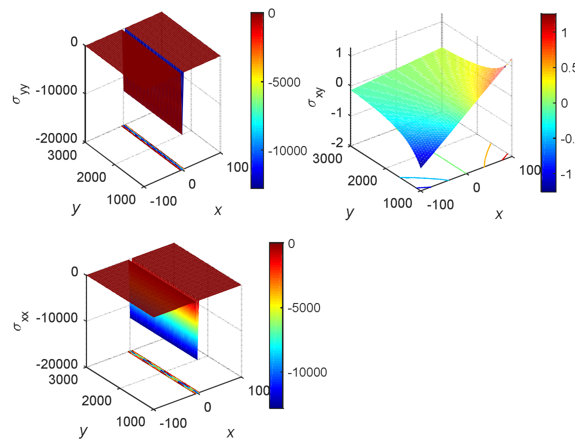

Fig. 10. Distribution of stresses in the part of the body of steel bordered by domain $-c < x < c$, $-30 < y < -10$, when $c = 1m$, $b = 30m$, $E = 2\times10^6\ kg/cm^2$, $\nu = 0.3$ (in displacements for the problem when $P_0(x) = P\cdot 10^{-|4x|}$)

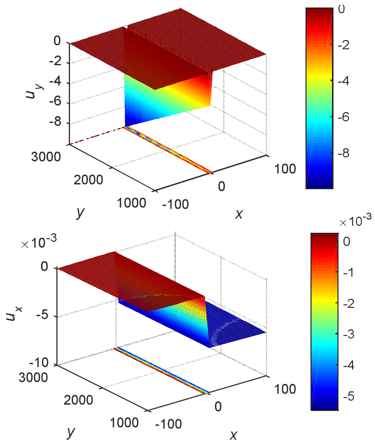

Fig. 11. Distribution of displacements in the part of the body of steel bordered by domain $-c<x<c,\ -30<y<-10$, when $c=1m$, $b=30m$, $E=2\times10^{6}\ kg/cm^{2}$, $\nu=0.3$ (in displacements for the problem when $P_0(x)=P\cdot10^{-|4x|}$)

5 Conclusion

The paper sets non-classical problems, and problems of localization of stress and displacement for a homogeneous isotropic elastic half-plane are formulated based on them. The essence of the problems is as follows: we must find the distribution of the normal stress along section A_1B_1 (see Fig. 1) of the border of the half plane so that normal stress σ_{yy} or normal displacement u_y along segment AB parallel to the border of a given length distanced from the border by b within the body should equal to the value of the given function. If we take the kind of this function, which describes the point-force applied to the middle point of section AB (e.g. $U_0(x)=C\cdot10^{-|4x|}$, $(C=\text{constant})$), we will obtain the problem of localization of stresses and displacements. The set problems are solved by BEM [15].

By using the MATLAB's software, we obtained the numerical results and plotted the corresponding graphs showing the values of normal stress to be applied to the part of the boundary of the half plane to obtain the point force or displacement in the middle point of a segment inside the body. The paper also presents 3D graphs of distribution of stresses and displacements within the parts of the bodies of steel and technical rubber bordered by domain $-c<x<c,\ -30<y<-10$.

The problems considered in the work can be used in practice, e.g. in soils and rocks, materials that are susceptible to cracking and faulting when sheared, as well materials used to demolish military structures or in underground facilities.

Appendix

Flamant problem

The problem of the plane elasticity when a point-force is applied to one point of the boundary of an elastic isotropic half-plane is known as Flamant problem. The solution of Flamant problem is known and is given in a number of text-books dedicated to the theory of elasticity (For an example, see [27, 28]). It is an example of a singular solution in the static theory of elasticity.

In case of distribution of constant normal stresses $p_y(x)=P_y$ along segment $-a\le x\le a$, $y=0$ with a finite length we will have [15].

$$u_x=\frac{P_y}{2\pi\mu}\left\{(1-2\nu)\left[(x-a)\theta_1-(x+a)\theta_2-\pi a\right]+(1-\nu)y\ln\left(r_1^2/r_2^2\right)\right\},$$

$$u_y=\frac{P_y}{2\pi\mu}\left\{-(1-2\nu)y(\theta_1-\theta_2)+(1-\nu)\left[(x-a)\ln r_1^2-(x+a)\ln r_1^2+\right.\right.$$
$$\left.\left.+(L+a)\ln(L+a)^2-(L-a)\ln(L-a)^2\right]\right\}, \qquad (A1)$$

$$\sigma_{xx}=\frac{P_y}{\pi}\left[\theta_1-\theta_2+y(x-a)/r_1^2-y(x+a)/r_2^2\right],$$

$$\sigma_{yy}=\frac{P_y}{\pi}\left[\theta_1-\theta_2-y(x-a)/r_1^2+y(x+a)/r_2^2\right],$$

$$\sigma_{xy}=\frac{P_y}{\pi}y^2\left(1/r_1^2-1/r_2^2\right)$$

where

$\theta_1=\arctan(y/(x-a))$, $\theta_2=\arctan(y/(x+a))$, $r_1^2=(x-a)^2+y^2$, $r_2^2=(x+a)^2+y^2$, L is any arbitrary constant and means that u_y displacement will be measured relatively to the displacement of any $x=\pm L$ point of the boundary of a half-plane (reference point).

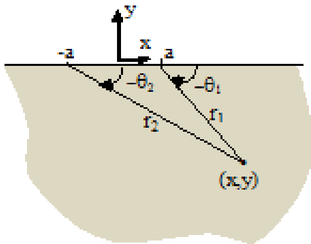

Fig.12. Distances to the extreme points and corresponding angles.

When $y=0$, σ_{yy} is as follows:

$$\sigma_{yy} = -P_y\left(\theta_1 - \theta_2\right)/\pi .$$

According to Fig.12, when $y=0$, then $\theta_1 = \theta_2$, with the exception of segment $|x|<a$, where $\theta_1 = -\pi$ and $\theta_2 = 0$. Thus, we obtain that $\sigma_{yy} = 0$ when $|x|>a$, and $\sigma_{yy} = P_y$ when $|x|<a$.

References:

[1] Eringen A.C. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *J. Appl. Phys.,* Vol. 54, 1983, pp. 4703–4710.

[2] Bitsadze A.V. On the theory of nonlocal boundary value problems, (Russian). *Dokl Akad Nauk SSSR*, Vol. 277, No 1, 1984, pp. 17–19.

[3] Gordeziani D.G. A class of nonlocal boundary value problems in elasticity theory and shell theory, (Russian) Theory and numerical methods of calculating plates and shells. *Tbilis Gos Univ II,* 1984, pp. 106–127.

[4] Obolashvili E.I. Solution of nonlocal problems in plane elasticity theory, (Russian) *Current problems inmathematical physics*. (Russian) *Tbilis Gos Univ II,* 1987, pp. 295–302.

[5] Dehghan Mehdi. Numerical solution of a non-local boundary value problem with Neumann's boundary conditions. *Commun. Numer. Methods Eng.*, Vol.19, No 1, 2002, pp. 1–12.

[6] Agalovyan L.A. The solution asymptotics of classical and nonclassical, static and dynamic boundary-value problems for thin bodies. *J. Int. Appl. Mech*., Vol 38, No7, 2002, pp. 765–782.

[7] Sladek J., Sladek V., Baant Z.P. Non-local boundary integral formulation for softening damage. *Int. J. Numer. Meth. Eng.*, Vol. 57, 1, 2003, pp. 103–116.

[8] Khachikyan A.S. On harmonic and biharmonic problems for equations with nonclassical boundary conditions, (Russian). *Izv. Nats. Akad. Nauk. Armen. Mekh.,* Vol. 59, No 4, 2006, pp. 24–31.

[9] Sarkisyan V.G. A plane elasticity problem with nonclassical boundary conditions. Actual problems of continuum mechanics. *Proceedings of International Conference devoted to the 95th anniversary of Academician NAN of Armenia N. Kh. Arutyunyan*, Yrevan 2007, pp. 373–376.

[10] Gordeziani D., Gordeziani E., Davitashvili T., Meladze G.. About the solution of some nonlocal boundary and initial-boundary value problems. (Russian) *GESJ Comput. Sci. Telecommun.*, Vol. 3, No 26, 2010, pp. 161–169.

[11] Avalishvili G., Avalishvili M., Gordeziani D. On some non-classical two-dimensional models for thermoelastic plates with variable thickness. *Bull. Georgian Natl. Acad. Sci. (N.S.)* Vol. 4, No 2, 2010, pp. 27–34.

[12] Avalishvili G., Avalishvili M., Gordeziani D. On integral nonlocal boundary value problems for some partial differential equations. *Bull. Georgian Natl. Acad. Sci. (N.S.)* Vol. 5, No 1, 2011, pp. 31–37.

[13] Ma H.M., Gao X .L., Reddy J. N. A non-classical Mindlin plate model based on a modified couple stress theory. *Acta Mechanica* Vol. 220, No 1, 2011, pp. 217–235.

[14] Khomasuridze N., Janjgava R., Zirakashvili N. Some non-classical thermoelasticity problems for a rectangular parallelepiped. *Meccanica*, Vol. 49. No 6, 2014, pp. 1337-1342.

[15] Crouch S.L., Starfield A.M. *Boundary element methods in solid Mechanics*. London-Boston-Sydney, GEORGE, ALLEN & UNWIN, 1983.

[16] Zirakashvili N. On the numerical solution of some two-dimensional boundary-contact delocalization problems. *Meccanica*, Vol. 48, No 7, 2013, pp. 1791–1804.

[17] Hadamard J. *Leçons sur la propagation des ondes et les équations de L'hydrodynamique*. Paris, Herman, 1903.

[18] Thomas T.Y. *Plastic Flow and Fracture in Solids*. New York, Academic Press, 1961.

[19] Hill R. Acceleration waves in solids. *J. Mech. Phys. Solids,* Vol. 10, 1962, pp. 1-16.

[20] Mandel J. Propagation des surfaces de discontinuite dans un milicu elasto-plastique, in : H. Kolsky and W. Prager, eds., Proceedings IUTAM Symposium on Stress

Waves in Inelastic Solids, Brown University, Springer, Berlin: 1963; 337-344.

[21] Rice J.R. The localization of plastic deformation, in: W. T. Koiter, ed., Theoretical and Applied Mechanics, Proceedings of the 14th IUTAM Congress, Delft, North-Holland, Amsterdam: 1976; 207-220.

[22] Loret B. and Prevost JH. Dinamic strain localization in elasto-(visco)-plastic solids, part 1. General formulation and ine-dimensional examples, *Comuter Methods in Applied Mechanics and Engineering*, Vol. 83, 1990, pp. 247-273

[23] R. de Borst and L.J. Sluys Localisation in a Cosserat continuum under static and dynamic loading conditions. *Computer Methods in Applied Mechanics and Engineering*, Vol. 90, pp. 1990;805-827.

[24] Ronaldo I. Borja. A finite element model for strain localization analysis of strongly discontinuous fields based on standard Galerkin approximation. *Comput. Methods Appl. Mech. Eng*., Vol. 190, 2000, pp. 1529-1549.

[25] Ronaldo I. Borja, Richard A. Regueiro. Strain localization in frictional material exhibiting displacement jumps. *Comput. Methods Appl. Mech. Eng*., Vol. 190, 2001, pp. 2555-2580

[26] Rabotnov Y.N. *Mechanics of deformable solids*. Moskow, Nauka, 1988.

[27] Timoshenko S.P., Goodier J.N. *Theory of elasticity*. 3rd ed. New York, McGraw-Hill, 1970.

[28] Sadd M. H. *Elasticity; Theory, Applications and Numerics*. Second Edition, Amsterdam. Boston. Heodelberg. London. New. York. Oxford. Paris. San Diego. San Francisco. Singapore. Sydney. Tokyo: Elsevier, 2009.