Avoidance of numerical singularities in free vibration analysis of Euler-Bernoulli beams using Green functions

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Abstract: This paper investigates the reliability of an algorithm that implements the Green function method in free vibration analysis of Euler-Bernoulli beams. The investigation is concerned with the robustness of the algorithm with respect to the occurrence of numerical singularities in the calculation procedure of mode shapes. The problem is studied for beams supported with an arbitrary number of intermediate translational springs, which can be understood as a generalization of the cases when the beam is without elastic supports and when the beam rests on intermediate rigid supports. The problem of numerical singularities arises from the fact that the elements of the modal vector have to be expressed in terms of an "arbitrarily" chosen referential element of that vector, whose value can vanish if it coincides closely enough with a node of the sought mode shape. The problem is generally tackled here with the introduction of a fictitious spring of a vanishingly small stiffness, and the robustness of the algorithm depends crucially on the appropriate placement of that spring. This paper presents several useful guidelines for the implementation of computer code based on these principles and its reliability is demonstrated through examples.

Key-Words: numerical singularity, Green functions, free vibrations, Euler-Bernoulli beam, spring support

1 Introduction

The methods of deriving closed form expressions for the frequency equation and the mode shape equation of vibrating beams are a well-studied field of research. For Euler-Bernoulli beam with various boundary conditions these analytical solutions can be found in standard textbooks [1], and their computational implementation generally does not entail any complications regarding accuracy or sensitivity to the input parameters. However, in more complex cases when the beam carries any kind of intermediate elements and restraints (springs, masses, dampers etc.), the mathematical complexity rises progressively with each element added into the model. This mathematical complexity is, in turn, often accompanied by algorithmic complexity which is required to implement such mathematical models.

This paper is concerned with the method of Green functions, whose primary weakness is the inability to produce some mode shapes when inappropriately implemented. We shall demonstrate that this problem can reliably be overcome with the introduction of a conveniently placed fictitious spring of negligible stiffness. By doing so, the full potential of the Green function method can be exploited practically without restrictions; with the two most prominent advantages of the Green function method being straightforward extension from free vibration to forced vibration analysis and the relative ease of incorporating point applied loads and/or restrictions into the model. For the purpose of this study, the beam is assumed to be supported with an arbitrary number of translational springs, which is a good approximation in a number of real structural problems.

A good general insight into the application of Green functions in beam vibration analysis can, for example, be found in [2-7], and this paper is a continuation of the work presented by the authors in [8]. This paper is structured as follows. Chapter 2 presents the mathematical model of the stated problem, and the analysis of algorithmic robustness of the presented approach and problems encountered in some characteristic cases is given in Chapter 3. Chapter 4 summarizes the results with concluding remarks.

2 Mathematical model

The subject under investigation is a beam (Fig. 1) supported with an arbitrary number of translational springs. In this case, the transverse displacement can be obtained from the differential equation:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}+k(x)\,w(x,t)=0\,, \qquad (1)$$

where $w(x,t)$ is the deflection, EI is flexural rigidity, ρ is the density and A is the cross-sectional area of the beam. The support stiffness at position x is designated as $k(x)$. If the beam is supported with N linear springs (at locations x_n, $n = 1\ldots N$), it can be written

$$k(x)=\sum_{n=1}^{N}k_n\delta(x-x_n), \qquad (2)$$

where k_n is the stiffness of the n-th spring, and δ is the Dirac delta function.

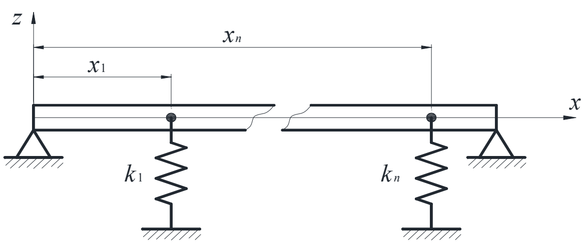


Fig. 1. A beam pinned at both ends supported with translational springs

2.1 Green functions

The natural frequencies ω and the mode shapes $W(x)$ of the analysed beam are determined from the homogeneous solution of the differential equation (1). Using the method of separation of variables, the solution is assumed in the following form:

$$w(x,t)=W(x)e^{i\omega t}\,. \qquad (3)$$

After substituting (3) into (1) and introducing the nondimensional variables with respect to the beam length L:

$$\xi=\frac{x}{L}\,,\ \xi_n=\frac{x_n}{L}\ \text{and}\ \tilde{W}=\frac{W}{L}\,, \qquad (4)$$

the left hand side of equation (1) is rewritten as:

$$\frac{\mathrm{d}^4\tilde{W}(\xi)}{\mathrm{d}\xi^4}-\varepsilon^4\tilde{W}(\xi)=-\tilde{W}(\xi)\sum_{n=1}^{N}K_n\,\delta(\xi-\xi_n)\,. \qquad (5)$$

Here, ε is the nondimensional natural frequency and K_n is the nondimensional stiffness, defined as:

$$\varepsilon^4=\rho A\omega^2L^4\big/EI\,, \qquad (6)$$

$$K_n=k_nL^3\big/EI\,. \qquad (7)$$

Moreover, by using the standard feature of the Dirac function:

$$\tilde{W}(\xi)\delta(\xi-\xi_n)=\begin{cases}\tilde{W}(\xi_n)\delta(\xi-\xi_n), & \xi=\xi_n\\ 0, & \xi\neq\xi_n,\end{cases} \qquad (8)$$

the differential equation can be written as:

$$\frac{\mathrm{d}^4\tilde{W}(\xi)}{\mathrm{d}\xi^4}-\varepsilon^4\tilde{W}(\xi)=-\sum_{n=1}^{N}\tilde{W}(\xi_n)K_n\,\delta(\xi-\xi_n)\,. \qquad (9)$$

The Green function method is applied here to solve the differential equation in (9). In the first step, the Green function $G(\xi,\xi_n)$ must be introduced, which by definition satisfies the differential equation when the forcing term is equal to the Dirac delta function, that is:

$$\frac{\mathrm{d}^4G(\xi,\xi_n)}{\mathrm{d}\xi^4}-\varepsilon^4G(\xi,\xi_n)=\delta(\xi-\xi_n)\,. \qquad (10)$$

Secondly, the Green function for a particular beam is obtained by solving equation (10) in conjunction with the set boundary conditions. The Green function for a pinned-pinned boundary condition can be shown to be:

$$G(\xi,\xi_n)=\frac{a}{2\varepsilon^3b}+\frac{1}{2\varepsilon^3}\big[\sinh(\varepsilon\xi-\varepsilon\xi_n)-\\ -\sin(\varepsilon\xi-\varepsilon\xi_n)\big]H(\xi-\xi_n), \qquad (11)$$

with

$$a=\sin(\varepsilon-\varepsilon\xi_n)\sin(\varepsilon\xi)\sinh(\varepsilon)-\\ -\sinh(\varepsilon-\varepsilon\xi_n)\sinh(\varepsilon\xi)\sin(\varepsilon),$$

$$b=\sin(\varepsilon)\sinh(\varepsilon).$$

Here, $H(\cdot)$ is the Heaviside function. For more details the reader is referred to [8].

2.2 Frequency equation

The solution of (9) can now be written as:

$$\tilde{W}(\xi)=-\sum_{n=1}^{N}\tilde{W}(\xi_n)K_nG(\xi,\xi_n)\,. \qquad (12)$$

In case when the beam is supported with one spring equation (12) is evaluated at position ξ_1 as:

$$\big[K_1G(\xi_1,\xi_1)+1\big]\tilde{W}(\xi_1)=0\,. \qquad (13)$$

In case when the beam is supported with two springs at positions ξ_1 and ξ_2, equation (12) is evaluated at ξ_1 and ξ_2, yielding:

$$\begin{aligned}\tilde{W}(\xi_1) &= -\tilde{W}(\xi_1)K_1G(\xi_1,\xi_1) - \tilde{W}(\xi_2)K_2G(\xi_1,\xi_2)\\ \tilde{W}(\xi_2) &= -\tilde{W}(\xi_1)K_1G(\xi_2,\xi_1) - \tilde{W}(\xi_2)K_2G(\xi_2,\xi_2)\end{aligned}, \quad (14)$$

or in matrix form:

$$\begin{bmatrix} K_1G(\xi_1,\xi_1)+1 & K_2G(\xi_1,\xi_2) \\ K_1G(\xi_2,\xi_1) & K_2G(\xi_2,\xi_2)+1 \end{bmatrix} \begin{bmatrix} \tilde{W}(\xi_1) \\ \tilde{W}(\xi_2) \end{bmatrix} = 0. \quad (15)$$

Generally speaking, by evaluating equation (12) for N spring positions ($\xi = \xi_m$, $m = 1 \ldots N$) a system of N linear equations is obtained, which can be assembled in matrix form as

$$\mathbf{KW} = 0, \quad (16)$$

where the mode shape vector $\mathbf{W}$ is defined as:

$$\mathbf{W} = \begin{bmatrix} \tilde{W}(\xi_1) \\ \tilde{W}(\xi_2) \\ \ldots \\ \tilde{W}(\xi_N) \end{bmatrix}. \quad (17)$$

The necessary condition for a nontrivial solution of the vector $\mathbf{W}$ is:

$$|\mathbf{K}| = 0. \quad (18)$$

This equation is known as the frequency equation. Its solutions are nondimensional frequencies ε, from which the natural frequencies are obtained as

$$\omega = \varepsilon^2 \sqrt{\frac{EI}{\rho A L^4}}. \quad (19)$$

Comparing expressions (13) and (15) with (16) and (18), the frequency equations for a beam supported with one and two springs are obtained as:

$$K_1G(\xi_1,\xi_1) + 1 = 0, \quad (20)$$

$$\begin{vmatrix} K_1G(\xi_1,\xi_1)+1 & K_2G(\xi_1,\xi_2) \\ K_1G(\xi_2,\xi_1) & K_2G(\xi_2,\xi_2)+1 \end{vmatrix} = 0. \quad (21)$$

2.3 Mode shape equation

For each of the calculated nondimensional natural frequencies ε, the mode shape can now be obtained from (12). The first step is to choose one element of the mode shape vector, for example $\tilde{W}(\xi_1)$, as the referential one and to express the remaining elements relative to this reference. From (12) this now yields:

$$\frac{\tilde{W}(\xi)}{\tilde{W}(\xi_1)} = -K_1G(\xi,\xi_1) - \sum_{n=2}^{N} \frac{\tilde{W}(\xi_n)}{\tilde{W}(\xi_1)} K_nG(\xi,\xi_n). \quad (22)$$

Afterwards, this equation is evaluated for all spring positions ($\xi = \xi_m$) and a system of $N-1$ linear equations is obtained, which is assembled in matrix form as:

$$\mathbf{K}_1\mathbf{W}_1 = \mathbf{R}, \quad (23)$$

where:

$$\mathbf{W}_1 = \begin{bmatrix} \tilde{W}(\xi_2) \\ \tilde{W}(\xi_3) \\ \ldots \\ \tilde{W}(\xi_N) \end{bmatrix}, \quad \mathbf{R} = \tilde{W}(\xi_1) \begin{bmatrix} -K_1G(\xi_1,\xi_1)-1 \\ -K_1G(\xi_2,\xi_1) \\ \ldots \\ -K_1G(\xi_N,\xi_1) \end{bmatrix}. \quad (24)$$

The matrix $\mathbf{K}_1$ is obtained by removing the first column and the last row from the matrix $\mathbf{K}$. The solutions of (23) are all members of the vector $\mathbf{W}_1$, expressed relative to the chosen one, in this case $\tilde{W}(\xi_1)$. For example, for $N = 2$, from equation (23) it follows:

$$\tilde{W}(\xi_2) = -\tilde{W}(\xi_1)\frac{K_1G(\xi_1,\xi_1)+1}{K_2G(\xi_1,\xi_2)}. \quad (25)$$

The mode shape equations for a beam supported with one and two springs are now obtained as:

$$\frac{\tilde{W}(\xi)}{\tilde{W}(\xi_1)} = -K_1G(\xi,\xi_1), \quad (26)$$

$$\frac{\tilde{W}(\xi)}{\tilde{W}(\xi_1)} = -K_1G(\xi,\xi_1) + \frac{K_1G(\xi_1,\xi_1)+1}{G(\xi_1,\xi_2)}G(\xi,\xi_2). \quad (27)$$

3 Model implementation, examples

To illustrate the presented theory and the problems associated with its implementation a pinned-pinned beam is studied, keeping in mind that the presented principles equally apply to other cases of boundary conditions. All examples presented in this paper are implemented in Matlab.

If the beam is supported with only one spring, the natural frequencies are obtained by solving the frequency equation (20), where the Green function G for a pinned-pinned beam is defined in (11). Following this, for $\xi < \xi_n$, it is obtained:

$$K_1\frac{a}{2\varepsilon^3 b} + 1 = 0. \quad (28)$$

In order to speed up numerical calculations, this equation is rewritten in the numerator/denominator form and the zeros are now obtained only for the numerator

$$f(\varepsilon) = K_1a + 2\varepsilon^3 b = 0, \quad (29)$$

which correspond to the zeros of equation (28). For example, if we take $K_1 = 1200$ and $\xi_1 = 0.1$ the natural frequencies are obtained from (28) as presented in Fig. 2 and for $K_1 = 1200$ and $\xi_1 = 0.1$ and $\xi_1 = 0.5$ from (29) as given in Fig. 3.

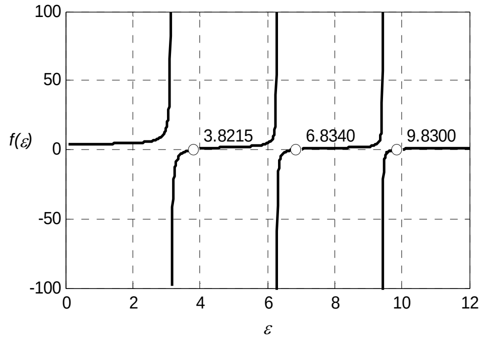
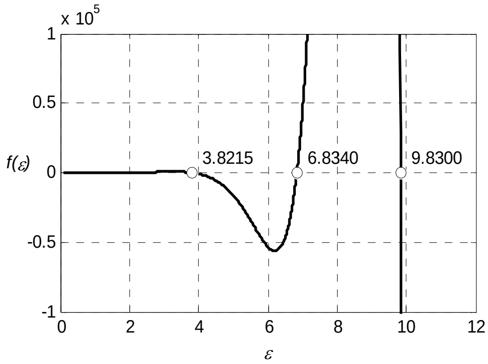


Fig. 2. Zeros of the frequency equation as defined in (28) for $K_1 = 1200$ and $\xi_1 = 0.1$

(a)



(b)

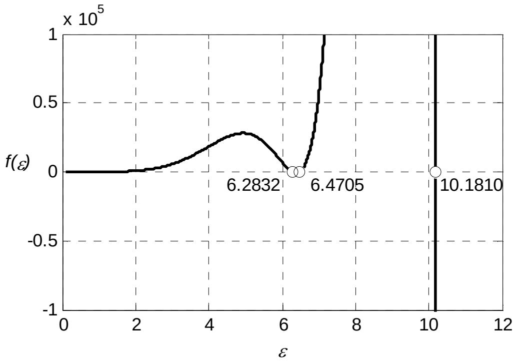


Fig. 3. Zeros of the frequency equation as defined in (29) for $K_1 = 1200$ and a) $\xi_1 = 0.1$, b) $\xi_1 = 0.5$

A more informative diagram is obtained if Eq. (29) is solved iteratively for various spring positions and stiffness values, Fig. 4. Here, each plotted curve shows the change of the non-dimensional natural frequency with respect to the position of the spring for a constant value of K_1.

Additional insight is obtained after plotting the nondimensional frequencies as functions of the non-dimensional stiffness while the position of the spring is held constant, as shown in Fig. 5. At $\xi_1 = 0.5$ the critical value of K_1 is found numerically as $K_{1\mathrm{cr}} = 996$. For $K_1 > K_{1\mathrm{cr}}$, no further increase in ε_1 is gained with increased stiffness. Hence, the original second mode becomes the new first mode. The same phenomenon can be seen in Fig. 4 at $\xi_1 = 0.5$, where for $K_1 > K_{1\mathrm{cr}}$ the curves related to the first mode shape touch the curves related to the second mode shape.

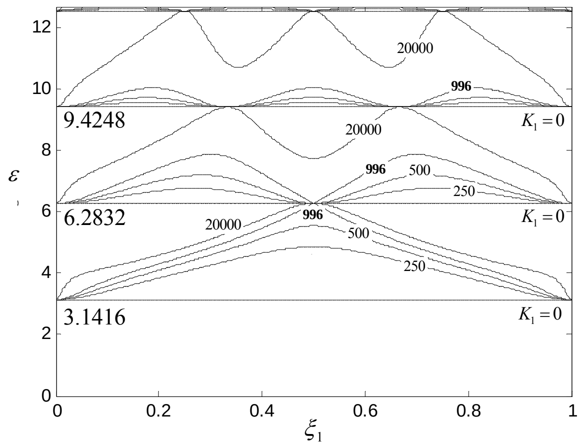
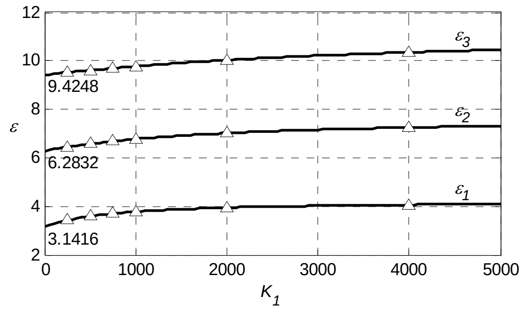


Fig. 4. First three values of ε for a pinned-pinned beam for different values of K_1 and ξ_1; the results are plotted for five values of K_1; 0, 250, 500, 996, 20000.

(a)



(b)

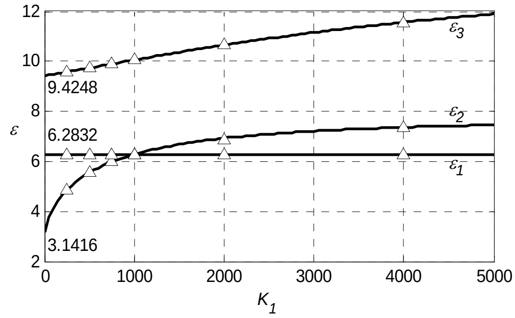


Fig. 5. Dependence of nondimensional frequencies ε_1, ε_2 and ε_3 on K_1 for pinned-pinned beam supported with a single spring at a) $\xi_1 = 0.1$, b) $\xi_1 = 0.5$ (Δ–FEM results)

The mode shapes can be obtained from equation (26) relative to the deflection at the point of application of the first spring. However, the problem arises if this deflection approaches zero and the modal deflections calculated relative to such referential value become infinite. This can generally happen in two cases: (1) when the spring stiffness is too high, and

(2) if the mode shape has a zero at the point where the spring is located (regardless of the spring stiffness).

The problem of zero deflection at the location of the spring can be solved with the introduction of a fictitious spring of negligible stiffness ($K_0 \approx 0$) placed at a convenient position to the left of the position of the spring (i.e. $\xi_0 < \xi_1 = 0.5$). The chosen position must not be in the immediate vicinity of the expected modal zeros of the analysed mode shapes. The system is now modelled as a system with two springs, where the first spring is fictitious.

Fig. 6 presents mode shapes with spring stiffness $K_1 = 1000$ at $\xi_1 = 0.5$ and for two cases: (a) with no fictitious and (b) with a fictitious spring at $\xi_0 = 0.1$.

(a)

(b)

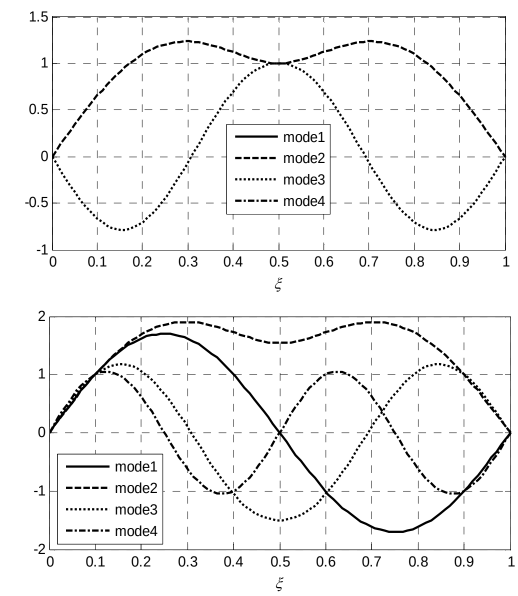


Fig. 6. Mode shapes for $K_1 = 1000$ at $\xi_1 = 0.5$; a) with no fictitious spring – the algorithm failed to produce the 1st and the 4th mode shape, b) with a fictitious spring at $\xi_0 = 0.1$

It can be observed that the algorithm failed to produce the 1st and the 4th mode shape, whose zeroes coincide with the position of the spring ($\xi_1 = 0.5$). In contrast, with the introduction of the fictitious spring all desired mode shapes are presented correctly. At the position of the real spring in Fig. 6a ($\xi_1 = 0.5$) as well as at the position of the fictitious spring in Fig. 6b ($\xi_0 = 0.1$) the modal amplitudes equal unity in all mode shapes. This happens because the modes are always scaled with respect to the displacements of the referential spring and this is in accordance with the definition of mode shapes in Eq. (22).

The mode shapes of a beam with spring stiffness $K_1 = 10^{40}$ at $\xi_1 = 0.5$ are shown in Fig. 7 for two cases: (a) with no fictitious spring and (b) with a fictitious spring at $\xi_0 = 0.2$.

(a)

(b)

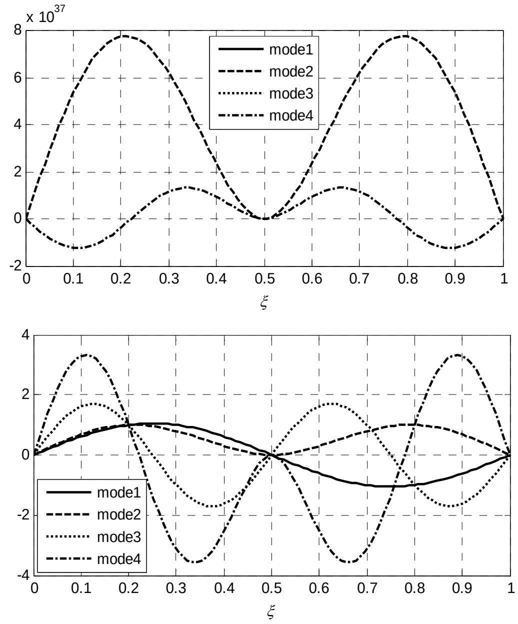


Fig. 7. Mode shapes with $K_1 = 10^{40}$ at $\xi_1 = 0.5$; a) with no fictitious spring – the algorithm failed to produce the 1st and the 3rd mode shape, b) with a fictitious spring at $\xi_0 = 0.2$

It must be noted that the presented approach does not rely on finding some universally applicable location of the fictitious spring, because that is impossible. An appropriate position can be deduced only in the context of the structural layout of the problem at hand and the number of modes that the analyst wishes to obtain. As the structural complexity rises, so does the possibility of unpredictable locations of nodes at higher order mode shapes, which might coincide with ξ_0.

A more convenient scaling of the mode shape amplitudes can be obtained with mass normalization, which enables the comparison with the finite element results, as shown in Fig. 8 for two positions $\xi_1 = 0.1$ and $\xi_1 = 0.5$, and both with $K_1 = 1000$. As expected these results do not depend on the position of the fictitious spring and a complete agreement with the results of the finite element analysis is observed. For that purpose all the required quantities had to be known, therefore a solid rectangular beam with the following geometrical and material properties was analysed: width 0.03 m, height 0.005 m, length 1 m, density $\rho = 7850$ kg/m^3 and the modulus of elasticity $E = 2.1 \cdot 10^{11}$ N/m^2.

If we assume that the beam is supported with two springs at positions ξ_1 and ξ_2, the frequencies of the beam can be obtained from the frequency equation (21), with the Green function G for a pinned-pinned beam defined in (11).

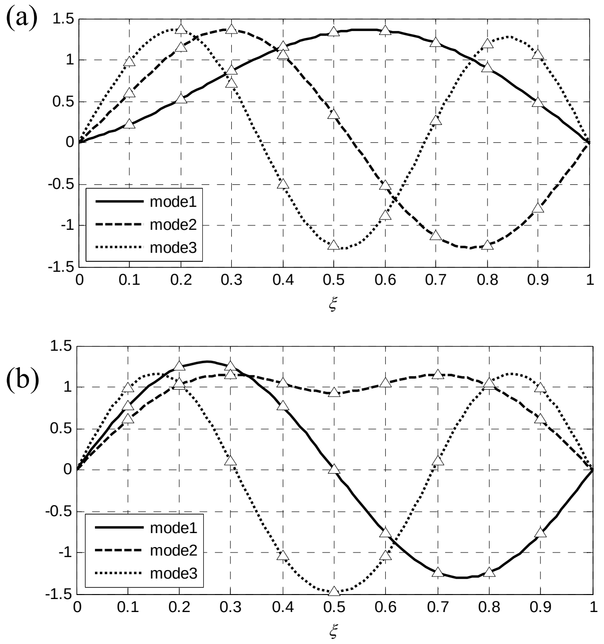


Fig. 8. Mass normalized mode shapes with spring stiffness K_1 = 1000 at position a) ξ_1 = 0.1, b) ξ_1 = 0.5

Fig. 9 presents the dependence of ε on spring stiffnesses K_1 and K_2 when the springs are located at ξ_1=1/3 and ξ_2=2/3 (at zero points of the third mode shape). The obtained surface has a plateau where an increase in either K_1 and/or K_2 no longer produces an increase in ε_1. This means that the original third mode becomes the new first mode.

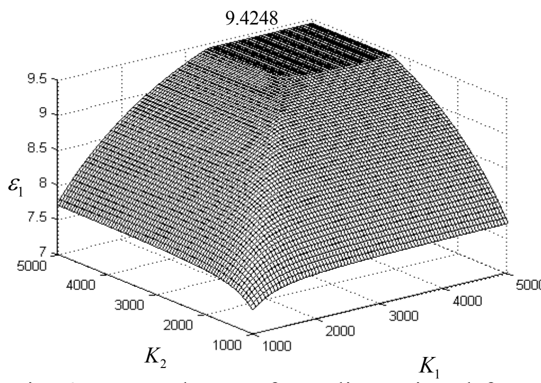


Fig. 9. Dependence of nondimensional frequency ε_1 on K_1 and K_2 for a pinned-pinned beam supported with two springs at ξ_1 = 1/3 and ξ_2 = 2/3

4 Conclusion

In this paper the robustness of the algorithm for free vibration analysis based on Green function method is investigated. The presented problem is especially concerned with numerical singularities which inevitably appear in the calculation procedure of mode shapes if the equations are formulated only based on the real structure. The problem is overcome with the introduction of a fictitious spring of a negligible stiffness which must always be positioned so as to avoid modal nodes. Some useful general guidelines on how to resolve these issues when implementing this method are demonstrated trough presented examples.

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