Upper bounds of transverse modulus for unidirectional fibrous composites by strength of materials approach

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Abstract: The objective of this work is to propose upper bounds of transverse elastic modulus for two – phase and three phase unidirectional fibrous composites according to strength of materials approach. To this end, the authors introduce at first a closed form expression the values of which are proved to be strictly greater than those arising from the well known Paul's formula for a two phase fibrous material. Next, the authors taking into account the interphase concept perform a theoretical formula for the transverse elastic modulus of three – phase unidirectional fibrous composites, yielding values strictly greater than those obtained from inverse mixtures law for a three phase unidirectional composite.
To verify the validity of the proposed expressions, their numerical results for various fiber contents were compared with theoretical values yielded by some reliable formulae derived from other workers, together with experimental values found in the literature and a reasonable agreement was observed.

Keywords: Fibrous composites, transverse modulus, upper bound, Paul's formula, interphase, inverse law of mixtures

1 Introduction

From the engineering viewpoint, a unidirectional fiber – reinforced composite material constitutes a fundamental structural member of composite structures with a v ariety of applications in building or naval engineering. In addition, according to mechanical standpoint the most trivial type of fiber reinforced material is an elastic one that consists of linearly elastic fibers and matrix. The investigation of the elastic properties of unidirectional fiber –reinforced materials on the basis of constituent elastic properties and the prediction of the elastic moduli is one of the main engineering problems that are still open.
In this context, several theoretical models have been appeared in the literature. In Ref. [1] Paul applied the principles of minimum energy and minimum complementary energy in order to designate the bounds on the elastic modulus of macroscopically isotropic two – phase fibrous composites. Yet, in Ref. [2] Hill derived the same bounds by the implementation of a different approach. Hashin and Rosen [3] achieved to constrain Paul's bounds to in order to result in more useful calculations of elastic moduli for isotropic heterogeneous materials. In this valuable work, an ideal model of random array of parallel hollow or solid fibers, is impacted in a matrix. This fundamental model of unidirectional fibrous composites is also known as composite cylinder assemblage model. In addition, Whitney and Riley [4] performed a work somewhat proportional to that of Hashin and Rosen, but less rigorous. Meanwhile, the fiber arrays have been extensively studied by Adams and Tsai [5]. In this considerable work, it was proved that the hexagonal array analysis agree better with experiments than to results of the square array analysis.
On the other hand, in Refs. [6,7] simplified expressions for the moduli were introduced, where different influential factors such as contiguity, fiber geometry, packing geometry and loading conditions have been taken into account. Further, Ekval [8] obtained a modification of Paul's lower bound in which the triaxial stress

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state in the matrix due to fiber restrained is accounted, whilst in Ref. [9] a valuable experimental study on the elastic constants of fibrous composites was performed.

Another considerable experimental investigation towards the estimation of the mechanical properties of fibrous composite materials was carried out by Clements and Moore [10].

On the other hand, in Ref. [11] an elasticity approach was made together with the interphase concept towards the estimation of the transverse modulus of unidirectional fibrous composites.

In the past years, there is a lot of recent research work carried out for the determination of elastic constants of unidirectional fibrous composites and for the investigation of the effect of many parameters such as filler – matrix interaction, adhesion efficiency, fiber arrangement and vicinity etc.

In Ref. [12] the influence of fiber packing on the elastic properties of a transversely random unidirectional glass/epoxy composite, was investigated, whereas Huang [13] gave a micromechanical prediction of ultimate strength of transversely isotropic fibrous composite materials.

Further, in Ref. [14] a micro – scale simulation and prediction of the mechanical properties of fibrous composites by means of the bridging micromechanics model was carried out, whilst for a thorough study on the effective properties of fibrous composite media of periodic structure, one may refer to Ref. [15].

In addition, Sideridis et al. [16] proposed strength of materials and elasticity approaches to evaluate the elastic constants of unidirectional three phase fibrous composites, by taking into consideration the concept of the boundary interphase. In above mentioned work, to approach the mode of variation of the variable interphase elastic properties an nth degree polynomial function with respect to interphase radius was initially considered, and for n = 2 it yielded a parabolic law.

Further, in Ref. [17] the strength properties of hybrid nylon-steel and polypropylene-steel fiber-reinforced high strength concrete at low volume fraction were examined, whilst the effect of size and stacking of glass fibers on the mechanical properties of the fiber – reinforced – mortars was investigated in Ref. [18].

In Ref. [19] the elastic constants of composites consisting of polymer matrix and unidirectional transversely isotropic fibers were estimated by a classical elasticity approach.

Moreover, an elasticity approach towards the evaluation of the transverse modulus of unidirectional fibrous composites, with the concurrent consideration of an irregular distribution of fibers was made in Ref. [20], whereas Shah et al. [21] proposed a detailed analysis on compressive properties of fibrous composites of polymeric m atrix via combined end and shear loading.

Concurrently, in Ref. [22] the influence of the statistical character of fiber strength on the predictability of tensile properties of polymer composites reinforced with natural filler was examined by comparing the well known linear and power – law Weibull models.

Finally, in Ref. [23] theoretical formulae to find the five elastic constants of a general class of three – phase fibrous composites reinforced with randomly oriented fibers were performed, on the basis of fiber vicinity concept, something that generally concerns all types of composites.

The present work aims at introducing upper bounds of transverse elastic modulus for two and three – phase unidirectional fibrous composites, according to strength of materials approach. In this context, the authors derive in a rigorous manner closed – form expressions the values of which are strictly greater when compared with those obtained from Paul's formula and inverse law pf mixtures respectively.

2 Designation of upper bounds of transverse modulus

Let us remark that Paul's lower bound [1] for the transverse modulus of unidirectional fibrous composites is given as

$$E_{T(LB)} = \frac{E_m \mathrm{E_f}}{E_m U_\mathrm{f} + \mathrm{E_f} U_m} \tag{1}$$

Evidently, the above relationship is in consensus with the principle of the complementary energy and coincides with the inverse rule of mixtures for unidirectional two – phase composite media, which of course is derived according to strength of materials approach.

Now, given that in general $E_m < \mathrm{E_f}$, the product $E_m \mathrm{E_f}$ is their geometric mean squared. Thus, the following inequality holds

$$E_{T(LB)} = \frac{\sqrt{E_m \mathrm{E_f}}}{E_m U_\mathrm{f} + \mathrm{E_f} U_m} \cdot \sqrt{E_m \mathrm{E_f}} \Leftrightarrow$$

$$E_{T\,(LB)} < \frac{\sqrt{E_m \mathrm{E_f}}}{E_m U_\mathrm{f} + \mathrm{E_f} U_m} \cdot \frac{E_\mathrm{f} - E_\mathrm{m}}{\ln(E_\mathrm{f} / E_\mathrm{m})} \qquad (2)$$

Here, the fraction $\frac{E_\mathrm{f} - E_\mathrm{m}}{\ln(E_\mathrm{f} / E_\mathrm{m})}$ appearing in the right hand side of the above inequality denotes the logarithmic mean of the strictly positive terms $E_m; \mathrm{E_f}$.

It is known from Calculus [24] that given two arbitrary strictly positive real numbers $\mathrm{a};\mathrm{b}$ such that $\mathrm{a} < \mathrm{b}$ the following fundamental inequality holds

$$\mathrm{a} < \frac{2\mathrm{ab}}{\mathrm{a}+\mathrm{b}} < \sqrt{\mathrm{ab}} < \frac{\mathrm{b}-\mathrm{a}}{\ln\mathrm{b} - \ln\mathrm{a}} < \frac{\mathrm{a}+\mathrm{b}}{2} < \sqrt{\frac{\mathrm{a}^2+\mathrm{b}^2}{2}} < \frac{\mathrm{a}^2+\mathrm{b}^2}{\mathrm{a}+\mathrm{b}} < \mathrm{b} \qquad (3)$$

Here, it is obvious that the contra – harmonic mean i.e. the term $\frac{\mathrm{a}^2+\mathrm{b}^2}{\mathrm{a}+\mathrm{b}}$ yields the maximum value when compared with the other types of mean between two strictly positive real numbers. However, the use of this quantity instead of logarithmic mean would yield unrealistic values for the upper bound of transverse modulus, i.e. greater than the stiffness of fiber. Apparently, should an upper bound for any composite property be above the corresponding property of any of its constituents, it has no phy sical sense and no use in engineering practice.

On the other hand, it can be proved [see Appendix A] that given four arbitrary strictly positive real numbers $\mathrm{a};\mathrm{b};\mathrm{c};\mathrm{d}$ the following inequality holds identically

$$\frac{(\mathrm{a}+\mathrm{b})\cdot\mathrm{c}\cdot\mathrm{d}}{\mathrm{a}\cdot\mathrm{d}+\mathrm{b}\cdot\mathrm{c}} \le \frac{\mathrm{a}\cdot\mathrm{c}+\mathrm{b}\cdot\mathrm{d}}{\mathrm{a}+\mathrm{b}} \qquad (4)$$

The equality holds if and only if $\mathrm{c} \equiv \mathrm{d}$.

An application of the above inequality for $\mathrm{a} = U_\mathrm{f}; \mathrm{b} = U_m; \mathrm{c} = \mathrm{E_f}; \mathrm{d} = E_m$ with the concurrent data that $U_\mathrm{f} + U_m = 1$ and $\mathrm{E_f} \neq E_m$ yields

$$\frac{1}{E_m U_\mathrm{f} + \mathrm{E_f} U_m} < \frac{\mathrm{E_f} U_\mathrm{f} + E_m U_m}{E_m \cdot \mathrm{E_f}} \qquad (5)$$

Evidently, the numerator of the fraction in the right hand side of the above inequality denotes the longitudinal modulus $\mathrm{E_L}$ of the two phase unidirectional fibrous composite as derived from strength of materials approach.

Next, inequality (2) can be combined with (5) to yield

$$E_{T\,(LB)} < \frac{\mathrm{E_f} U_\mathrm{f} + E_m U_m}{E_m \cdot \mathrm{E_f}} \sqrt{E_m \mathrm{E_f}} \cdot \frac{E_\mathrm{f} - E_\mathrm{m}}{\ln(E_\mathrm{f} / E_\mathrm{m})} \Leftrightarrow$$

$$E_{T(LB)} < \frac{\mathrm{E_f} U_\mathrm{f} + E_m U_m}{\sqrt{E_m \mathrm{E_f}}} \cdot \frac{E_\mathrm{f} - E_\mathrm{m}}{\ln(E_\mathrm{f} / E_\mathrm{m})} \qquad (6)$$

or equivalently

$$E_{T\,(LB)} < \frac{\mathrm{E_L}}{G(E_\mathrm{m}, E_\mathrm{f})} \cdot L(E_\mathrm{m}, E_\mathrm{f}) \qquad (7)$$

Here, the quantities $L(E_\mathrm{m}, E_\mathrm{f})$; $G(E_\mathrm{m}, E_\mathrm{f})$ are the logarithmic and geometric mean of $E_\mathrm{m}, E_\mathrm{f}$ respectively.

In this context, one may observe that inequality (7) signifies an upper bound for the transverse modulus of a two – phase unidirectional fibrous composite which can be obtained from the following formula

$$E_{T\,(UB)} = \mathrm{E_L} \cdot \frac{L(E_\mathrm{m}, E_\mathrm{f})}{G(E_\mathrm{m}, E_\mathrm{f})} \qquad (8)$$

Here, since $\mathrm{E_f} > E_m$ it follows that the ratio $\frac{L(E_\mathrm{m}, E_\mathrm{f})}{G(E_\mathrm{m}, E_\mathrm{f})}$ is strictly greater than unity and therefore the upper bound of transverse modulus is strictly greater than the longitudinal modulus $\mathrm{E_L}$ when the latter is obtained from strength of materials approach.

Next, by taking into account the inverse rule of mixtures with the concurrent consideration of interphase concept, one may estimate the lower bound of transverse modulus as follows

$$\frac{1}{E_T} = \frac{U_\mathrm{f}}{\mathrm{E_f}} + \frac{U_m}{E_m} + \frac{U_i}{E_i} \qquad (9)$$

Again, one may point out that eqn. (9) is in consistency with the principle of the complementary energy and arises from strength of materials approach.

Solving for E_{TC} one finds

$$E_T = \frac{\mathrm{E_f} E_m E_i}{\mathrm{E_f} E_m U_i + \mathrm{E_f} E_i U_m + E_i E_m \mathrm{U_f}} \qquad (9a)$$

or equivalently

$$E_T = \frac{\sqrt{E_m \mathrm{E_f}} \cdot \sqrt{E_i \mathrm{E_f}} \cdot \sqrt{\mathrm{E_m} E_i}}{\mathrm{E_f} E_m U_i + \mathrm{E_f} E_i U_m + E_i E_m \mathrm{U_f}} \quad (9b)$$

Then, eqn. (9b) can be combined with inequality (3) to yield

$$E_T < \frac{L(E_m, \mathrm{E_f}) \cdot L(E_i, \mathrm{E_f}) \cdot L(\mathrm{E_m}, E_i)}{\mathrm{E_f} E_m U_i + \mathrm{E_f} E_i U_m + E_i E_m \mathrm{U_f}} \quad (10)$$

where $L(E_m, \mathrm{E_f}) \cdot L(E_i, \mathrm{E_f}) \cdot L(\mathrm{E_m}, E_i)$ are the logarithmic means of the three moduli

On the other hand, it can be proved [see Appendix A] that given six strictly positive real numbers $a_1, a_2, a_3, a_4, a_5, a_6$ the following inequality holds

$$\frac{a_1 + a_2 + a_3}{a_4 a_5 a_3 + a_4 a_6 a_2 + a_5 a_6 a_1} \leq \frac{a_4 a_1 + a_5 a_2 + a_6 a_3}{a_4 a_5 a_6 (a_1 + a_2 + a_3)} \quad (11)$$

The equality holds if and only if $a_4 \equiv a_5 \equiv a_6$

An application of inequality (11) for $a_1 = \mathrm{U_f}; a_2 = U_m; a_3 = U_i; a_4 = \mathrm{E_f}; a_5 = \mathrm{E_m}; a_6 = \mathrm{E}_i$ with the concurrent data that $a_1 + a_2 + a_3 = 1$ and $\mathrm{E_f} \neq E_m$ yields

$$\frac{1}{\mathrm{E_f} E_m U_i + \mathrm{E_f} E_i U_m + E_i E_m \mathrm{U_f}} < \frac{\mathrm{E_f} \mathrm{U_f} + E_m U_m + E_i U_i}{\mathrm{E_f} E_m E_i} \quad (12)$$

Here one may observe that the sum $\mathrm{E_f} \mathrm{U_f} + E_m U_m + E_i U_i$ appearing in the right hand side of inequality (12) denotes the longitudinal modulus $\mathrm{E_L}$ of the three – phase fibrous composite as obtained from strength of materials approach. Now, inequality (10) can be combined with (12) and (3) to yield

$$E_T < \mathrm{E_L} \cdot \frac{L(E_m, \mathrm{E_f}) \cdot L(E_i, \mathrm{E_f}) \cdot L(\mathrm{E_m}, E_i)}{G(E_m, \mathrm{E_f}) \cdot G(E_i, \mathrm{E_f}) \cdot G(\mathrm{E_m}, E_i)} \quad (13)$$

where $G(E_i, E_m); G(\mathrm{E_f}, E_i); G(\mathrm{E_f}, E_m)$ are the geometric means of the strictly positive terms $\mathrm{E_f}, E_m, E_i$

Hence, one may observe that inequality (13) signifies an upper bound for the transverse modulus of a three – phase unidirectional fibrous composite which can be obtained from the following formula

$$E_{T(UB)} = \mathrm{E_L} \cdot \frac{L(E_m, \mathrm{E_f}) \cdot L(E_i, \mathrm{E_f}) \cdot L(\mathrm{E_m}, E_i)}{G(E_m, \mathrm{E_f}) \cdot G(E_i, \mathrm{E_f}) \cdot G(\mathrm{E_m}, E_i)} \quad (14)$$

3 Determination of the Elastic Modulus of the Interphase

The elastic modulus of the interphase E_i can be generally approached as a nth degree polynomial of the polar variable r of a c oaxial three phase cylinder model [16].

$$E_i(r) = Ar^n + Br^{n-1} + Cr^{n-2} + ... \quad (15)$$

with $r_f \leq r \leq r_i$

and $E_m \leq E_i(r) \leq E_\mathrm{f}$

The following boundary conditions hold

At $r=r_f$: $E_i(r) = \eta E_\mathrm{f}$

At $r=r_i$: $E_i(r) = E_m$

Let us consider the maximum influence of interphase by supposing that the coefficient η equals unity.

To facilitate our derivations, let us take into consideration the linear, hyperbolic two degree parabolic and third degree variation of the interphase stiffness.

a) Linear variation of $E_i(r)$

$$E_i(r) = A + Br \quad (16)$$

with $r_f \leq r \leq r_i$

According the above boundary conditions we have

$$E_i(r) = \frac{E_f r_i - E_m r_f}{r_i - r_f} - \frac{E_f - E_m}{r_i - r_f} r \quad (17)$$

b) Hyperbolic variation

$$E_i(r) = A + B/r \quad (18)$$

with $r_f \leq r \leq r_i$

According to the same boundary conditions we have

$$E_i(r) = \frac{E_m r_i - E_f r_f}{r_i - r_f} + \frac{(E_f - E_m) r_f r_i}{(r_i - r_f)\, r} \quad (19)$$

c) Two degree parabolic variation

$$E_i(r) = Ar^2 + Br + C \quad (20)$$

with $r_f \le r \le r_i$

To estimate the constants A, B, C, along the above boundary conditions i.e. $E_i(r_f) = E_f; E_i(r_i) = E_m$, one may also request without violating the generality that the value $r = r_f$ constitutes a critical point for the single valued function $E_i(r)$ a fact that implies

$$\left.\frac{d}{dr} E_i(r)\right|_{r=r_f} = 0 \quad (21)$$

In this context, one obtains the following expression

$$E_i(r) = \frac{(E_f - E_m) r^2 - 2(E_f - E_m) r_f r + E_f r_i^2 + E_m r_f^2 - 2 E_m r_f r_i}{(r_i - r_f)^2} \quad (22)$$

or equivalently

$$E_i(r) = \frac{E_f - E_m}{(r_i - r_f)^2} \cdot r^2 - \frac{2(E_f - E_m) r_i}{(r_i - r_f)^2} r + \frac{E_f r_i^2 + E_m \cdot r_f^2 - 2 \cdot E_m r_f r_i}{(r_i - r_f)^2} \quad (23)$$

d) Logarithmic Variation

According to a general logarithmic variation, the term $E_i(r)$ arises from the following expression:

$$E_i(r) = A \ln r + \frac{B}{r} \quad (24)$$

$$\forall r > 0 : r_f \le r \le r_i$$

By taking into consideration the same boundary conditions as previously, we obtain

$$E_i(r) = \frac{E_f - E_m}{\ln \frac{r_i}{r_f}} \ln \frac{r_i^{E_f/(E_f - E_m)}}{r \cdot r_i^{E_m/(E_f - E_m)}} \quad (25)$$

e) Exponential Variation

Finally, let us assume that the quantity $E_i(r)$ varies according to the following general exponential law

$$E_i(r) = A \cdot e^{Br} \quad (26)$$

By the application of the same boundary conditions as before, we obtain

$$E_i(r) = \frac{E_f}{\ln\left(\frac{E_m}{E_f}\right)} e^{\frac{\ln \frac{E_m}{E_f}}{r_i - r_f} r} \quad (27)$$

Suggestively let us select the two degree parabolic variation, to approximate the interphase elastic modulus with respect to polar radius.

Next, to accommodate our derivations let us estimate the average values of interphase thermal conductivity by the following relationship

$$\overline{E_i}(r) = \frac{1}{U_i(r)} \int_{r_f}^{r_i} E_i(r)\, dU_i(r) \Leftrightarrow$$

$$\overline{E_i}(r) = \frac{2U_f}{U_i \cdot r_f^2} \int_{r_f}^{r_i} E_i(r)\, dr \quad (28)$$

4 Experimental Work

The unidirectional glass-fiber composites used in the experimental part of our investigation consisted of an epoxy matrix (Permaglass XE5/ 1, Permali Ltd., U.K.) reinforced with long E-glass fibers. The matrix material was based on a diglycidyl ether of bisphenol A together with an aromatic amine hardener (Araldite MY 750/ HT972, Ciba – Geigy, U.K.). The glass fibres had a diameter of $1.2 \cdot 10^{-5}\, m$ and were contained at a volume fraction of about 65%. The fiber content was determined, as customary, by igniting samples of the composite and weighting the residue, which gave the weight fraction of glass as: $w_f = 79.6 \pm 0.28\%$. This and the measured values of the relative densities of permaglass (p_f = 2.55 gr/cm3) and of the epoxy matrix (p = 1.20 gr/cm3) gave the value U_f = 0.65. Furthermore, chip specimens with a 0.004 m diameter and thicknesses varying between 0.001 and 0.0015 m made either of the fiber composite of different filler contents, or of the matrix material, were tested by the authors on a differential scanning calorimetry (DSC) Thermal Analyzer at the zone of the glass transition temperature for each mixture, in order to determine the specific heat capacity values.

Apparently, as the filler volume fraction is increased the proportion of macromolecules characterized by a reduced mobility is also increased. This fact is synonymous with an augmentation in interphase volume fraction [25]. Lipatov [26] has shown that, if calorimetric measurements are performed in the neighborhood of the glass transition zone of the composite, energy jumps are observed. These jumps are too sensitive to the amount of filler added to the matrix and can be used to evaluate the boundary layers developed around the inclusions. This fact supports the empirical conclusion presented in Ref. [26], according to which the extent of the interphase expressed by its thickness Δr motivates the variation of the amplitudes of heat capacity jumps appearing at the glass transition zones of the matrix material and the composite with various filler – volume fractions. Moreover, the size of heat capacity jumps for unfilled and filled materials is directly related to Δr by an empirical relationship given by Lipatov [26]

This expression defines the thickness Δr corresponding to the interphase and is written out below

$$\left(\frac{r_{\mathrm{f}}+\Delta r}{r_{\mathrm{f}}}\right)^{2}-1=\frac{\lambda}{1-U_{\mathrm{f}}} \qquad (29)$$

where the coefficient λ is given by

$$\lambda=1-\frac{\Delta C_{p}^{\mathrm{f}}}{\Delta C_{p}^{0}} \qquad (30)$$

Here, the numerator and the denominator of the fraction appearing in the right hand side of Eqn. (30) are the sudden changes of the heat capacity for the filled and unfilled polymer respectively.

Next, the volume fraction of the interphase layer can be estimated as follows [26]:

$$U_{i}=\frac{2U_{\mathrm{f}}\Delta r}{r_{\mathrm{f}}} \qquad (31)$$

Also, the calculations for the transverse elastic modulus were carried out with E_f = 70 GN/m2 and E_m = 3.5 GN/m^2 for the fiber and matrix moduli respectively.

5 Results and Discussion

Table 1, contains the numerical results occur

concerning the upper bounds of the composite transverse modulus with respect to fiber content, for a two and three phase material as obtained from (8) and (14) respectively. Also, in the same Table the corresponding values of this property after the consideration of interphase concept appear, arising from eqn. (23) in combination with (28) and (31). Besides, in the same table one may observe that the radius ri varies from 6 μm, when Uf = 0 to 6.235 μm, when the value of the fiber content equals 0.65. Here, in order to illustrate the physical meaning of the interphase thickness, we emphasize that in reality any polymer composite, (fibrous or particulate), necessarily consists of three different phases (matrix, filler and interphase). Besides, the interphase content varies from 0 when Uf = 0, to 0.051967 when the value of the fiber content equals 0.65. Roughly speaking one may notice that this value constitutes the optimum fiber volume fraction above which the reinforcing action of the fibers is upset.

U_f E_T (GN/m2)	Paul [1]	Three-phase inverse law of mixtures	Whitney and Riley [4]	Halpin and Tsai [7]	Ekvall [8] Eq. A4	Ekvall [8] Eq. A5	Sideridis [11]	Eqn. (8)	Eqn (14)	Sih et al (Exp.) [9]	Clements and Moore (Exp.) [10]
0	3.5	3.5	3.500	3.500	4.640	3.500	3.5	4.963679	5.007	3.5	
0.1	3.87	3.866556	5.130	4.500	5.120	4.210	5.334	14.39467	17.003	4.13	
0.2	4.32	4.316393	6.080	5.700	5.700	4.780	6.573	23.82566	27.71	4.55	
0.3	4.9	4.880637	6.980	7.190	6.450	5.450	7.902	32.85665	37.174		
0.4	5.65	5.607404	8.020	9.080	7.410	6.310	9.526	42.68764	46.673	9.16	
0.5	6.68	6.584081	9.310	11.540	8.740	7.460	11.734	52.11862	55.332	11.71	
0.6	8.16	7.956872	11.040	14.880	10.570	9.110	14.931	61.54961	63.664		12.22
0.65	9.24	8.87625	12.250	17.618	11.960	10.350	17.096	66.26511	67.965		16.03

Table 1 Transverse modulus of the composite with respect to fiber volume fraction

Fig. 1 illustrates the variation of the transverse modulus against the fiber content as derived from the proposed formulas for two and three phase fibrous composites respectively, see Eqns. (8) and (24), along with other theoretical expressions, see Refs. [1,4,7,11] and Appendix 2. In addition, the

experimental results obtained from Sih et al., Ref. [9], and Clements and Moore [10] also appear.

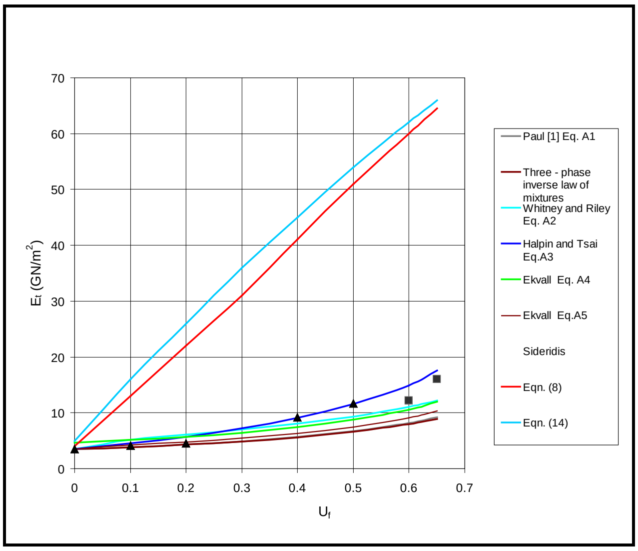

Fig. 1 Theoretical and experimental values for transverse modulus versus fiber content

Indeed, the theoretical predictions yielded by eqn. (8) and (14) are well above those arising from the other theoretical formulae along with the experimental results obtained from Sih et al. and Clements and Moore. Besides, they are below the Young modulus of the fiber. In this context, one may conclude thus these upper bounds for transverse modulus designated by eqn. (8) and (14) hold both for low and medium filler contents and marginally they are valid for higher filler volume concentrations up to 65%.

On the other hand, it can be observed that the values obtained from Sideridis based on Elasticity approach [11] together with those arising from Halpin – Tsai f ormula [17] are well above the rest theoretical and experimental values. This is attributed to the fact that our overall methodology towards the two proposed formulae for the lower bound of transverse modulus for two and three phase fibrous composites, i.e. Eqns. (8) and (24), is actually in consensus with the strength of materials approach something that also concerns Paul's expression and three – phase inverse mixing law.

In addition, the graphical representations of the values arising from eqns. (8) and (14) which are close to each other present a similar behavior with those emerging from Paul's formula and three – phase inverse rule of mixtures. This is regarded to the previous ascertainment t hat all these formulae are consistent with strength of materials approach. H owever, some discrepancies should be expected because some assumptions and conceptions of the interphase theory of filled polymers cannot be fulfilled in praxis. In the meanwhile, by returning to inequality (11) it can be observed that no restriction is imposed for the summation of the strictly positive numbers a_1, a_2, a_3. In this context, one may derive the following generalized form of inequality (12).

$$\frac{1}{\dfrac{E_f E_m a_3 + E_f E_i a_2 + E_i E_m a_1}{a_1 + a_2 + a_3}} < \frac{E_f a_1 + E_m a_2 + E_i a_3}{a_1 + a_2 + a_3} \cdot \frac{1}{E_f E_m E_i} \Leftrightarrow$$

$$\frac{E_f a_1 + E_m a_2 + E_i a_3}{a_1 + a_2 + a_3} \cdot \frac{E_f E_m a_3 + E_f E_i a_2 + E_i E_m a_1}{a_1 + a_2 + a_3} > E_f E_m E_i \quad (32)$$

Here one may point out that the product of the weighted arithmetic means of $E_f E_m; E_f E_i; E_i E_m$ and $E_f; E_m; E_i$ with respect to the same three weight functions $a_j \big|_{j=1,2,3}$ where $a_j : R \to R_+$, which are arbitrarily selected, is strictly greater than the product of the moduli of the three constituent phases.

Evidently, one may extend the above inequality in a similar way to address multilayer fibrous composites.

6 Conclusion

In this theoretical work, upper bounds of transverse modulus of two and three phase unidirectional fibrous composites were rigorously estimated in a unified manner, according to strength of materials approach, in terms of the constituents' moduli and the corresponding longitudinal modulus of the overall material. The latter was obtained from standard rule of mixtures for two and three phases respectively. Yet, it was observed that the theoretical predictions obtained from these proposed formulae are well above when compared with those arising from other theoretical expressions derived from elasticity approach for two and three phase fibrous composites. Also, these values are greater than experimental results concerning the transverse modulus of this type of composites that were found in the literature. In this context, and given that these bounds yielded theoretical results below the fiber stiffness it is the authors' opinion that they may be useful from the engineering viewpoint for conceptual or embodiment design procedures.

Appendix A

At first, we shall give a complete proof of the following inequality

$$\frac{(a+b)\cdot c\cdot d}{a\cdot d+b\cdot c}\leq\frac{a\cdot c+b\cdot d}{a+b}$$

where a,b,c,d are strictly positive real numbers

Since all involved quantities are strictly positive, it is enough to show that

$$(a+b)^2\cdot c\cdot d\leq(a\cdot c+b\cdot d)(a\cdot d+b\cdot c)$$

The above inequality is equivalently written as

$$(a+b)^2\leq\left(\frac{a}{d}+\frac{b}{c}\right)\cdot(a\cdot d+b\cdot c)\Leftrightarrow$$

$$a^2+b^2+2ab\leq a^2+\frac{a}{d}b\cdot c+\frac{b}{c}a\cdot d+b^2\Leftrightarrow$$

$$2ab\leq ab\left(\frac{c}{d}+\frac{d}{c}\right)\Leftrightarrow 2\leq\frac{c}{d}+\frac{d}{c}$$

Now, let x be a strictly positive real number.

Then it is well known that the following inequality holds identically

$$x+\frac{1}{x}\geq 2$$

The equality holds if and only if $x=1$

It this context, it is evident that the proof of the initial inequality is complete.

Next, we shall give a complete proof of the following inequality

$$\frac{a_1+a_2+a_3}{a_4a_5a_3+a_4a_6a_2+a_5a_6a_1}\leq\frac{a_4a_1+a_5a_2+a_6a_3}{a_4a_5a_6(a_1+a_2+a_3)}$$

where a_1,a_2,a_3,a_4,a_5,a_6 are strictly positive real numbers

Since all involved quantities are strictly positive, it is enough to show that

$$\frac{(a_4a_5a_3+a_4a_6a_2+a_5a_6a_1)(a_4a_1+a_5a_2+a_6a_3)}{a_4a_5a_6}\geq(a_1+a_2+a_3)^2$$

By multiplying the summations of the two parentheses, the fraction appearing in the left hand side of the above inequality is equivalently written as

$$\frac{(a_4^2a_5a_3a_1+a_4a_5^2a_3a_2+a_4a_5a_6a_3^2)}{a_4a_5a_6}+$$
$$+\frac{(a_4^2a_6a_1a_2+a_4a_5a_6a_2^2+a_6^2a_4a_2a_3)}{a_4a_5a_6}+$$
$$+\frac{(a_4a_5a_6a_1^2+a_5^2a_6a_1a_2+a_6^2a_5a_1a_3)}{a_4a_5a_6}$$

or

$$\left(\frac{a_4a_3a_1}{a_6}+\frac{a_5a_3a_2}{a_6}+a_3^2\right)+$$
$$\left(\frac{a_4a_1a_2}{a_5}+a_2^2+\frac{a_6a_2a_3}{a_5}\right)+$$
$$\left(a_1^2+\frac{a_5a_1a_2}{a_4}+\frac{a_6a_1a_3}{a_4}\right)=$$

$$a_1^2+a_2^2+a_3^2+\left(\frac{a_4}{a_6}+\frac{a_6}{a_4}\right)a_3a_1+$$
$$\left(\frac{a_5}{a_6}+\frac{a_6}{a_5}\right)a_3a_2+\left(\frac{a_4}{a_5}+\frac{a_5}{a_4}\right)a_1a_2$$

Now, let $x$be a strictly positive real number.

Then it is well known that the following inequality holds identically

$$x+\frac{1}{x}\geq 2$$

The equality holds if and only if $x=1$

In this context, it is evident that

$$a_1^2 + a_2^2 + a_3^2 + \left(\frac{a_4}{a_6} + \frac{a_6}{a_4}\right)a_3 a_1 + \left(\frac{a_5}{a_6} + \frac{a_6}{a_5}\right)a_3 a_2 + \left(\frac{a_4}{a_5} + \frac{a_5}{a_4}\right)a_1 a_2 \geq a_1^2 + a_2^2 + a_3^2 + 2a_3 a_1 + 2a_3 a_2 + 2a_1 a_2 \Leftrightarrow$$

$$a_1^2 + a_2^2 + a_3^2 + \left(\frac{a_4}{a_6} + \frac{a_6}{a_4}\right)a_3 a_1 + \left(\frac{a_5}{a_6} + \frac{a_6}{a_5}\right)a_3 a_2 + \left(\frac{a_4}{a_5} + \frac{a_5}{a_4}\right)a_1 a_2 \geq (a_1 + a_2 + a_3)^2$$

Thus we proved that the initial inequality holds identically.

Appendix B

Let us present some accurate formula concerning transverse modulus of fibrous composites that were used for comparison.

Paul [1]

$$\frac{1}{E_2} = \frac{U_f}{E_f} + \frac{U_m}{E_m} \qquad \text{(A1)}$$

Whitney- Riley [4]

$$E_2 = \frac{[2K_c(1-\nu_{TT})E_1]}{E_1 + 4K_c \nu^2{}_{12}} \qquad \text{(A2)}$$

with

$$\nu_{12} = \nu_m - \frac{2\cdot(\nu_m - \nu_f)\cdot(1-\nu_m{}^2)\cdot \mathrm{E}_f \cdot \lambda}{\mathrm{E}_m \cdot (1-\lambda)\cdot L' + [L\cdot\lambda + (1+\nu_m)]\cdot \mathrm{E}'}$$

Halpin – Tsai [7]

$$E_2 = E_m \frac{(1+\xi\cdot\eta\cdot U_f)}{(1-\eta\cdot U_f)} \qquad \text{(A3)}$$

Ekvall [8]

$$E_2 = \frac{E_f \cdot E'_m}{U_f \cdot E'_m + U_m \cdot E_f \cdot (1-\nu^2{}_m)} \qquad \text{(A4)}$$

Ekvall [8]

$$\frac{1}{E_2} = \frac{U_m}{E_m} + \frac{U_f}{E_f} - \frac{[(E_f\cdot\nu_m / E_m) - \nu_f]^2}{E_f \cdot [(\nu_f \cdot E_f / \nu_m \cdot E_m) + 1]} \qquad \text{(A5)}$$

Sideridis [11]

$$\frac{1-\nu_{\tau\tau}}{\mathrm{E}_T} = \frac{2\nu^2{}_{12}}{E_1} + \frac{(1-\nu_f - 2\nu^2{}_f)U_f}{E_f} + \frac{(1-\nu_m - 2\nu^2{}_m)\cdot U_f}{E_m} + \frac{2U_f}{r^2{}_f}\cdot\int_{r_f}^{r_i}\frac{[1-\nu_i(r) - 2\nu^2{}_i(r)]\cdot r}{E_i(r)}\,dr$$

where the major Poisson ratio for linear variation of the corresponding property of interphase is given as

$$\nu_{12} = \nu_f \cdot U_f + \nu_m(1-U_f-U_i) + \frac{1}{3}\{(\nu_f + 2\cdot\nu_m)(U_f+U_i) - (2\cdot\nu_f+\nu_m)U_f + (\nu_f - \nu_m)[U_f(U_f+U_i)]^{0.5}\}$$

whereas the minor Poisson ratio arises from three – phase inverse law of mixtures as follows

$$\frac{1}{\nu_{\mathrm{TT}}} = \frac{U_f}{\nu_f} + \frac{U_m}{\nu_m} + \frac{2\cdot U_f}{r^2{}_f}\cdot\int_{r_f}^{r_i}\frac{r}{\nu_i(r)}\,dr$$

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