Dynamical behavior of thin cylindrical shell: An asymptotic approach

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Abstract: - Enlightened by the asymptotic behavior of the thin wall structured body, the present study deals with the novel Legendre polynomials to investigate the dynamic behavior of an isotropic thin cylindrical shell by using asymptotic approach under cylindrical symmetry. It is found that the motion in n-th mode of the cylinder does not depend on higher or lower modes and variants of each mode satisfies same set of equations. Here we set some special assumption to make the problem simple and make an attempt to give an analytic expression of radial vibration when a semi-infinite cylinder of large radius is subjected to large longitudinal stimuli. Excellent analytical expressions are demonstrated by using the Duhamel's principle for non-homogeneous differential equations and due to the presence of Legendre Polynomials, and cylindrical symmetry also.

Key-Words: - Radial and longitudinal vibration; asymptotic approach; cylindrical shell; cylindrical symmetry; linear classical theory.

1 Introduction

From the pioneering work of Love [1], theories of elastic vibration and waves are getting enormous importance in the field of continuum mechanics. In 1959, three dimensional investigation on harmonic wave propagation in hollow circular cylinder has been studied by Gazis [2]. In 1965, wave propagation in transversely isotropic circular cylinder of infinite length has been investigated by Mirsky [3]. Later on, in the year 1992, Sinha [4] have discussed the axisymmetric wave propagation in circular cylindrical shell immersed in fluid in two parts. In the first part they have discussed the theoretical analysis of the propagating modes and the axisymmetric modes excluding torsional modes that are obtained theoretically and experimentally has been compared in the second part. A rather detailed account of diverse recent theoretical advances and applications of vibration of shells in the various fields can be found in the monograph of Leissa [5].

In last two decades, dissimilar forms of vibration depending on the material properties/arrangements in various kinds of plates and shells have been investigated by several authors. Among them we can mention; composite Vibration of functionally graded multi-layered orthotropic cylindrical panel under thermo-mechanical load has been discussed by Wang [6]. Three dimensional vibrations of a homogeneous transversely isotropic thermo-elastic cylindrical panel have been investigated by Sharma [7] and the free vibration of transversely isotropic piezoelectric circular panels was analyzed by Ding [8]. Soldatos and Hadhgeorgian [9] have studied the frequency of vibration in isotropic cylindrical shell and panel by iterative approach. Free vibration of composite cylindrical panels with random material properties have been developed by Sing [10], in this work they have modeled the mechanical properties of laminated composite cylindrical panel on its natural frequencies as random variables. Zhang [11] have showed a wave propagation method to analyze the frequency of cylindrical panels. Loy and Lam [12] have studied on the vibrations of thin cylindrical panels of simply supported boundary conditions with Flugge's theory and also investigated the vibration of rotating cylindrical panel. Three dimensional solution of a simply supported rectangular hybrid plate has been obtained by Kapuria [13]. Natural frequency of a cylindrical panel on a Kerr foundation has been analyzed by Chen [14]. Free vibrations of thin cylindrical shells having finite lengths with freely supported and clamped edges have been discussed by Yu and Syracuse [15]. The static and dynamic analysis of plates supported on elastic foundations (see Ref. [16]) is an interesting problem in engineering. The dynamic response of isotropic cylindrical shell buried at a depth below the free surface of the ground from the point of view of the three-dimensional elastic theory have given by Wong et al [17]. Paliwal et al [18] presented a clear investigation on the coupled free vibrations of isotropic circular cylindrical shell on Winkler and Pasternak foundations by employing a membrane theory. Upadhyay and Mishra [19] have studied the non-axisymmetric dynamic behavior of buried orthotropic cylindrical shells excited by a combination of P-wave, SV- and SH- waves.

Classical plate theory really develops after the pioneering work of Kirchoff. After that thousands of publications are presented which try to give the foundations and methods of deduction of Kirchoff-Love theory and its possible improvements. Books of Ciarlet [20, 21] can be mentioned in this context. This two dimensional linear model which are in fact, the two-dimensional approximation of three dimensional theories of elastic plate involves a priori assumptions regarding the variations of unknowns (i.e. displacements and the stresses) across the thickness of the plates. The assumptions on which the theory of small deflection of thin elastic plate is based can be found in [22]. Another method which has been used to obtain two dimensional model of thin elastic plates is the so-called asymptotic expansion method. In this method, a formal power series expansion of three-dimensional solution is used by considering the thickness of the plate as the small parameter and the Kirchoff model of linear elastic isotropic plates is obtained as the leading term of formal asymptotic expansion. The early works of Goldenveizer [23], Freidrich and Dressler [24] etc., are representative example of this approach. A rigorous mathematical reformulation of the asymptotic approach has been given by Ciarlet and Destuynder [25] in which the three-dimensional problem is posed in variational form and a functional framework is used. There is another popular classical theory by taking into account shear effect that is ignored in Kirchoff-Love plate theory, which is known Reissner-Mindlin plate theory.

In this present article we have concentrated on the arguments to analyze the free vibration of an isotropic thin cylindrical shell by asymptotic approach under cylindrical symmetry.

2 Problem Formulation

2.1 Basic equations and notations

Balance of linear momentum yields the following equation

$$t^{ij},_j + \rho(f^j - \ddot{u}^j) = 0 \tag{1}$$

Balance of moment of momentum yields

$$t^{ij} = t^{ji} \tag{2}$$

where ρ is the density; t^{ij}, f^i, u^i are the contra-variant component of sress-tensor, body-force density, displacement of the continuum along x_i-direction respectively, 'dot' denotes the differentiation with respect to time and 'comma' denotes the covariant differentiation with respect to x_i-co-ordinate $(i = 1,2,3)$.

In linear elasticity we have the following constitutive equation for isotropic body

$$t^{ij} = \lambda \varepsilon^{kk} g^{ij} + G(\varepsilon^{ij} + \varepsilon^{ji})$$
(3)

λ, G are the two elastic constant in classical theory for isotropic body and $\varepsilon^{ij} = u^i_{,j}$, g^{ij} is fundamental metric-tensor ; $i, j, k = 1,2,3$.

Here we use cylindrical co-ordinate and let $\hat{r}, \hat{\theta}, \hat{z}$ be the unit vector forming orthonormal triad at the point (r, θ, z). $t_{rr}, t_{\theta\theta}, t_{zz}, t_{rz}, t_{\theta z}, t_{r\theta}$ are the components of stress tensor and u_r, u_θ, u_z are the components of displacement vector with respect to unit orthonormal triad mentioned above and are given by

$$g_{11}t^{11} = t_{rr},\; g_{22}t^{22} = t_{\theta\theta}, g_{33}t^{33} = t_{zz}, \sqrt{g_{11}g_{22}}\,t^{12} = t_{r\theta}, \sqrt{g_{33}g_{22}}\,t^{32} = t_{z\theta}, \sqrt{g_{11}g_{33}}\,t^{13} = t_{rz}$$

and

$$\sqrt{g_{11}}u^1 = u_r, \sqrt{g_{22}}\,u^2 = u_r, \sqrt{g_{33}}u^3 = u_z.$$
(4)

For cylindrical co-ordinate

$$x^1 = r,\; x^2 = \theta,\; x^3 = z$$

$g_{11} = 1, g_{22} = r^2, g_{33} = 1$ and

$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -r, \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{r}$ are the only non-zero Christoffel symbol of second kind in cylindrical co-ordinate system.

2.2 Mathematical formulation:

We consider here a thin right circular cylindrical shell and use cylindrical co-ordinate system under which a point in space is represented by (r, θ, z), where r, θ, z have usual meaning, to measure its deformation.

Let the shell is existing above the plane $z = 0$ and is infinitely long along z-direction with one end lying at $z = 0$ plane. It occupies the region between $r = a$ and $r = b$, where $b > a$ and $b - a$ is small. We take z-axes along the axes of the cylinder and want to discuss its vibration under axial symmetry.

We introduce a new variable R defined by

$$R = \frac{2r - (a+b)}{b - a}$$

Therefore, as r runs over a to b; R runs over -1 to 1.

Now we make the following assumption

$$u^i = \sum_{n=0}^{N} u^i_{(n)}(z,t) P_n(R)$$
(5.a)

$$t^{kl} = \sum_{n=0}^{N} [t^{kl}_{(n)}(z,t)/I_n] P_n(R)$$
(5.b)

where $i, k, l = 1,2,3;$ N is some natural number denotes order of the theory in asymptotic approach, $P_n(R)$ denotes the Legendre polynomial of degree n and $I_n = 2/(2n+1)$ .we further assume that the thickness of the shell, i.e. $b - a$ does not change due to deformation and for that we must have $u^1_{(n)} = 0$ if n is odd.

We know that $\{P_n(R) : n \in \mathrm{N}\}$ forms a orthogonal basis of the inner-product space of ℓ^2-measurable function in [-1, 1] with the inner-product which is defined by

$$\langle f \,|\, g \rangle = \int_{-1}^{1} fg\, dx,$$

where $f, g \in \ell^2[-1,1]$. It is well-known that such space is Hilbert-space and any function $f \in \ell^2[-1,1]$ can be expressed as $\sum_{n=0}^{\infty} c_n P_n(R)$, where $c_n = \langle f \,|\, P_n(R) \rangle / I_n$ with $I_n = \|P_n\|^2$. c_n is called the co-ordinate of f in L^2-space with respect to the basis $\{P_n(R) : n \in \mathrm{N}\}$.

Now we can reduce the problem to a one dimensional problem by usage of integration theory systematically.

Following the asymptotic theory we assume that it is possible to choose N, so that dependence of $u_{(n)}^{k}, t_{(n)}^{kl}$ on $(b-a)$ is ignorable. Since $(b-a)$ is small we can ignore $(b-a)^m$ for some natural number m and in that case we choose $N = m.$

Taking the body-force $f^{j} = 0$, equation (1) can be written in the following manner:

$$\frac{\partial t^{ij}}{\partial x^{i}} + \begin{Bmatrix} j \\ ik \end{Bmatrix} t^{ik} + \begin{Bmatrix} i \\ ik \end{Bmatrix} t^{kj} - \rho \ddot{u}^{j} = 0$$

or, $$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{i}} \left\{ \sqrt{g} t^{ij} \right\} + \begin{Bmatrix} j \\ ik \end{Bmatrix} t^{ik} - \rho \ddot{u}^{j} = 0,$$

where $\sqrt{g} = \det\left(g^{ij}\right)$

Now from the above equation we get

$$\left\langle \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{i}} \left\{ \sqrt{g} t^{ij} \right\} + \begin{Bmatrix} j \\ ik \end{Bmatrix} t^{ik} - \rho \ddot{u}^{j} \middle| P_n(R) \right\rangle = 0$$

Here, $g = r^2$, $\partial/\partial x^2 = \partial/\partial\theta = 0$ [due to cylindrical symmetry], $i, j, k = 1,2,3.$

Using (5.a), (5.b) and by making suitable approximation we get from the above equation

$$\frac{\partial t_{(n)}^{3j}}{\partial z} + \overline{\begin{Bmatrix} j \\ ik \end{Bmatrix}} t_{(n)}^{ik} + \int_{-1}^{+1} \frac{1}{r} \frac{\partial}{\partial r} \left[r\, t_{(n)}^{1j} \right] P_n(R)\, dR - I_n \rho \ddot{u}_{(n)}^{j} = 0 \tag{6}$$

Replacing r by $\frac{b+a}{2}$ in the expression of $\begin{Bmatrix} j \\ ik \end{Bmatrix}$, we get $\overline{\begin{Bmatrix} j \\ ik \end{Bmatrix}}$.

Now we consider the following expression

$$\int_{-1}^{+1} \frac{1}{r} \frac{\partial}{\partial r} \left[r\, t^{1j} \right] P_n(R)\, dR$$

$$= \frac{dR}{dr} \left[t^{1j} P_n(R) \right]_{-1}^{+1} - \frac{dR}{dr} \int_{-1}^{+1} r t^{1j} \frac{d}{dR} \left[\frac{P_n(R)}{r} \right]$$

$$\approx \frac{dR}{dr} \left[t^{1j} P_n(R) \right]_{-1}^{+1} - 2\frac{dR}{dr} \sum_{i \geq 0}^{n-2i1\geq 1} \bar{t}_{(n-2i-1)}^{1j} + \frac{2}{b+a} t_{(n)}^{1j}$$

[Replacing r by $(b+a)/2$ we get]

$$= -\frac{4}{b-a} \sum_{i \geq 0}^{n-2i1\geq 1} \bar{t}_{(n-2i-1)}^{1j} + \frac{2}{b+a} t_{(n)}^{1j},$$ where $\bar{t}_{(n.)}^{1j} = t_{(n)}^{1j} / I_n$.

Consider the boundary surface $R = \pm 1$, is free of traction.

Now using this we get from (6)

$$\frac{\partial t_{(n)}^{3j}}{\partial z} + \overline{\begin{Bmatrix} j \\ ik \end{Bmatrix}} t_{(n)}^{ik} - \frac{4}{b-a} \sum_{i \geq 0}^{n-2i1\geq 1} \bar{t}_{(n-2i-1)}^{1j} + \frac{2}{b+a} t_{(n)}^{1j} - I_n \rho \ddot{u}_{(n)}^{j} = 0$$

Now from our assumption that $t_{(n)}^{ij}, u_{(n)}^{i}$ does not depend on $b-a$, we get

$$\frac{\partial t_{(n)}^{3j}}{\partial z} + \overline{\begin{Bmatrix} j \\ ik \end{Bmatrix}} t_{(n)}^{ik} + \frac{2}{b+a} t_{(n)}^{1j} - I_n \rho \ddot{u}_{(n)}^{j} = 0$$
(7)

$i, j, k \in \{1,2,3\}.$

From Equation (7) it is to be noted that the motion in n-th mode does not depend on higher or lower modes and variants of each mode satisfies same set of equations.

From general plate theory we assume

$$t^{11} = 0,$$

$$or, \quad \varepsilon_{11} = -\lambda/(\lambda + 2G)\varepsilon_{KK}\,; \;\; K = 2,3$$

From equations (3), (5.a) and (5.b) we obtain

$$t_{(n)}^{KL} = g^{KL} \frac{E\nu}{1-\nu^2} I_n u_{(n)}^{3},_3 + G I_n \left[g^{LL} u_{(n)}^{K},_L + g^{KK} u_{(n)}^{L},_K \right]$$
,

$$t_{(n)}^{L1} = GI_n\left[g^{LL}u_{(n)}^1,_L + g^{11}u_{(n)}^L,_1\right] + \frac{4}{(b-a)}\sum_{i\geq 0}^{n+2i+1\leq N} u_{(n+2i+1)}^L \tag{8}$$

where
$\lambda = E/(1-2\nu)(1+\nu), E/2(1+\nu) = G; K, L = 2,3;$ E, ν are called Young modulus and Poisson's ratio respectively.

From our assumption we should have the expression of $t_{(n)}^{L1}$ which is independent of the terms involving $(b-a)$.

So we assume

$$\sum_{i\geq 0}^{n+2i+1\leq N} u_{(n+2i+1)}^L = 0. \tag{9}$$

From (7), (8) and (9) we get

$$\left(G\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2} - 2\right)u_{(n)}^1 - \frac{E\nu}{1-\nu^2}\frac{2}{b+a}\frac{\partial}{\partial z}u_{(n)}^3 = 0 \tag{10}$$

$$\left(G\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)u_{(n)}^2 = 0 \tag{11}$$

$$\left(\frac{E\nu}{1-\nu^2}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)u_{(n)}^3 + \frac{2G}{b+a}\frac{\partial}{\partial z}u_{(n)}^1 = 0 \tag{12}$$

From (10), (11) and (12) it is seen that the motion of torsion (u^2) of the cylinder is independent of longitudinal (u^3) and radial (u^1) motion but longitudinal and radial motion are coupled.

Consider $(b+a)$ is large enough to ignore the term $\frac{1}{b+a}\frac{\partial}{\partial z}u_{(n)}^1$ and then equation (12) is reduced to

$$\left(\frac{E\nu}{1-\nu^2}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)u_{(n)}^3 = 0 \tag{13}$$

3 Analytical discussion of solution under certain consideration:

From (11) it is seen that $u_{(n)}^2$ satisfies a classical wave equation propagating along z-axes with a wave speed $c_0 = \sqrt{G/\rho}$.

Now we are going to solve (10) and (13) to construct a relation between radial and longitudinal disturbance.

Let us consider

$$u_{(n)}^1\Big|_{t=0} = \dot{u}_{(n)}^1\Big|_{t=0} = 0. \tag{14}$$

Applying Laplace transform $L[t \to s]$ to equation (10) we get

$$\left[G\frac{d^2}{dz^2} - \left(\rho s^2 + 2\right)\right]\overline{u}_{(n)}^1 = \overline{F}_n(z,s)$$

$$\text{or, } \left[\frac{d^2}{dz^2} - \left(s^2 + \frac{2}{\rho}\right)\frac{1}{c_0^2}\right]\overline{u}_{(n)}^1 = \overline{F}_n(z,s)$$

with boundary condition, (15)

$$u_{(n)}^1\Big|_{z=0} = 0$$

$$u_{(n)}^1\Big|_{z\to\infty} = 0.$$

Where,

$$\overline{u}_{(n)}^1(z,s) = L\left[u_{(n)}^1(z,t); t \to s\right], \tag{15.a}$$

$$F_n(z,t) = \frac{2}{(b+a)}\frac{E\nu}{1-\nu^2}\frac{\partial u_{(n)}^3}{\partial z},$$
(15.b)

$$\overline{F}_n(z,s) = L[F_n(z,t); t \to s].$$
(15.c)

Now to solve (15) which is a non-homogeneous equation, we use Duhamel's construction of solution of non-homogeneous differential equation, which is known as Duhamel's principle.

We expect the solution of (15) can be given by

$$\overline{u}_{(n)}^1 = \int_0^z \upsilon_n(z;\xi,s)\,d\xi,$$
(16)

where for each ξ satisfying $0 < \xi < z$, υ_n satisfies the equation,

$$\left[\frac{d^2}{dz^2} - \left(s^2 + \frac{2}{\rho}\right)\frac{1}{c_0^2}\right]\upsilon_n = 0,$$

$$\frac{\partial}{\partial z}\upsilon_n(z;\xi,s)\Big|_{z=\xi} = \overline{\psi}_n(\xi,s);\ \upsilon_n(z;\xi,s)\Big|_{z\to\infty} = 0.$$
(17)

Solution of (16) is given by

$$\upsilon_n(z;\xi,s) = -c_0\overline{\psi}_n(\xi,s)\left(s^2 + \frac{2}{\rho}\right)^{-1/2} e^{-\sqrt{s^2+\frac{2}{\rho}}\,\frac{z-\xi}{c_0}}$$
(18)

From (17) we can say $\upsilon_n(z;z,s)\big|_{z\to\infty} = 0$ and for that we must have

$$\psi_n(z,s)\big|_{z\to\infty} = 0$$
(19)

From (16) we get

$$\left[\frac{d^2}{dz^2} - \left(s^2+\frac{2}{\rho}\right)\frac{1}{c_0^2}\right]\overline{u}_{(n)}^1 = \upsilon_n(z;z,s) + \frac{d}{dz}[\upsilon_n(z;z,s)] + \int_0^z\left[\frac{d^2}{dz^2} - \left(s^2+\frac{2}{\rho}\right)\frac{1}{c_0^2}\right]\upsilon_n(z;\xi,s)\,d\xi$$

$$= \upsilon_n(z;z,s) + \frac{d}{dz}[\upsilon_n(z;z,s)]$$

$$= \overline{\psi}_n(z,s)\left[1 - c_0\left(s^2 + \frac{2}{\rho}\right)^{-1/2}\right]$$ [By using equation (18)]

From the above equation we can say $\overline{u}_{(n)}^1$ can be a solution of (15) if

$$\overline{\psi}_n(z,s)\left[1 - c_0\left(s^2+\frac{2}{\rho}\right)^{-1/2}\right] = \overline{F}_n(z,s)$$
(20)

Let , $\psi_n(z,t) = L^{-1}[\overline{\psi}_n(z,s); s \to t]$

Then by applying Laplacian inverse on (19) we obtain the following integral equation:

$$\psi_n(z,t) = F_n(z,t) + c_0\int_0^t \psi_n(z,\tau) J_0\left[\sqrt{2/\rho}\,(t-\tau)\right]d\tau \tag{21}$$

where J_0 denotes Bessel's function of 1st kind of order zero.

From (19) we get

$$F_n(z,t)\big|_{z\to\infty} = 0$$
(22)

Equation (21) is a Voltera type integral equation and solution by iterative scheme is given in the following:

$$\psi_n(z,t) = F_n(z,t) + \sum_{m=0}^{\infty} c_0^m \int_0^t F_n^m(z,\tau) J_0\left[\sqrt{2/\rho}\,(t-\tau)\right]d\tau, \tag{23}$$

where,

$$F_n^0 = F_n,$$

$$F_n^m = \int_0^t F_n^{m-1}(z,\tau) J_0\left[\sqrt{2/\rho}\,(t-\tau)\right] d\tau.$$

From equations (16) and (18) we get

$$\bar{u}_n^1(z,s) = -c_0 \int_0^z \bar{\psi}_n(\xi,s)\left(s^2+\frac{2}{\rho}\right)^{-1/2} e^{-\sqrt{s^2+\frac{2}{\rho}}\frac{z-\xi}{c_0}} d\xi$$

Fortunately we have the following tabulated result

$$L^{-1}\left[\frac{e^{-b\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}; s\to t\right] = J_0\left(a\sqrt{t^2-b^2}\right)H(t-b)$$

Using this result we finally get

$$u_n^1(z,t) = -c_0\int_{\xi=0}^{z}\left\{\int_{\tau=0}^{t-\frac{z-\xi}{c_0}} \psi_n(\xi,\tau) J_0\left(\sqrt{\frac{2}{\rho}\left[(t-\tau)^2-\left(\frac{z-\xi}{c_0}\right)^2\right]}\right) d\tau\right\} d\xi, \quad if \quad t>\frac{z}{c_0},$$

$$= -c_0\int_{\xi=z-c_0 t}^{z}\left\{\int_{\tau=0}^{t-\frac{z-\xi}{c_0}} \psi_n(\xi,\tau) J_0\left(\sqrt{\frac{2}{\rho}\left[(t-\tau)^2-\left(\frac{z-\xi}{c_0}\right)^2\right]}\right) d\tau\right\} d\xi, \quad if \quad t<\frac{z}{c_0}.$$

(24)

Now we solve (13) subject to the following initial and boundary conditions:

$$u_{(n)}^3(z,0)=0;\; u_{(n)}^3(0,t)= f_n^+(t); \quad \partial u_{(n)}^3(0,t)/\partial z = g_n^+(t)$$

. (25)

From (22) we get another condition

$$\left.\frac{\partial u_{(n)}^3}{\partial z}\right|_{z\to\infty} = 0$$

(26)

Now solution of (13) satisfying (25) and (26) can be given by

$$u_{(n)}^3(z,t) = \frac{c}{2}\int_{t-\frac{z}{c}}^{t+\frac{z}{c}} g_n(\tau)d\tau + \frac{1}{2}\left[f_n\left(t+\frac{z}{c}\right)+f_n\left(t-\frac{z}{c}\right)\right],$$

(27)

In which

$c = \sqrt{E\nu/\rho(1-\nu^2)}$,

$f_n : [-\infty,\infty]\to[-\infty,\infty]$ and $g_n : [-\infty,\infty]\to[-\infty,\infty]$ is defined as follows,

$$f_n(t) = f_n^+(t), \quad if \quad t>0,$$
$$= -f_n^+(-t) \quad if \quad t<0.$$

$$g_n(t) = g_n^+(t), \quad if \quad t>0,$$
$$= -g_n^+(-t) \quad if \quad t<0.$$

To satisfy the equation (26) we need

$$\left. f_n^+(t)\right|_{t\to\infty} = 0,$$

$$\left. g_n^+(t)\right|_{t\to\infty} = 0.$$

So we have

$$F_n(z,t) = \frac{2}{(b+a)}\frac{E\nu}{1-\nu^2}\frac{\partial u_{(n)}^3}{\partial z}$$
$$= \frac{1}{(b+a)}\frac{E\nu}{1-\nu^2}\left[\left\{g_n\left(t+\frac{z}{c}\right)+g_n\left(t-\frac{z}{c}\right)\right\}+\frac{1}{c}\left\{f_n'\left(t+\frac{z}{c}\right)-f_n'\left(t-\frac{z}{c}\right)\right\}\right]$$

(28)

Here we take f_n^+, g_n^+ in a way so that $f_n^+(t)/(b+a),\; g_n^+(t)/(b+a)$ are finite and can't be ignorable for all t, otherwise we obtain the trivial case where $F_n(z,t)\approx 0$,which implies $u_{(n)}^1 \approx 0$.

4. Conclusion:

We have discussed above the radial vibration of a thin isotropic semi-infinite cylindrical cell under cylindrical symmetry in asymptotic approach when the cell is disturbed longitudinally at one end. Equation (24) gives the expression of $u^{1}_{(n)}(z,t)$ in terms of $\psi_n(z,t)$. Equation (23) gives the expression for $\psi_n(z,t)$ in terms of $F_n(z,t)$. $F_n(z,t)$ is defined in (15.b) and it is given explicitly in terms of known function $f_n^+(t)$, $g_n^+(t)$ which are the given longitudinal disturbance on boundary introduced in Equation (25).

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