The stiffness of short and randomly distributed fiber composites

E. SIDERIDIS, J. VENETIS, V. KYTOPOULOS
School of Applied Mathematics And Physical Sciences, Section Of Mechanics
National Technical University of Athens
5 Heroes Of Polytechnion Avenue, Gr – 15773 Athens
GREECE
Telephone: +302107721251
Fax: +302107721302
Email: siderem@mail.ntua.gr, johnvenetis4@gmail.com, victor@central.ntua.gr

Abstract: In this work, analytical calculations are described to estimate the elastic moduli of polymer composite materials consisting of short fibers, by extending and generalizing a reliable model. The fibers may have a finite length and an orientation characterized as random. An effort was made to complete and improve a previous procedure concerning transversely isotropic composites. To this end, a cylinder model with a short fiber in the centre of the matrix is considered. The elastic moduli were expressed in terms of the fiber content and fiber length. The obtained representations were used to evaluate the moduli of a randomly oriented short fiber composite. A comparison was made with theoretical values derived from two other authors who established trustworthy and accurate models. Finally, our theoretical values were also compared with obtained experimental results.

Keywords: Fiber composite materials, short fibers, unit cell, elastic constants, model of equivalent fiber, fiber contiguity

1 Introduction

It is well known that from the commercial standpoint the composite materials are made from matrices of epoxy, unsaturated polyester and in addition of some other thermosets and a few thermoplastics. The reinforcements are glass, graphite, aramid, thermoplastic fibers, metal and ceramic. The reinforcement may be continuous, woven, or chopped fiber and a typical commercial composite contains from 20% to 50% by weight glass or other reinforcement. The percentage of reinforcement in advanced composites can be as high as 70%; these materials usually use epoxy as the matrix material, and graphite is the most common reinforcement. Glass fibers are usually used for low cost and wherever high stiffness is not necessary.

The purpose of adding reinforcements to polymers is usually to enhance mechanical properties.

One of the most useful forms of composites for the construction of high – performance structural elements, is the type of panels made from aligned fibers containing polymerized matrix. They can also be made with woven fibers or randomly oriented chopped fibers; these have inferior stiffness and cannot have such high fiber materials. In order to perform stress and stiffness analyses of the chopped – fiber resin composites, it is essential that the elastic properties be known. In reality, these materials are anisotropic and heterogeneous. Evidently, their properties depend on the elastic properties and volume fractions of the constituent materials, the fiber length or aspect ratio, the degree of the alignment, the adhesion between fibers and matrix and finally they are affected by the fabrication techniques.

Nevertheless, one should primarily elucidate that chopped fibers, flakes, particles, and similar discontinuous reinforcements may enhance short – term mechanical properties, but these types of reinforcements are usually not as effective as continuous reinforcements in increasing creep strength and similar long – term strength characteristics, since continuous reinforcements serve to distribute applied loads and strain throughout the entire structure.

If the fibers are randomly distributed with respect to orientation and position, then samples of material which contain statistically significant numbers of fibers seem to be isotropic. If several such samples are compared to one another, the material can be considered as homogeneous. If the fibers have a specific orientation, then the composite will be anisotropic. Besides, in case the statistical distribution of fillers' orientation varies with position, then the composite becomes inhomogeneous even if the fiber density is uniform.

Mostly, theoretical studies for the stiffness and strength of such materials have been concerned with the aligned case, or with the situation of cross – ply laminates of continuous unidirectional fibers. The application of materials reinforced with short randomly – oriented f ibers is not relatively new. For some reasons, the fibers may be arranged such that they have partially aligned whereby they are constrained to lie in a plane but are otherwise randomly oriented.

Generally, the elastic moduli of a chopped – fiber composite can be derived from those of a unidirectional discontinuous fiber composite through a normalized integration process. There have been several previous studies of the random fiber case. T he first such study was apparently that due to Cox [1] who studied the stiffness of cellulose fiber materials by assuming that the external load is carried entirely by the fibers. The result was the Cox formula i.e.

$$E_{3D} = \frac{E_f U_f}{3}$$

where E_f is the fiber modulus and U_f the its volume fraction. The corresponding result for the two dimensional case was found to be

$$E_{2D} = \frac{E_f U_f}{3}$$

The first consideration of fiber – matrix effects was provided by Tsai and Pagano [2], in the context of laminated plates. Nielson and Chen [3] used computational techniques to predict the modulus of the composite. Further work along the lines using the "Laminate analogy" was given by Halpin, Jerina and Whitney [4].

Lees [5], in order to explain his experimental data concerning the longitudinal modulus derived a formula for the elastic modulus of short fiber composites assuming that the fiber and the matrix have different longitudinal strain under axial loading. One of the most important theories developed is that of Christensen and Waals [6], who provided a method for determining random orientation properties using a geometric averaging method. Later, Christensen [7], thought that there is a need to obtain analytical forms which directly admit physical interpretation. Assuming that the fiber phase is much stiffer than the matrix one he simplified the expressions based on the classical results of Hill [8,9] and Hashin [10,11] on unidirectional transversely isotropic composites which after a normalized integration process yield the elastic moduli of a randomly oriented fiber composite. The results for the three and two dimensional case respectively are represented as follows

$$E_{3D} = \frac{E_f}{6} U_f + \frac{1}{1-U_f}\left(1 + \frac{U_f}{4} + \frac{U_f^2}{6}\right) E_m;$$

$$E_{2D} = \frac{E_f}{3} U_f + \frac{1-U_f}{3} E_m + \frac{19}{27}\frac{E_f(1+U_f)+E_m(1-U_f)}{E_f(1-U_f)+E_m(1+U_f)} E_m$$

where E_m is the matrix modulus

The procedure presented in Ref. [6] was used by Eisenberg [12] in order to derive expressions which take into consideration the fabrication induced anisotropy of chopped – fiber resin composites. Except for the theory of Halpin and Pagano [2, 4], the above discussed theories do not consider the influence of fiber length or aspect ratio and therefore they are applicable for continuous fiber composites.

There are several theoretical formulae appearing in the literature to study the elastic moduli of unidirectional short fiber composites. Ogorkievicz and Weidman [13] idealized the composite consisting of a polymeric matrix containing unidirectionally aligned discontinuous glass fibers into a prism of the polymeric material and within it, a prism of glass. The volume of the glass prism, expressed as a fraction of the volume of the larger prism is equal to the volume fraction of the glass fibers and its proportions are fixed by the aspect ratio of the fibers. Some assumptions and a strength of materials approach led to Eq. (4) for the modulus.

A theory to enable prediction of moduli from basic matrix and fiber properties is due to Krenchel [14] who considered the composite to comprise a number of plies of uniaxially aligned fibers with the fibers of each ply being aligned in a different direction. The fiber – ply contributions are summed together with a contribution due to the matrix to give the continuous fibers length correction factors have been proposed to account for the reduction in composite stiffness that occurs with fibers of finite length. The factor proposed by Cox [1] is presented in the previous equations in a form which takes into account the wide distribution of fiber lengths. Halpin et al [15] adopted a more rigorous approach towards the prediction of the anisotropy of stiffness due to fiber orientation. They considered the composite to consist of a number of plies, each one containing matrix and uniaxially aligned fibers, again with the fibers of each ply to be aligned in a different direction. The moduli of each ply can be estimated. The simple relationships in these

equations are based on a generalization of the classical Rayleigh and Maxwell results, which may also describe short fibers with finite aspect ratio ℓ_f/d_f were used by Charrier and Sudlow [16] to calculate the moduli of short fiber systems. The fibers may have a finite aspect ratio and their orientation is characterized by one of the several parameters, intermediate between randomness and complete alignment in one direction.
The longitudinal modulus can be evaluated via an expression derived in Ref. [17] and has been used by Berthelot [18] along with Hashin and Hill expressions for the other moduli to estimate the effective elastic properties of a u nidirectional fiber composite with misaligned discontinuous fibers. It was assumed that they are planar uniformly oriented between directions θ_1 and -θ_1 where θ_1 denotes the misalignment angle.

Amongst the proposed formulas concerning unidirectional fiber composites, one can distinguish Halpin – Tsai formulas since they appear to be capable to account analytically for the influence of fiber length on elastic properties.
A successful theoretical investigation on the elastic moduli of randomly oriented short – fiber composites was carried out by Weng and Sun [20]. In this work, the longitudinal modulus and Poisson ratio of such a composite were found in terms of fiber content and the tip to tip spacing of the fibers. The proposed simulation was a composite cylinder model with a short cylindrical fiber, embedded in the centre of the matrix.
Here, these expressions are modified to account for the influence of volume fraction and aspect ratio of the filler. These results together with Hashin – Hill formulas for the other moduli were used in Christensen and Waal's normalized expressions to calculate the elastic modulus and Poisson ratio of such a composite in terms of the fiber content and its aspect ratios.
Facca et al [21] applied six micromechanical composite models (theoretical and semi – empirical) to predict the properties of fibre reinforced composites. Kalaprasad et al [22] also selected a number of micromechanical composite models to predict the properties of the composites with longitudinally as well as randomly oriented fibres. The Hirsch and the Bowyer – Bader models were found to forecast the rates of elastic modulus of the composites with both types of fiber distribution most accurately, whereas both the rule of mixtures or parallel model and the inverse rule of mixtures or series model failed to predict it.
The Halpin – Tsai model [23,21] is also a theoretical model, which except the elastic models of the constituent materials includes a geometrical parameter i.e. aspect ratio of the fiber. The model has a complicated mathematical structure, but still fails to meet the observations made in Refs. [21,22] satisfactorily. The developed semi – empirical modified Halpin – Tsai model [22, 23] however, is in good agreement with the experimental results of Facca et al.
On the other hand, the experimental results of Kalaprasad et al [22] were described satisfactorily by another semi – empirical model namely Bowyer – Badel model [24].
From the literature data [21,22,25], it has become evident that the theoretical models which do n ot contain any adjustable parameter, usually fail to predict the modulus of elasticity of fiber reinforced composites, and to predict the modulus satisfactorily and therefore one has to apply a relation with at least one adjustable parameter, which provide the model with semi-empirical nature. As there is no escape from the use of an empirical relation, it is easier to adopt some predictive model expressed in terms of mass fraction instead of volume fraction.
A few years ago Mirbagheri et al. [26] conducted intensive research on hybrid composites consisting of ternary mixture of wood flour, kenaf fiber and polypropylene, and found that the rule of mixtures could successfully describe the modulus of elasticity of the polymer composites. Meanwhile, Fu et al. [27] applied two approaches i.e. the rule of mixtures along with laminate analogy approach in order to describe the elastic modulus of a ternary mixture of particle – fiber – polymer. Besides, Islam et al [28] performed a thorough analysis on t he existing theoretical models to predict the elastic modulus of short fiber reinforced polymer composites, whereas in Ref. [29] an Elasticity approach was adopted to predict the elastic constants of fibrous composites, reinforced with transversely isotropic fibers.
In the present article, by extending and completing the theory of equivalent fiber based on a composite – cylinder model with a short cylindrical matrix the moduli of a unidirectional, short fiber composite were derived in terms of fiber content and aspect ratio of the fibers. These results were used to calculate elastic modulus and Poisson ratio of a randomly oriented short – fiber composite in terms of fiber content and aspect ratio. The theoretical results were compared with those derived from two other dominant theories on this subject. Moreover, tensile experiments were carried out on p olyester resin – randomly

oriented short fiber composites to determine mechanical properties and compare experimental results with theoretical values.

2 Theoretical Considerations

In this unit, by adopting a composite cylinder model in accordance with the theory of elasticity the longitudinal and the transverse elastic modulus, along with the longitudinal Poisson ratio of a composite with unidirectional continuous fibers will be derived. The theoretical analysis is based on the following assumptions:

i) T he matrix and the fibers are isotropic and linearly elastic.

ii) The composite reinforced with unidirectional fibers is linearly elastic, macroscopically homogeneous, transversely isotropic and without voids.

iii) The adhesion between the matrix and the fibers is perfect.

iv) The possible interaction amongst the fibers is neglected.

Here we should elucidate that the aforementioned suppositions will be taken into account in the derivation of the elastic moduli of a unidirectional continuous fibrous reinforced composite material.

a) **Longitudinal elastic modulus**

In order to evaluate the longitudinal elastic modulus, let us consider a representative layer of the composite material reinforced with continuous fibers, as it can be seen in Fig. 1.

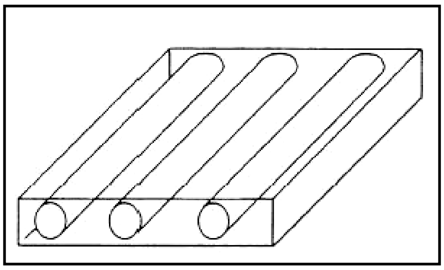

Fig.1 Representative layer of the composite

Then, let us concentrate on a cross – sectional area of the above model as it can be illustrated in Fig. 2.

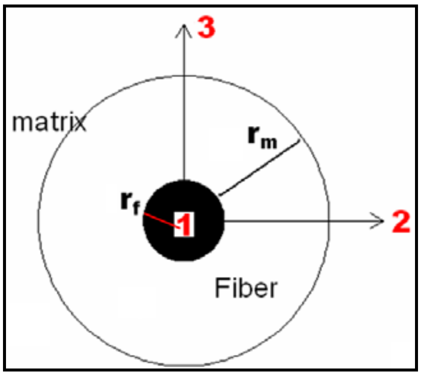

Fig. 2 Cross section of the model

By the use of Airy stress – function the compatibility equation in cylindrical coordinates is expressed as follows

$$\nabla^2\Phi = \frac{d^4\Phi}{dr^4} + \frac{2}{r}\frac{d^3\Phi}{dr^3} - \frac{1}{r^2}\frac{d^2\Phi}{dr^2} + \frac{1}{r^3}\frac{d\Phi}{dr} = 0 \quad (1)$$

Evidently, this ordinary differential equation belongs to Euler's type and the its solution is given as

$$\Phi = c_1 \ln r + c_2 r^2 \ln r + c_3 r^2 + c_4 \quad (2)$$

Each one of the phases of the composite material is characterized by a corresponding stress function.

Thus, the expressions for the stresses in each one of the aforementioned phases are

$$\sigma_{r,f} = \frac{1}{r}\frac{d\Phi_f}{dr} = \frac{A}{r^2} + B(1 + 2\ln r) + 2C \quad (3)$$

$$\sigma_{\theta,f} = \frac{d^2\Phi_f}{dr^2} = -\frac{A}{r^2} + B(3 + 2\ln r) + 2C \quad (4)$$

$$\sigma_{r,m} = \frac{1}{r}\frac{d\Phi_m}{dr} = \frac{F}{r^2} + G(1 + 2\ln r) + 2H \quad (5)$$

$$\sigma_{\theta,m} = \frac{d^2\Phi_m}{dr^2} = -\frac{F}{r^2} + G(3 + 2\ln r) + 2H \quad (6)$$

To avoid infinite stresses at $r = 0$ both constants A και B should vanish.

Thus, Eqns. (3) and (4) respectively, become

$$\sigma_{r,f} = \sigma_{\theta,f} = 2C$$

For the matrix material, it can be shown that $G = 0$ and therefore

$$\sigma_{r,m} = \frac{F}{r^2} + 2H \quad \text{and} \quad \sigma_{\theta,m} = -\frac{F}{r^2} + 2H$$

Next, let us apply a tensile stress σ_c, which is exerted at the direction of z axis.

The equilibrium of forces at this direction yields the following relationship

$$\sigma_f \cdot A_f + \sigma_m \cdot A_m = \sigma_c \cdot A_c \tag{7}$$

with

$$\sigma_{z,f} = \sigma_f, \sigma_{z,m} = \sigma_m$$

where

A_f, A_m, A_c is the area of the fiber, matrix, and composite respectively.

Now, if one puts $\sigma_f = s_1$, $\sigma_m = s_2, \sigma_c = s$ and divide by the term A_c the above relationship becomes

$$s_1 \cdot U_f + s_2 \cdot U_m = s \tag{8}$$

where

$U_f = \dfrac{r_f^2}{r_m^2}$ and $U_m = \dfrac{r_m^2 - r_f^2}{r_m^2}$ are the fiber and matrix content respectively

Evidently, $U_f + U_m = 1$

Moreover, from the stress – strain relationships we deduce that

$$\varepsilon_{r,f} = \frac{1}{E_f}\left[2C - U_f(2C + s_1)\right] \tag{9}$$

$$\varepsilon_{r,m} = \frac{1}{E_m}\left[\frac{F}{r^2}(1 + U_m) - 2H(1 - U_m) - U_m s_2\right] \tag{10}$$

$$\varepsilon_{\theta,f} = \frac{1}{E_f}\left[2C - U_f(2C + s_1)\right] \tag{11}$$

$$\varepsilon_{\theta,m} = \frac{1}{E_m}\left[\frac{F}{r^2}(1 + U_m) - 2H(1 - U_m) - U_m s_2\right] \tag{12}$$

$$\varepsilon_{z,f} = \frac{1}{E_f}\left[s_1 - 4U_f C\right] \tag{13}$$

$$\varepsilon_{z,m} = \frac{1}{E_m}\left[s_2 - 4U_m H\right] \tag{14}$$

where E_f and E_m are the elastic moduli of fiber and matrix respectively

Meanwhile, the radial displacements are given as

$$u_{r,f} = r\varepsilon_{\theta,f} \quad \text{and} \quad u_{r,m} = r\varepsilon_{\theta,m}$$

In continuing, the boundary conditions are formulated as follows

At $r = r_f$

$$\sigma_{r,f} = \sigma_{r,m} \to 2C = \frac{F}{r_f^2} + 2H \tag{15}$$

At $r = r_m$

$$\sigma_{r,m} = 0 \to \frac{F}{r_f^2} + 2H = 0 \tag{16}$$

At $r = r_f$

$$u_{r,f} = u_{r,f} \Rightarrow$$

$$E_m\left[2C(1 - U_f) - U_f s_1\right] = E_f\left[-\frac{F}{r_f^2}(1 + U_m) + 2H(1 - U_m) - U_m s_2\right] \tag{17}$$

Since the axial strains in matrix and fiber coincide, it implies that

$$E_m(s_1 - 4CU_f) = E_f(s_2 - 4HU_m) \tag{18}$$

The solution of the system of eqns. (17) and (18) for the terms s_1 and s_2 respectively yields

$$s_1 = \frac{sE_f + 4(CE_m U_f - HE_f U_m)}{E_f U_f + E_m U_m} U_f \tag{19}$$

$$s_2 = \frac{sE_f - 4(CE_m U_f - HE_f U_m)}{E_f U_f + E_m U_m} U_f \tag{20}$$

A substitution of the above data back into Eqn. (18) gives

$$(1 - v_f)(E_f U_f + E_m U_m) - v_f(E_f v_f U_f + E_m v_m U_m)E_m = -\frac{F}{2Cr_f^2}(1 + v_m)(E_f U_f + E_m U_m)E_f$$

$$+\frac{H}{C}\left[(1 - v_m)(E_f U_f + E_m U_m) - v_m(E_f v_m U_f + E_m v_f U_m)\right]E_f + \frac{sW}{2C} \tag{21}$$

where ν_f and ν_m are the Poisson ratios of fiber and matrix respectively

From the solution of the system of eqns. (15), (16) and (21) one estimates the rates of the constants F , H , C as follows

$$F=\frac{Wr_f^2}{(K+L)+(P-K)U_f}s \tag{22}$$

$$2H=\frac{WU_f}{(K+L)+(P-K)U_f}s \tag{23}$$

$$2C=\frac{W(1-U_f)}{(K+L)+(P-K)U_f}s \tag{24}$$

with

$$K=((1-v_f)(E_fU_f+E_mU_m)-2v_f(E_fU_mv_f+E_fU_mv_m))E_f \tag{25}$$

$$L=(1+v_m)(E_fU_f+E_mU_m)E_f \tag{26}$$

$$P=((1-v_m)(E_fU_f+E_mU_m)-2v_m(E_fU_fv_m+E_mU_mv_f))E_f \tag{27}$$

$$W=\left(v_f-v_m\right)E_fE_m \tag{28}$$

A combination between the system of eqns. (25) to (28) and the system of eqns. (19) and (20) results in the following explicit expressions for the quantities s_1 and s_2 respectively

$$s_1=\left(\frac{E_f}{E_fU_f+E_mU_m}+\frac{2W\cdot\left(U_m(1-U_f)E_mv_f+E_fU_fv_m\right)}{(E_fU_f+E_mU_m)\left((K+L)+(P-K)U_f\right)}\right)s\Rightarrow$$

$$s_1=\eta\cdot s \tag{29}$$

and

$$\frac{s_2}{s}=\frac{E_m}{E_fU_f+E_mU_m}-\frac{2W\cdot U_f\left((1-U_f)E_mv_f+E_fU_fv_m\right)}{(E_fU_f+E_mU_m)\left((K+L)+(P-K)U_f\right)}\Rightarrow$$

$$s_2=\xi\cdot s \tag{30}$$

The longitudinal modulus E_L of the composite can be estimated by the equalization of the overall deformation energy with the sum of corresponding ones for the constituent materials. Hence we can write out

$$\frac{1}{2}\int_{V_c}\frac{s^2}{E_L}dV_c=\frac{1}{2}\int_{V_f}\left(\sigma_{r,f}\varepsilon_{r,f}+\sigma_{\theta,f}\varepsilon_{\theta,f}+\sigma_{z,f}\varepsilon_{z,f}\right)dV_f+$$

$$\frac{1}{2}\int_{V_m}\left(\sigma_{r,m}\varepsilon_{r,m}+\sigma_{\theta,m}\varepsilon_{\theta,m}+\sigma_{z,m}\varepsilon_{z,m}\right)dV_m \tag{31}$$

Since we have initially proposed a modified version of Hashin cylinder model, the last relationship can be equivalently recasted as follows

$$\begin{aligned}\frac{1}{2}\int_{V_c}\frac{s^2}{E_L}2\pi rhdr&=\frac{1}{2}\int_0^{r_f}\left(\sigma_{r,f}\varepsilon_{r,f}+\sigma_{\theta,f}\varepsilon_{\theta,f}+\sigma_{z,f}\varepsilon_{z,f}\right)2\pi rhdr\\&+\frac{1}{2}\int_{r_f}^{r_m}\left(\sigma_{r,m}\varepsilon_{r,m}+\sigma_{\theta,m}\varepsilon_{\theta,m}+\sigma_{z,m}\varepsilon_{z,m}\right)2\pi rhdr\end{aligned} \tag{32}$$

Finally, by substituting the above expressions for both stresses and strains back into eqn. (32) one finds

$$\begin{aligned}\frac{1}{E_L}&=\frac{1}{E_f}\left(8C^2(1-v_f)-8Cv_f\eta+\eta^2\right)U_f\\&+\frac{1}{E_m}\left(2F^2U_f+8H^2(1-v_m)-8Hv_m\xi+\xi^2\right)(1-U_f)\end{aligned} \tag{33}$$

b) **Transverse elastic modulus**

Next, in order to evaluate the transverse elastic modulus of the composite let us consider a cross – sectional area of this material under the following loading condition, as it can be seen in Fig.3, where, p_2 is the applied external pressure and p_1 denotes the common stress at the interface. Let us consider the stress function in eqn. (1) and the described relationship in eqns. (2) – (6).

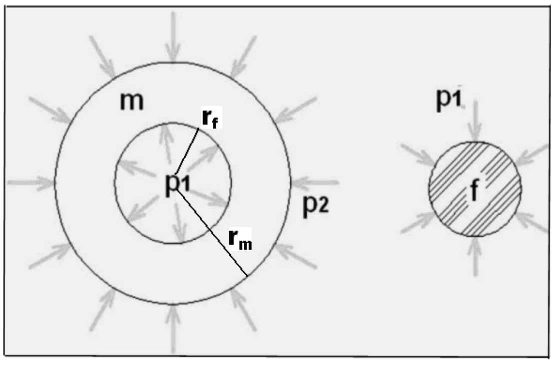

Fig. 3 Loading for the specification of transverse modulus

The boundary conditions are formulated as follows

$$r = r_f : \sigma_{r,f} = \sigma_{r,m} = -p_1 \rightarrow \frac{F}{r_f^2} + 2H = -p_1 \quad (34)$$

$$r = r_m : \sigma_{r,m} = -p_2 \rightarrow \frac{F}{r_m^2} + 2H = -p_2 \quad (35)$$

Hence the constants F and H are estimated as

$$F = \frac{(p_2 - p_1)(r_f r_m)^2}{r_m^2 - r_f^2} \quad (36)$$

and

$$2H = \frac{\left(p_1 r_f^2 - p_2 r_m^2\right)}{r_m^2 - r_f^2} \quad (37)$$

Moreover, in the present analysis we have also assumed that the axial strains are a priori negligible, and therefore

$$\varepsilon_{z,f} = 0 \rightarrow \sigma_{z,f} = v_f\left(\sigma_{r,f} + \sigma_{\theta,f}\right) \quad (38a)$$

$$\varepsilon_{z,m} = 0 \rightarrow \sigma_{z,m} = v_m\left(\sigma_{r,m} + \sigma_{\theta,m}\right) \quad (38b)$$

Besides, from the stress – strain relationships we have

$$\varepsilon_{\theta,f} = \varepsilon_{r,f} = \frac{2C\left(1 - v_f - 2v_f^2\right)}{E_f} \quad (39a)$$

$$\varepsilon_{\theta,m} = \frac{1}{E_m}\left[-\frac{F}{r^2}\left(1 + v_m\right) + 2H\left(1 - v_m - 2v_m^2\right)\right] \quad (39b)$$

$$\varepsilon_{r,m} = \frac{1}{E_m}\left[\frac{F}{r^2}\left(1 + v_m\right) + 2H\left(1 - v_m - 2v_m^2\right)\right] \quad (39c)$$

Concurrently, the following relationships hold

$$u_{r,f} = r\varepsilon_{\theta,f}, \quad u_{r,m} = r\varepsilon_{\theta,m}$$

Also, at $r = r_f$ it implies that

$$u_{r,f} = u_{r,m} \quad (40)$$

Consequently, we infer

$$2C\left(1 - v_f - 2v_f^2\right)E_m = E_f\left[-\frac{F}{r_f^2}\left(1 + v_m\right) + 2H\left(1 - v_m - 2v_m^2\right)\right] \quad (41)$$

Apparently, if one substitutes the rates of the quantities F and $2H$ back into eqn. (41) a linear algebraic expression between the magnitudes of the pressures p_1 and p_2 arises.

Hence we can write out

$$p_1 = \lambda p_2$$

with

$$\lambda = \frac{2(1+v_m)(1-v_m)E_f}{(1+v_m)\left((1-2v_m)U_f + 1\right)E_f + (1 - v_f - 2v_f^2)(1-U_f)E_m} \quad (42)$$

The transverse elastic modulus is calculated from the eq. (31), by setting on its left member instead of the integral $\frac{1}{2}\int_{V_c}\frac{s^2}{E_\ell}dV_c$ the terms in the quantity $\frac{1}{2}\int_{V_c}\frac{p_2^2}{K_c}dV_c$, where p_2 is the initially imposed pressure and K_c denotes the bulk modulus of the composite, which is evaluated by means of the following expression

$$K_c = \frac{1}{2\left(\frac{1 - v_{TT}}{E_T} - \frac{2v_{LT}{}^2}{E_L}\right)}$$

Since

$$\varepsilon_x = \frac{1}{E_T}\left(\sigma_x - v_{TT}\sigma_y\right) - \left(\frac{v_{LT}}{E_L}\right)\sigma_z$$

$$\varepsilon_y = \frac{1}{E_T}\left(\sigma_y - v_{TT}\sigma_x\right) - \left(\frac{v_{LT}}{E_L}\right)\sigma_z$$

$$\varepsilon_z = \frac{1}{E_L}\left[\sigma_z - v_{LT}\left(\sigma_x + \sigma_y\right)\right]$$

$\sigma_x = \sigma_y = -p_2$ and $\varepsilon_z = 0$

where E_T denotes the transverse elastic modulus.

and v_{LT}, v_{TT} are the longitudinal and transverse Poisson ratio respectively

Thus, by substituting the resultant relationships for the strains, the stresses and the unknown constants together with the auxiliary term λ back into eqn. (32) which is modified properly in order to include at the left member the expression with K_c and p_2 one obtains the following explicit expression for the transverse elastic modulus

$$\frac{1}{E_T}=\frac{U_f(1-v_f-2v_f^2)\lambda^2}{E_f(1-v_{TT})}+\frac{(1-v_m-2v_m^2)(\lambda U_f-1)^2}{E_m(1-U_f)^2(1-v_{TT})}+\frac{U_f^2(1+v_m)(1-\lambda^2)}{E_m(1-U_f)^2(1-v_{TT})}+\frac{2v_{LT}^2}{E_L(1-v_{TT})} \tag{43a}$$

c) **Longitudinal Poisson ratio**

Next, the longitudinal Poisson ratio can be evaluated by means of the following reasoning.

It is valid that

$$v_{LT}=-\frac{d_r/r_m}{\varepsilon_z}$$

where d_r denotes the radial displacement in the cylindrical area of the composite and ε_z is the axial displacement.

Thus

$$v_{LT}=-\frac{(u_{r,m})_{r=r_m}/r_m}{\varepsilon_{z,m}}=\frac{\left[-F/r_m^2(1+v_m)+2H(1-v_m)-v_m s_2\right]}{s_2-4Hv_m} \tag{43b}$$

A substitution of eqns. (22), (23), (30), back into (43b), leads to the following closed – form relationship

$$v_{LT}=\frac{2WU_f(E_fU_f+E_mU_m)+\left\{E_m\left[(K+L)+(P-K)U_f\right]-2WU_f\left[(1-U_f)E_mv_f+U_fE_fv_m\right]\right\}v_m}{E_m\left[(K+L)+(P-K)U_f\right]-2WU_f\left[(1-U_f)E_mv_f+U_fE_fv_m\right]+2WU_f(E_fU_f+E_mU_m)v_m}$$

(43c)

d) **Transverse Poisson ratio**

Moreover, the transverse Poisson ratio of the composite can be estimated by means of Halpin – Tsai equation which is presented as follows:

$$v_{TT}=v_m\frac{1+\xi_1\cdot\eta_1\cdot U_f}{1-\eta_1\cdot U_f} \tag{44}$$

where:

$$\eta_1=\frac{v_f-v_m}{v_f+\xi_1 v_m}$$

and

$$\xi_1=1,2,3...$$

e) **Shear Modulus**

On the other hand, to define shear modulus one can take into account Halpin – Tsai formula i.e.

$$G_{LT}=G_m\frac{1+\xi_2\cdot\eta_2\cdot U_f}{1-\eta_2\cdot U_f} \tag{45}$$

where

$$\eta_2=\frac{G_f-G_m}{G_f+\xi_2 G_m}$$

and ξ_2 is a constant term which depends on the fiber volume fraction and arises from the following empirical relationship

$$\xi_2=1+40\cdot U_f^{10}$$

Here G_f, G_m are the shear moduli of fiber and matrix respectively.

Besides, Hashin and Rosen proposed the following expression concerning the shear modulus of a composite was proposed

$$G_{LT}=G_m\frac{[(G_f+G_m)+(G_f-G_m)\cdot U_f]}{[(G_f+G_m)-(G_f-G_m)\cdot U_f]} \tag{46}$$

In continuing, to introduce the discontinuity of fibers we will use the model of equivalent fiber, which is described as follows:

Let us suppose a composite material consisting of unidirectional discontinuous fibers

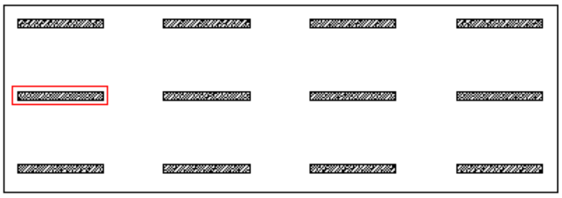

Fig. 4 Arrangement of the short fibers in the matrix

Next, let as focus on the cross – sectional area of a representative volume element of this material.

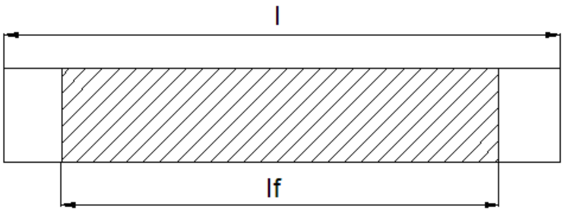

Fig. 5 Cross – sectional area of an arbitrary short fiber of the composite

We transform the above volume element in order to create another volume element consisting of an equivalent fiber

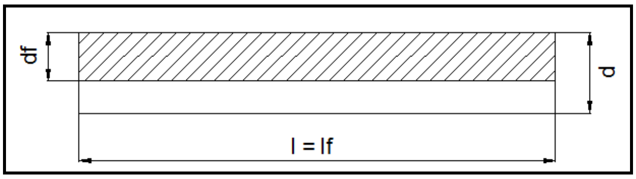

Fig. 6 Cross – sectional area of the equivalent fiber

Here, we should emphasize that the real system consisting of short fibers and matrix is characterized by the elastic constants E_f, E_m, U_f, U_m whereas the equivalent system consisting equivalent fiber and matrix, is described by the constants $E_{feq}, E_m, U_{feq}, U_{meq}$.

In this context, let as calculate the fundamental elastic properties of the composite.

Primarily, we have to estimate the filler content which in the real system is given as

$$U_f = \frac{\pi \cdot \frac{d_f^2}{4} \cdot l_f}{\pi \cdot \frac{d^2}{4} \cdot l} \Rightarrow U_f = \frac{d_f^2}{d^2} \cdot \frac{l_f}{l} \qquad (47)$$

and in the equivalent system

$$U_f = \frac{\pi \cdot \frac{d_f^2}{4} \cdot l}{\pi \cdot \frac{d^2}{4} \cdot l} \Rightarrow U_{feq} = \frac{d_f^2}{d^2} \qquad (48)$$

Hence,

$$U_{feq} = \frac{l}{l_f} \cdot U_f \qquad (49)$$

To evaluate the elastic modulus of equivalent fiber, one may adopt the Strength of Materials approach according to which the system filler – matrix is equivalent with a system of two ideal springs connected in series. Evidently, this implies that the exerted force is the same both for matrix and filler, while their lengthening differs, as it can be seen in Fig. 7

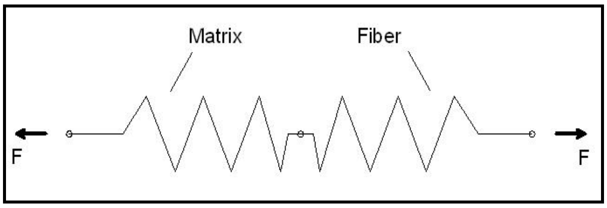

Fig. 7 Strength of materials approach for the simulation of equivalent fiber

$$F = F_f = F_m \quad \text{and} \quad \Delta l = (\Delta l)_f + (\Delta l)_m$$

Moreover the following relationships hold,

$$F = \sigma \cdot \mathrm{A}, \qquad F_f = \sigma_f \cdot \mathrm{A}_f$$

$$F_m = \sigma_m \cdot \mathrm{A}_m, \quad \sigma = E_{feq} \cdot \varepsilon$$

$$\sigma_f = E_f \cdot \varepsilon_f, \quad \sigma_m = E_m \cdot \varepsilon_m$$

$$(\Delta l)_f = \varepsilon_f \cdot l_f \quad (\Delta l)_m = \varepsilon_m \cdot l_m$$

and therefore,

$$\varepsilon \cdot l = \varepsilon_f \cdot l_f + \varepsilon_m \cdot l_m \Rightarrow \varepsilon \cdot l = \varepsilon_f \cdot l \cdot V_f + \varepsilon_m \cdot l \cdot V_m \Rightarrow \varepsilon = \varepsilon_f \cdot V_f + \varepsilon_m \cdot V_m \Rightarrow$$

$$\frac{\sigma}{E_{feq}} = \frac{\sigma_f \cdot l_f}{E_f \cdot l} + \frac{\sigma_m \cdot l_m}{E_m \cdot l} \quad (50)$$

Besides, referring to the equivalent fiber the following equality holds

$\sigma_f = \sigma_m = \sigma$

Consequently we find

$$E_{feq} = \frac{E_m \cdot E_f}{\frac{l_f}{l} \cdot E_m + \frac{l_m}{l} \cdot E_f} \quad (51)$$

To calculate Poisson ratio of the equivalent fiber we may consider without violating the generality of our presented mathematical formalism, that the overall lengthening and the dilatation of this hypothetical fiber are equal with the sum and the average of the real fiber and matrix respectively.

The overall lengthening is

$$\varepsilon_1 \cdot l = \varepsilon_f \cdot l_f + \varepsilon_m \cdot l_m \Rightarrow \varepsilon_1 = \varepsilon_f \cdot \frac{l_f}{l} + \varepsilon_m \cdot \frac{l_m}{l} \quad (52)$$

whereas the dilatation is given as

$$\varepsilon_2 = -v_f \cdot \varepsilon_f \cdot \frac{l_f}{l} - v_m \cdot \varepsilon_m \cdot \frac{l_m}{l} \quad (53)$$

Since $v_{feq} = -\frac{\varepsilon_2}{\varepsilon_1}$ we have

$$v_{feq} = \frac{E_m \cdot v_f + E_f \cdot v_m \cdot \left(\frac{l}{l_f} - 1\right)}{E_m + E_f \cdot \left(\frac{l}{l_f} - 1\right)} \quad (54)$$

Here, let us introduce a parameter named R which performs the ratio of the distance between two neighboring fibers and their length as it can be seen in Fig. 8

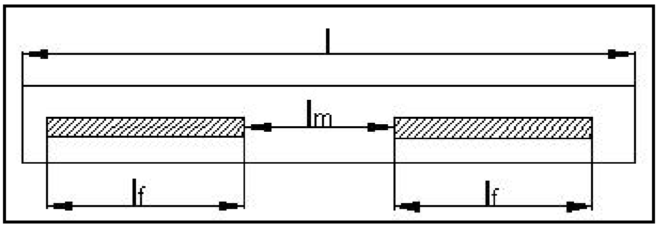

Fig. 8 Dimension of the fibers in the composite

This aforementioned parameter arises from the following expression

$$R = \frac{l_m}{l_f} = \frac{l}{l_f} - 1 \quad (55)$$

Hence, after the necessary algebraic manipulation we find

$$v_{feq} = \frac{E_m \cdot v_f + E_f \cdot v_m \cdot R}{E_m + E_f \cdot R} \quad (56)$$

$$E_{feq} = \frac{E_f \cdot E_m \cdot (1+R)}{E_m + R \cdot E_f} \quad (57)$$

$$U_{feq} = U_f (1+R) \quad (58)$$

Moreover, to examine the influence of aspect ratio of the fibers we introduce another parameter named a and given as

$$\mathrm{a} = \frac{l_f}{d_f} = \frac{l_f}{2r_f} \quad (59)$$

However, we have already found that

$$U_{feq} = \frac{d_f^2}{d^2}$$

$$U_f = \frac{d_f^2}{d^2} \cdot \frac{l_f}{l} \Rightarrow U_f = \frac{d_f^2}{d^2} \cdot S$$

where $S = \frac{l_f}{l}$

Therefore the latter relationship yields

$$U_f = \frac{d_f^2}{l_f^2}\cdot\frac{l_f^2}{d^2} \Rightarrow^2 \mathrm{a}\cdot U_f^2 = \frac{l_f^2}{d^2}\cdot S \Rightarrow$$

$$\mathrm{a}^2\cdot U_f^2 = \frac{l_f^2}{l^2}\cdot\frac{l^2}{d^2}\cdot S \Rightarrow$$

$$\mathrm{a}^2\cdot U_f^2 = S^3\left(\frac{l}{d}\right)^2 \qquad (60)$$

Here, one may additionally suppose, without violating the initial hypothesis concerning the randomness of fiber distribution and orientation, that the distance of two neighboring fibers in the longitudinal direction coincides with the distance of two neighboring fibers in the transverse direction as it can be seen in Fig.9.

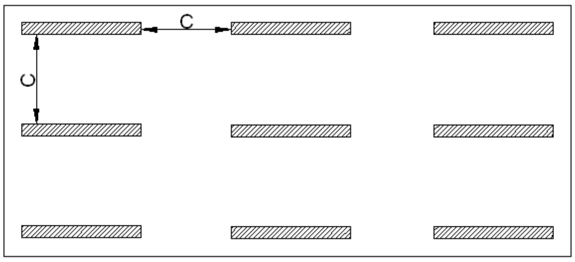

Fig. 9 Simplified approach of the fiber spacing

Thus we can write out

$$l - l_f = d - d_f = C$$

Hence it follows

$$l = d - d_f + l_f \Rightarrow \frac{l}{d} = 1 + \frac{l_f}{d} - \frac{d_f}{d} \qquad (61)$$

Since $\frac{l_f}{d} = \frac{l_f}{d_f}\cdot\frac{d_f}{d} = \mathrm{a}\sqrt{\frac{U_f}{S}}$, eqn. (61) yields

$$\frac{l}{d} = \frac{\sqrt{S} + \sqrt{U_f}\cdot(\mathrm{a}-1)}{\sqrt{S}} \qquad (62)$$

After an elementary algebraic manipulation, eqn. (62) results in the following quadratic equation

$$U_f\cdot\left(S^2-1\right)\cdot\mathrm{a}^2 - 2\cdot S^2\cdot\left(U_f - \sqrt{S\cdot U_f}\right)\cdot\mathrm{a} + \left(S + U_f - 2\cdot\sqrt{S\cdot U_f}\right) = 0 \qquad (63)$$

Evidently, the roots of the above polynomial equation are

$$\mathrm{a} = \frac{S\cdot\left(U_f - \sqrt{S\cdot U_f}\right)}{U_f\cdot(S+1)}$$

or

$$\mathrm{a} = \frac{S\cdot\left(U_f - \sqrt{S\cdot U_f}\right)}{U_f\cdot(S-1)}$$

According to the nature of the examined physical problem, it is obvious that one should reject beforehand the first solution since it leads to unrealistic results.

Thus, it im plies that the quantities S and R should be related as follows

$S = \frac{1}{R+1}$ and therefore

$$\mathrm{a} = \frac{\frac{1}{R+1}\cdot\left(U_f - \sqrt{\frac{1}{R+1}\cdot U_f}\right)}{U_f\left(\frac{1}{R+1} - 1\right)} \qquad (64)$$

3 Evaluation of the moduli by applying the model of equivalent fiber

Now, we apply the model of equivalent fiber in the relationships of the elastic constants which were previously derived, using Airy stress function. By this way, we introduce the discontinuity of fibers through the parameters R and a .

Here, one may distinguish and examine the following three cases which have been motivated by the consideration of the aspect ratio $\mathrm{a} = \frac{l_f}{d_f}$

$\mathrm{a} = \infty$ (continuous fibers)

- $\mathrm{a} = 10$ (discontinuous fibers)
- $\mathrm{a} = 0$ (particles)

For each case, one can estimate the elastic constants of the composite material by substituting the quantities E_f, v_f, U_f, G_f, U_m which appear in eqns. (33), (43a), (43c) and (46) back into eqns. (51), (54) (56), (57), (58) which describe the simulation of equivalent fiber and contain the terms $E_{feq}, v_{feq}, U_{feq}, G_{feq}, U_{meq}$. Consequently, in regards to the longitudinal elastic modulus one obtains

$$\frac{1}{E_{Leq}}=\frac{1}{E_{feq}}\left(8C_{eq}^{\ 2}(1-v_{eq})-8C_{eq}v_{eq}\eta_{eq}+\eta_{eq}^2\right)U_f \qquad (65)$$

$$+\frac{1}{E_m}\left(2F_{eq}^2U_{feq}(1+v_m)+8H_{eq}^2(1-v_m)-8H_{eq}v_m\xi_{eq}+\xi_{eq}^2\right)(1-U_{feq})$$

According to the same reasoning, the transverse elastic modulus arises from the following expression

$$\frac{1}{E_{Teq}}=\frac{U_{feq}(1-v_f-2v_{feq}^2)\lambda_{eq}^2}{E_{feq}(1-v_{Teq})}+\frac{(1-v_m-2v_m^2)(\lambda_{eq}U_{feq}-1)^2}{E_m(1-U_{feq})^2(1-v_{Teq})} \qquad (66)$$

$$+\frac{U_{feq}^2(1+v_m)(1-\lambda_{eq}^{\ 2})}{E_m(1-U_f)^2(1-v_{Teq})}+\frac{2v_{LTeq}^2}{E_L(1-v_{Teq})}$$

Moreover, the longitudinal Poisson ratio and the shear modulus are respectively estimated as

$$v_{LTeq}=\frac{2W_{eq}U_{feq}(E_{feq}U_{feq}+E_mU_m)+E_m((K_{eq}+L_{eq})+(P_{eq}-K_{eq})U_{feq})-2W_{eq}U_{feq}((1-U_{feq})E_mv_{feq}+U_{feq}E_{feq}v_m)\,v_m}{E_m((K_{eq}+L_{eq})+(P_{eq}-K_{eq})U_{feq})-2W_{eq}U_{feq}((1-U_{feq})E_mv_{feq}+U_{feq}E_{feq}v_m)+2W_{eq}U_{feq}(E_{feq}U_{feq}+E_mU_m)v_m}$$
(67)

and

$$G_{LTeq}=G_m\frac{[(G_{feq}+G_m)+(G_{feq}-G_m)\cdot U_{feq}]}{[(G_{feq}+G_m)-(G_{feq}-G_m)\cdot U_{feq}]} \qquad (68)$$

4 Numerical Examples

The numerical results of the above theoretical expressions are presented in the following tables, using the following fiber and matrix properties E_f= 72 G Pa, E_m=3.5 GPa, v_m=0.36, G_f=30 Gpa, G_m=1.3 Gpa

First case: $R=0 \Rightarrow a=\infty$ (continuous fibers).

$U_f=0.65$				
E_{feq}	v_{feq}	G_{feq}	U_{feq}	U_{meq}
$7.2\ 10^{10}\ N/m^2$	0.2	$3\cdot 10^{10}\ N/m^2$	0.65	0,35

Table 1 Elastic constants for $R=0 \Rightarrow a=\infty$

U_f	E_{Leq} (N/m²)	E_{Teq} (N/m²)	G_{LTeq} (N/m²)	v_{LTeq}
0	3.5E+9	3.5E+9	1.287E+9	0.36
0.1	1.038E+10	5.541E+9	1.547E+9	0.341
0.2	1.725E+10	6.976E+9	1.865E+9	0.322
0.3	2.412E+10	8.493E+9	2.265E+9	0.304
0.4	3.097E+10	1.031E+10	2.779E+9	0.287
0.5	3.782E+10	1.262E+10	3.469E+9	0.271
0.6	4.467E+10	1.575E+10	4.44E+9	0.256
0.7	5.15E+10	2.026E+10	5.91E+9	0.241
0.8	5.834E+10	2.739E+10	8.395E+9	0.227
0.9	6.517E+10	4.043E+10	1.35E+10	0.213
1	7.2E+10	7.2E+10	3E+10	0.2

Table 2 Elastic constants for $a=\infty$

The above numerical results concerning the three elastic constants $E_{Leq}; E_{Teq}; GL_{Teq}$ versus the filler content of the composite material are illustrated in Fig. 10

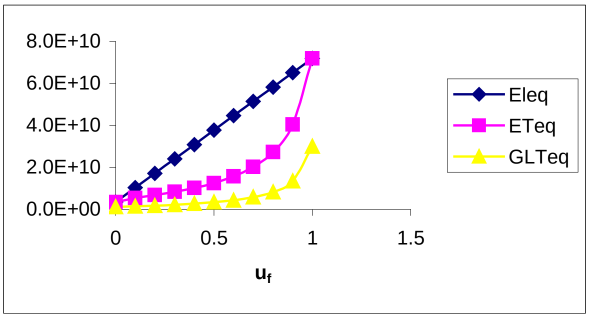

Fig. 10 Variation of $E_{Leq}, E_{Teq}, G_{LTeq}$ vs U_f for $a=\infty$ (N/m²)

U_f

Second case $a=10$ (discontinuous fibers).

$U_f=0.65$				
E_{feq}	v_{feq}	G_{feq}	U_{feq}	U_{meq}
5.023	0.251	2.008	0.665	0.335

10^{10} N/m^2		10^{10} N / m^2		

Table 3 Elastic constants for $a = 10$

U_f	R	E_{Leq} (N/m^2)	E_{Teq} (N/m^2)	G_{LTeq} (N/m^2)	v_{LTeq}
0	9.6[1]	3.5E+9	3.5E+9	1.287E+9	0.36
0.1	0.19	5.163E+9	4.404E+9	1.511E+9	0.355
0.2	0.112	8.115E+9	5.627E+9	1.806E+9	0.348
0.3	0.076	1.214E+10	7.05E+9	2.184E+9	0.337
0.4	0.054	1.72E+10	8.77E+9	2.676E+9	0.324
0.5	0.039	2.332E+10	1.098E+10	3.339E+9	0.309
0.6	0.027	3.054E+10	1.399E+10	4.278E+9	0.292
0.7	0.018	3.893E+10	1.84E+10	5.705E+9	0.272
0.8	0.011	4.857E+10	2.549E+10	8.134E+9	0.251
0.9	5.139E-3	5.956E+10	3.875E+10	1.318E+10	0.227
1	0	7.2E+10	7.2E+10	3E+10	0.2

Table 4 Elastic Constants vs U_f for $a = 10$

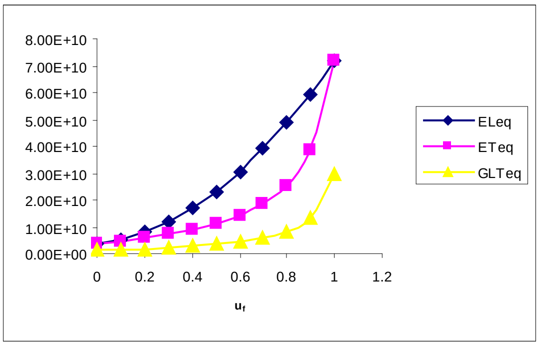

Fig.11 Variation of E_{Leq}, E_{Teq}, G_{LTeq} vs U_f for $a = 10$

Third case $a = 1$ (particles)

((N/m^2)			U_f	
E_{feq} (N/m^2)	v_{feq}	G_{feq} (N/m^2)	U_{feq}	U_{meq}
1,99 10^{10} N/m^2	0,322	7,529 10^9 N / m^2	0,75	0,25

Table 5 Elastic Constants vs U_f for $a = 1$

[1] The value of r which arises for U_f=0.00001

U_f	r	E_{Leq} (N/m^2)	E_{Teq} (N/m^2)	G_{LTeq} (N/m^2)	v_{LTeq}
0	45.41[2]	3.5E+9	3.5E+9	1.287E+9	0.36
0.1	1.154	4.096E+9	3.984E+9	1.455E+9	0.358
0.2	0.71	5.001E+9	4.653E+9	1.681E+9	0.356
0.3	0.494	6.252E+9	5.529E+9	1.976E+9	0.353
0.4	0.357	7.955E+9	6.684E+9	2.367E+9	0.349
0.5	0.26	1.03E+10	8.249E+9	2.903E+9	0.343
0.6	0.186	1.361E+10	1.047E+10	3.676E+9	0.335
0.7	0.126	1.852E+10	1.388E+10	4.877E+9	0.324
0.8	0.077	2.631E+10	1.974E+10	6.985E+9	0.305
0.9	0.036	4.03E+10	3.204E+10	1.162E+10	0.273
1	0	7.2E+10	7.2E+10	3E+10	0.2

Table 6 Elastic constants vs U_f for $a = 1$

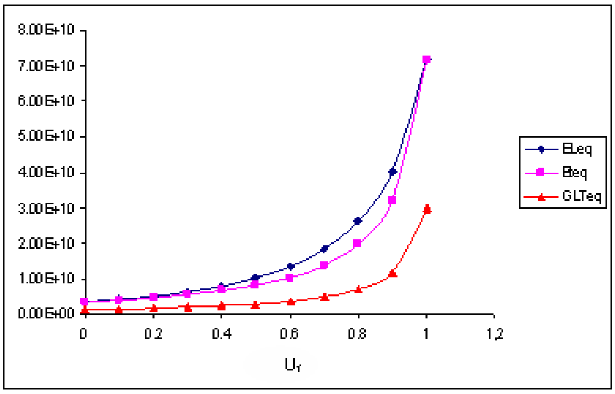

Fig. 12 Variation of E_{Leq}, E_{Teq}, G_{LTeq} vs U_f for $a = 1$

Normally, in a particulate composite material the elastic constants E_{Leq} and E_{Teq} should be equal. The discrepancies in the values of these elastic constants in this case, are due to the fact that while the fiber volume fraction increases, keeping constant the aspect ratio a and equal to unity, the parameter R (which as stated previously expresses the ratio of the distance between two neighboring fibers and their length) diminishes. As a r esult, there exists a transition from the

[2] The value of r which emerges for U_f=0.00001

particulate situation to the fibrous one although we have kept a=1. On the other hand, particulate composites are usually considered as spheres and not as short cylinders. From hence it is evident, that if we observe the results of the above table we can point out that for high values of the parameter R the rates of E_{Leq} and E_{Teq} are too close. This can be seen well in the following table, where the fiber contents are much closer.

U_f	R	E_{Leq} (N/m^2)	E_{Teq} (N/m^2)	G_{LTeq} (N/m^2)	v_{LTeq}
0	214[3]	3.5E+9	3.5E+9	1.287 E+9	0.36
0.01	3.642	3.542 E+9	3.538 E+9	1.301 E+9	0.36
0.02	2.684	3.59E +9	3.58E +9	1.315 E+9	0.359
0.03	2.218	3.642 E+9	3.623 E+9	1.331 E+9	0.358
0.04	1.924	3.697 E+9	3.669 E+9	1.347 E+9	0.357
0.05	1.714	3.756 E+9	3.717 E+9	1.364 E+9	0.356
0.06	1.554	3.818 E+9	3.767 E+9	1.381 E+9	0.354
0.07	1.426	3.883 E+9	3.819 E+9	1.399 E+9	0.353
0.08	1.321	3.952 E+9	3.872 E+9	1.417 E+9	0.352
0.09	1.231	4.022 E+9	3.927 E+9	1.436 E+9	0.35
0.1	1.154	4.096 E+9	3.984 E+9	1.455 E+9	0.349

Table 7 Elastic constants for vs U_f $a = 1$

Next, let us perform the variation of elastic moduli versus filler content for the three distinct rates of the parameter a

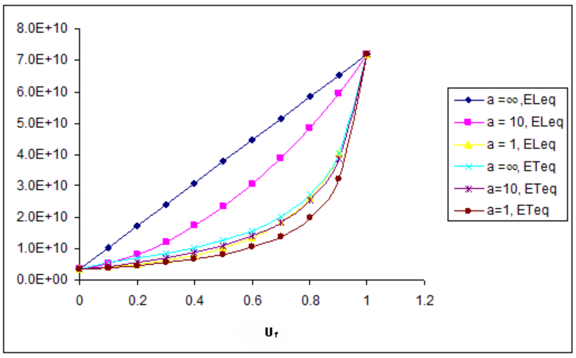

Fig. 13 Variation of E_{Leq}, E_{Teq} vs U_f for the three different values of a

On the other hand, the variation of Poisson ratio v_{LTeq} versus filler content for the three distinct rates of the parameter a is presented and illustrated at Table 8 and Fig. 14 respectively.

v_{LTeq}			
U_f	$a = \infty$	$a = 10$	$a = 1$
0	0.36	0.36	0.36
0.1	0.341	0.355	0.358
0.2	0.322	0.348	0.356
0.3	0.304	0.337	0.353
0.4	0.287	0.324	0.349
0.5	0.271	0.309	0.343
0.6	0.256	0.292	0.335
0.7	0.241	0.272	0.324
0.8	0.227	0.251	0.305
0.9	0.213	0.227	0.273
1	0.2	0.2	0.2

Table 8 Variation of Poisson ratio vs filler content for the three different values

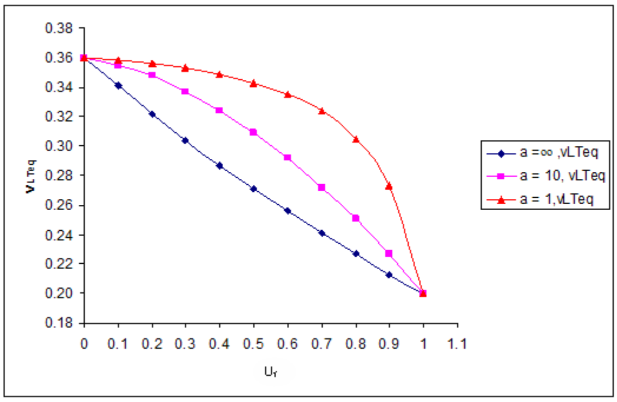

Fig. 14 Variation of Poisson ratio vs filler content for the three different values

Moreover, Table 9 a nd Fig. 15 describe the variation of shear modulus G_{LTeq} versus filler content for the three distinct rates of the parameter a .

G_{LTeq} (N/m^2)			
U_f	$a = \infty$	$a = 10$	$a = 1$
0	1.287E+9	1.287E+9	1.29E+09
0.01	1.547E+9	1.511E+9	1.46E+09
0.02	1.865E+9	1.806E+9	1.68E+09
0.03	2.265E+9	2.184E+9	1.98E+09
0.04	2.779E+9	2.676E+9	2.37E+09
0.05	3.469E+9	3.339E+9	2.90E+09
0.06	4.44E+9	4.278E+9	3.68E+09

[3] The value of r for U_f= 0.0000001

0.07	5.91E+9	5.705E+9	4.88E+09
0.08	8.395E+9	8.134E+9	6.98E+09
0.09	1.35E+10	1.318E+10	1.16E+10
0.1	3E+10	3E+10	3.00E+10

Table 9 variation of shear modulus

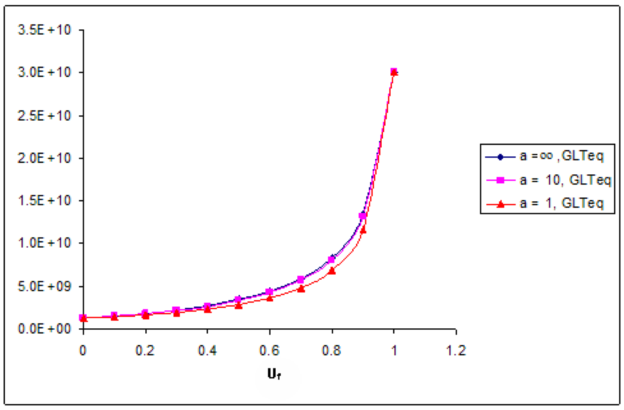

Fig. 15 Variation of shear modulus vs U_f

To estimate the elastic constants in the case of a random orientation of the fibers in the composite material the procedure described in Eisenberg approximation [12] can be adopted. Thus, for each case we apply the stages described in the appendix and after all the obtained results for elastic modulus and Poisson ratio are presented at the Tables 10 and 11 and illustrated at Figs. 15 and 16 respectively.

Equivalent Fiber Equivalent fiber	a = ∞	a = 10	a = 1
U_f	$\frac{\overline{E}}{E_m}$	$\frac{\overline{E}}{E_m}$	$\frac{\overline{E}}{E_m}$
0	1	1	1
0.1	1.878	1.292	1.145
0.2	2.731	1.75	1.349
0.3	3.614	2.348	1.623
0.4	4.554	3.101	1.988
0.5	5.583	4.042	2.489
0.6	6.755	5.23	3.201
0.7	8.175	6.776	4.278
0.8	10.082	8.932	6.077
0.9	13.16	12.412	9.666
1	20.571	20.571	20.571

Table 10 Values of the fraction $\frac{\overline{E}}{E_m}$ vs filler content

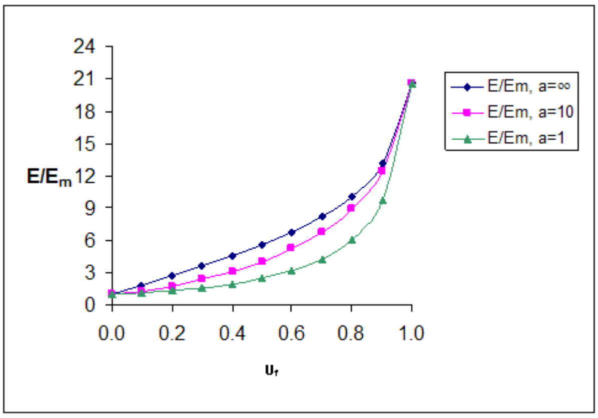

Fig. 16 Variation of the fraction $\frac{\overline{E}}{E_m}$ vs filler content

Equivalent Fiber Equivalent fiber	a = ∞	a = 10	a = 1
U_f	$\overline{v}$	$\overline{v}$	$\overline{v}$
0	0.36	0.36	0.36
0.1	0.37386	0.3645	0.35899
0.2	0.36235	0.36428	0.35773
0.3	0.3521	0.35875	0.35571
0.4	0.34313	0.35098	0.35263
0.5	0.33453	0.34221	0.34824
0.6	0.3253	0.33237	0.3421
0.7	0.31411	0.32043	0.33325
0.8	0.29837	0.30381	0.31917
0.9	0.27117	0.27523	0.29138
1	0.2	0.2	0.2

Table 11 Values of Poisson ratio vs filler content

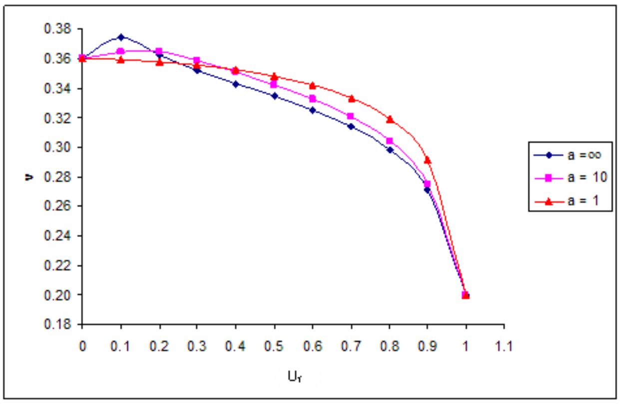

Fig. 17 Variation of Poisson ratio vs filler content

The values of the mean elastic modulus as resulted from our analysis which was actualized with the aid of Airy Stress Function and the model of equivalent fiber are higher that those obtained from Charrier and Sudlow approach [16] which appear in the Table 12.

Sudlow formula [16]	$\frac{l}{d}=\infty$	$\frac{l}{d}=10$	$\frac{l}{d}=1$
U_f	$\frac{\overline{E}}{E_m}$	$\frac{\overline{E}}{E_m}$	$\frac{\overline{E}}{E_m}$
0	1	1	1
0.1	1.603	1.424	1.281
0.2	2.162	1.882	1.621
0.3	2.759	2.408	2.038
0.4	3.439	3.038	2.566
0.5	4.253	3.820	3.254
0.6	5.277	4.836	4.191
0.7	6.648	6.230	5.544
0.8	8.654	8.310	7.672
0.9	12.09	11.902	11.515
1	20.571	20.571	20.571

Table 12 Values of the quotient $\frac{\overline{E}}{E_m}$ vs filler content according to Charrier and Sudlow approach. Suggestively, by calculating the difference between the values which arise from the first and second approximation respectively, we obtain the results of the following table.

U_f	(Equivalent Fiber) – (Sudlow [16]) for $\frac{l}{d}=\infty$	(Equivalent Fiber) – (Sudlow [16]) for $\frac{l}{d}=10$	(Equivalent Fiber) – (Sudlow [16]) for $\frac{l}{d}=1$
0	0	0	0
0.1	0.275	-0.132	-0.136
0.2	0.569	-0.132	-0.272
0.3	0.855	-0.06	-0.415
0.4	1.115	0.063	-0.578
0.5	1.33	0.222	-0.765
0.6	1.478	0.394	-0.99
0.7	1.527	0.546	-1.266
0.8	1.428	0.622	-1.595
0.9	1.07	0.51	-1.849
1	0	0	0

Table 13 C omparative representation of the two approximations vs filler content

At the following diagram the values of the ratio $\frac{\overline{E}}{E_m}$ are illustrated for the cases of continuous fibers, discontinuous fibers and spherical particles both for Charrier – Sudlow [16] and equivalent fiber approaches.

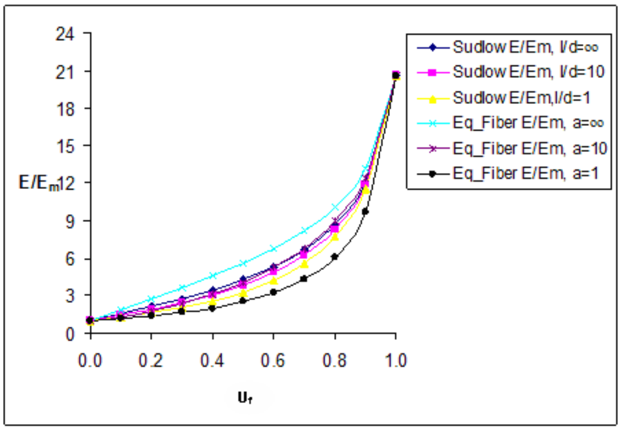

Fig. 18 C hange of the quotient $\frac{\overline{E}}{E_m}$ vs filler content for distinct rates of the ratio a and $\frac{l}{d}$

Next, the values of the averaging elastic modulus obtained for a random orientation of fibers and for the three aforementioned approximations are presented at the following table.

	Sudlow approach [16]	Eisenberg approach [12]	Equivalent fiber approach
	$\frac{l}{d}=\infty$, $C_l=1$		$a=\infty$
U_f	$\frac{\overline{E}}{E_m}$	$\frac{\overline{E}}{E_m}$	$\frac{\overline{E}}{E_m}$
0	1	1	1
0.1	1.603	1.689	1.878
0.2	2.162	2.418	2.731
0.3	2.759	3.171	3.614
0.4	3.439	3.958	4.554
0.5	4.253	4.794	5.788
0.6	5.277	5.714	7.139
0.7	6.648	6.788	8.724
0.8	8.654	8.201	10.783
0.9	12.09	10.643	13.984
1	20.571	20.571	20.571

Table 14 Values of $\frac{\overline{E}}{E_m}$ vs filler content for the case of random fibers according to the three approximations

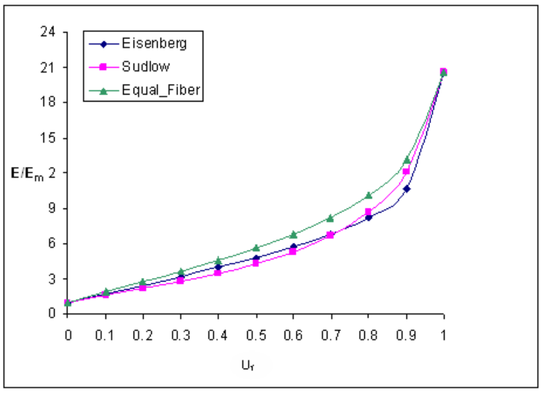

Fig. 19 Variation of $\frac{\overline{E}}{E_m}$ vs filler content for the case of random fibers according to the three approximations

Here, one can observe that the approximation of equivalent fiber yields higher values with respect to the two other analytical approaches. This is caused by the fact that the change of the aspect ratio i.e. the quotient: length to diameter of the cylindrical fiber can be actually achieved by the choice of the value of the ratio of fiber distance to the length.

5. Determination of the properties of randomly oriented fiber composites

5.1 Determination of fiber content and density

The weight of fibers contained in the material used for the experiments was determined using the burn – off test method which consists in to burn off the resin from a measured portion of a specimen. This gives the percentage weight of glass fibers by means of the following expression:

$$U_f=\frac{M_f/\rho_f}{M_f/\rho_f+M_m/\rho_m}\Leftrightarrow$$

$$U_f=\frac{1}{1+(M_f/M_m+(\rho_f/\rho_m)}\Leftrightarrow$$

$$U_f=\frac{1}{1+(\rho_f/\rho_m)(1/M_f-1)} \tag{68}$$

where

M_f weight fraction of fibers

M_m weight fraction of resin

ρ_f specific density of fibers

ρ_m specific density of resin

Meanwhile, the composite density ρ_c can be expressed in terms of ρ_f and ρ_m by the following approximate equation

$$\rho_c=\rho_f U_f+\rho_m(1-U_f) \tag{69}$$

5.2 Theoretical predictions of the variation of properties with glass content

To estimate the overall elastic modulus E_c of a glass fiber resin, the simple rule of mixtures can be taken into consideration, which is usually stated in the following form:

$$E_c = \eta_e E_f U_f + E_m U_m \tag{70}$$

where η_e is an efficiency factor depending on the type and the fabrication of the reinforcement. This simplified equation ignores the presence of voids and of low modulus mat binding material but usually is of adequate accuracy at this instance.

This equation, although by its nature only approximate gives a feel of the qualitative effect of the various parameters. Since, usually $E_f >> E_m$, it is evident that the modulus of the composite from the simple rule of mixtures will increase with increasing filler content. Therefore, with eq. (70) in hand and taking into account that $U_f + U_m = 1$ we obtain the following equation

$$E_c = E_m + \frac{(\eta_e E_f - E_m) M_f}{\frac{\rho_f}{\rho_m} + \left(1 - \frac{\rho_f}{\rho_m}\right) M_f} \tag{71}$$

On the other hand, Christensen equation [7] which has already been referred in the Introduction Unit has been derived from a more rigorous analysis of the micromechanics involved than the simple rule of mixtures.

$$E_{2D} = \frac{E_f}{3} U_f + \frac{1 - U_f}{3} E_m + \frac{19}{27} \left(\frac{E_f (1 + U_f) + E_m (1 - U_f)}{E_f (1 - U_f) + E_m (1 + U_f)} \right) E_m \tag{72}$$

The theory employs a quasi – isotropic model together with a geometric averaging technique to predict an asymptotic value for the elastic modulus of randomly reinforced short fiber composites, for the two – dimensional case, (i.e. when the fibers are aligned only in the plane of the laminate).

6. Experimental Work

6.1 Material and Method of Construction

The material was manufactured by a commercial fabricator. The resin was a pre – accelerated isophthalic polyester used with a peroxide catalyst. The reinforcement was powder bound, E glass fiber, chopped strand mat (CSM) of various weights per layer with glass tissue. The panels were laid up on the plate so that the lower surface of the panel was always flat and relatively smooth. A gel coat consisting of resin with two layers of surface tissue was first applied to the glass, which had previously been coated with release agent. The layers of CSM were then added ensuring that each layer was well wetted – out with resin. Finally, a second resin – rich layer (RRL) with a single piece of surface tissue was applied, mainly to improve the external appearance of the laminate. Most of the panels were post – cured for 48 hours at 50 ^{0}C. To evaluate the volume fraction of fibers, the burn – off test method was performed.

6.2 Testing of Materials

At first, tests were carried out to determine the fiber content and the density of the composite materials used. A prismatic sample was cut from each unbroken specimen or from each broken tension specimen as close as possible to the site of fracture.

The samples were measured and weighed before being placed in a furnace for several hours at 620 ^{0}C + 20^{0}C to "burn off" the resin. From the weigh of the residues assumed to be all glass, the fiber content by weight M_f and the volume fraction can be calculated for each specimen. The obtained value is the average of the three measurements. Similarly, prismatic samples of given dimensions and volume were cut from specimens and were weighed in order to determine the density of the materials tested. Tension tests were carried out on an Instron universal testing machine using Istron measuring and recording equipments. Load was measured by a strain gauge extensometer of 25 mm gauge length. The extensometer was attached to one surface of the specimen which was loaded to approximately 0.3% strain and unloaded. Then, without removing the extensometer, the specimen the specimen was loaded to failure at a rate of 0.2 cm/min. Thus the main information sought, the initial stress data and the ultimate strength data, was obtained. In each case, the properties were expressed in terms of the elastic modulus and ultimate tensile stress. Values of ultimate strain were also measured. Again, the obtained values are the average of the three measurements. Specimens were cut from material panels with nine layers of CSM having a nominal thickness of 1 cm and they were taken from each of the two perpendicular directions using a band – saw. The edges of the specimens were machined in a milling machine to the shape of a dog bone specimen having total length 150 mm and width 25.4 mm whereas at the narrow measuring area gauge length 60 mm and width 12.4 mm respectively according to BS 2782. Before testing, the width and thickness of each specimen were measured with a micrometer at three points inside the gouge length. From these measurements mean values of thickness and width were estimated of each specimen.

7. Results and Discussion

The procedure of the burn – off test yielded the value of the mean weight fraction of the fiber as 0.31. In continuing, by the aid of eq. (68) and using the values $\rho_f = 2560\,Kg/m^3$ and $\rho_m = 1100\,Kg/m^3$ the mean fiber volume fraction was evaluated as 0.172. As to the density of the composite, it has been estimated experimentally by weighting pieces of given dimensions and dividing the weight by the volume. The mean value was $\rho_c = 1419\,Kg/m^3$. Besides, the theoretical value of the composite density calculated from eq. (69) using the above mentioned numerical values, can be found as $\rho_c = 1426\,Kg/m^3$. Here, one can pinpoint that there exists a slight discrepancy between these two values which partly can be attributed to the difficulty of determining the mean thickness of the sample pieces since, as mentioned previously, they varied significantly. The main properties of the CSM composite are presented in the Table 15.

Material	t(cm)	M_f	U_f	σ_u (MN/m)	ε_u	E_{exp}(GN/m^2)	Rule of mixtures E(GN/m^2)	Christensen formula E(GN/m^2)
1	0.87	0.32	0.181	105	0.015	9.31	7.75	8.66
2	0.96	0.29	0.161	110	0.018	8.19	7.46	8.30
3	0.94	0.30	0.167	114	0.018	8.51	7.62	8.50
4	0.90	0.31	0.174	104	0.015	8.77	7.75	8.66
Mean Value	0.92	0.31	0.171	108	0.017	8.70	8.70	8.70

Table 15 Main properties of the CSM composite

Also, Fig. 20 illustrates the variation of the tensile modulus of the composite material versus fiber content according to the above experimental data together with the numerical values obtained from the rule of mixtures and Christensen formula [6].

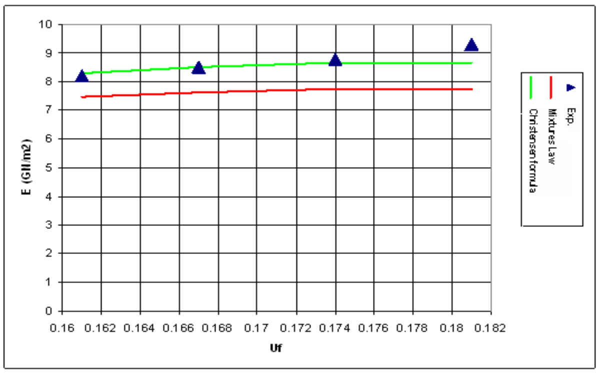

Fig. 20 Variation of the tensile modulus vs fiber content

Here, we should clarify that the material properties were calculated by considering the material as homogeneous and therefore they can be supposed as effective property values for the laminate as overall and consequently called laminate properties. In the first column, the specimen reference number is given whereas in columns two to four, mean thickness, fiber weight fraction and fiber volume fraction for each specimen appear. In the next columns, the results of the tensile experiments appear. The values at failure from the load cell and extensometer gave the ultimate stress σ_u, and ultimate strain, ε_u, respectively whereas the elastic modulus, E was obtained from the mean slope of the load – extension curves at a level strain below 0.2%. From Table 15 it can observed that between the specimens of the same nominal thickness there exists a remarkable variation of the thickness and the fiber volume fraction tended to vary significantly between different regions of the material as it was observed for the mean measures properties. Although this latter could also constitute an indication of orthology as specimens were cut from perpendicular directions, a consistent relation could be observed between properties in the two directions. Apparently, fiber volume fraction is an important factor for the strength and stiffness characteristics of glass fiber reinforced composites. Both ultimate stress and elastic modulus should increase with the concurrent increment of the glass fiber volume fraction. As it can be seen from Table 15 there is a tendency for these properties to increase with fiber content. On the other hand, the theoretical value for the elastic modulus derived from eq. (70) using the fixed rate 0.375 for the coefficient η_e which is given for randomly distributed short fibers and with E_f = 72GPa, E_f =3.5 GPa and U_f = 0.172 yields the value E_c = 7.54 GPa. Meanwhile, Christensen formula for the two – dimensional case given by eq. (72) yields the value E_c = 8.61 GPa which evidently is greater than the rule of mixtures. The comparison between theory and experiment shows that the values of E_c calculated by the Christensen formula is in a very good coincidence with experimental result whereas that arising from the rule of mixtures is lower than the experimental one and thus presents a l arge discrepancy.

8. Conclusions

In this manuscript an improved model to predict the elastic constants of short fibrous polymer composites was described.

The theoretical values obtained from modified micromechanical model looking at both the effect of fiber aspect ratio and content were compared with theoretical values obtained from other models.

It was found that both parameters have an important influence on the stiffness.

In general, this modulus increases with the augmentation of fiber length together with the volume fraction. Then by the use of an averaging approach the elastic constants of this type of materials were calculated. T he theoretical predictions were compared with experimental results, as well as with theoretical values yielded by some reliable formulae derived from other workers, and a reasonable agreement was found.

References

[1] Cox H.L The elasticity and strength of paper and other fibrous materials *British Journal of Applied Physics*, Vol. 3 No.3, 1952, pp. 72-78

[2] Tsai, S.W. and Pagano, N.J., *Invariant Properties of Composite Materials*, Technomic Publishing Company, 1968.

[3] Nielsen LE, Chen PE. *Young's modulus of composites filled with randomly oriented fibers*. Journal Materials Vol. 3 N o.2, 1968, pp.352–358

[4] J.C. Halpin, K. Jerine and J.M. Whitney *The Laminate Analogy for 2 and 3 D imensional Composite Materials*, Journal of Composite Materials Vol. 5 No. 1, 1971, pp. 36 – 49

[5] Lees, J. K., *A study of the tensile modulus of short fiber reinforced plastics*. Polymer Engineering Science, Vol. 8 No. 3, 1968, pp. 186–194.

[6] R.M. Christensen and F.M. Waals, *Effective Stiffness of Randomly Oriented Fibre Composites* Journal of Compsite Materials Vol. 6 No. 3, 1972, p.p. 518-532

[7] R.M. Christensen *Asymptotic modulus results for composites containing randomly oriented fibers* International Journal of Solids and Structures, Vol. 12 No. 7,1976, pp. 537 - 544

[8] H ill, R. *Theory of mechanical properties of fibre-strengthened materials; 1 elastic behaviour.* Journal of Mechanics and Physics of Solids Vol. 12, No. 4, 1964, pp. 199-212.

[9] Hill, R. *A self-consistent mechanics of composite materials*, Journal of Mechanics and Physics of Solids, Vol. 13, No. 4, 1964, pp. 213-222.

[10] Z. Hashin, *On elastic behaviour of fibre reinforced materials of arbitrary transverse phase geometry*, Journal of Mechanics and Physics of Solids Vol. 13, No. 3, 1965, pp.119-134

[11] Z. Hashin. *Viscoelastic fiber reinforced materials*, AIAA Journal, Vol. 4, No. 8, 1966), pp. 1411-1417.

[12] Eisenberg M.A,*Theory of fabrication-induced anisotropy of choppedfibre/resin panels* , Fibre Science and Technology, Vol. 12 N o.2, 1979, pp. 83-95

[13] R.M. Ogorkievicz and Weidmann, *Tensile Stiffness of a Thermoplastic Reinforced with Glass Fibres or Spheres*, Journal of Mechanical Engineering Science, Vol. 16 No. 1,1974, pp. 10-17

[14] H. Krenchel Fibre reinforcement: theoretical and practical investigations of the elasticity and strength of fibre-reinforced materials, Akademisk Forlag, 1964

[15] Ashton, J.E., Halpin, J.C. and Petit, P.H. Primer on Composite Materials Analysis. Technomic Publishing Company, 1969

[16] J.M. Charrier and M. J.Sudlow *Structure – Properties relationships for short –fibre composites*, Fibre Science and Technology, Vol. 6, No. 4, 1973, pp. 249-266

[17] J.M. Berthelot, A. Cupcic, J.M. Maufras *Experimental flexural strength-deflection curves of oriented discontinuous fibre composites* Fibre Science and Technology Vol. 11, No. 5 1978, pp. 367-398

[18] J.M. Berthelot, *Effect of fibre misalignment on the elastic properties of oriented discontinuous*

fibre composites, Fibre Science and Technology Vol. 17, No. 1, 1982, pp. 25 – 39

[19] Halpin, JC, and Tsai, SW. *Environmental factors in composite materials*. Technical Report AFML-TR. United State Air Force Materials Laboratory, 1969, 67-423.

[20] Weng, G. J. and Sun, C. T., *Effects of Fiber Length on Elastic Moduli of Randomly-Oriented Chopped-Fiber Composites*, Composite Materials: Testing and Design (Fifth Conference), ASTM STP 674, S. W. Tsai, Ed., American Society for Testing and Materials, 1979, pp. 149-162.

[21] A. G. Facca, M. T. Kortschot, and N. Yan, *Predicting the elastic modulus of natural fibre reinforced thermoplastics*,Composites Part A: Applied Science and Manufacturing, Vol. 37, No. 10, 2006, pp. 1660-1671

[22] G. Kalaprasad, K. Joseph, and S. Thomas, *Theoretical modelling of tensile properties of short sisal fibre-reinforced low-density polyethylene composites*, Journal of Materials Science, Vol.32, No. 16, 1997, pp 4261–4267

[23] J. C. Halpin and J. I. Kardos, *The Halpin-Tsai equations: A review, Polymer Engineering and Science* Vol. 16, No. 5, 1976 , pp. 344–352

[24] W.H. Bowyer, MG. Bader, *On the reinforcement of thermoplastics by imperfectly aligned discontinuous fibres,* Journal of Materials Science, Vol. 7, No. 11, 1972, pp 1315–1321

[25] L. Peponi, J. Biagiotti, M. Kenny, and I. Mondragon, *Statistical analysis of the mechanical properties of natural fibers and their composite materials. II. Composite materials*, Polymer Composites Vol. 29 No. 3, 2008, pp. 321–325.

[26] J. Mirbagheri, M. Tajvidi, I. Ghasemi, and J. C. Hermanson, *Prediction of the Elastic Modulus of Wood Flour/Kenaf Fibre/Polypropylene Hybrid Composites*, Iranian Polymer Journal Vol. 16 No. 4, 2007, pp. 271-278

[27] S. Y. Fu, G. Xu, and Y.W. Mai, *On the elastic modulus of hybrid particle/short-fiber/polymer composites*, Composites Part B: Engineering Vol. 33, No. 4, 2002, pp. 291-299

[28] M. A. Islam and K. Begum, *Prediction Models for the Elastic Modulus of Fiber-reinforced Polymer Composites: An Analysis*, Journal of Scientific Research, Vol. 3, 2011, No. 2, pp. 225-238

[29] J. Venetis and E. Sideridis *Elastic constants of fibrous polymer composite materials reinforced with transversely isotropic fibers*, AIP Advances Vol. 5, No. 3, 2015, Article ID 037118

Appendix

Here, we shall present a brief remark of the two aforementioned reliable models [12, 16] aiming at the prediction of the elastic properties of fibrous composites, whose numerical values were compared with our presented formulas.

Charrier and Sudlow method [16], focuses on the prediction of the elastic properties of randomly distributed short fiber composites.

To this end, a thin layer of this material was considered such that the direction of its continuous fibers to coincide with axis L as it can be seen in Fig. A_1.

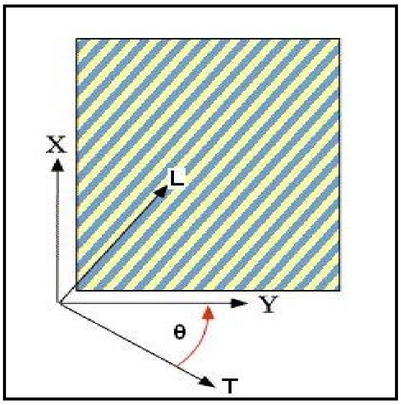

Fig. A_1 Thin layer of the composite

The flexibility matrix with respect to axes LT is formulated as follows

$$\left[S^{LT}\right]=\begin{bmatrix} S_{11}^{LT} & S_{12}^{LT} & 0 \\ S_{21}^{LT} & S_{22}^{LT} & 0 \\ 0 & 0 & S_{66}^{LT} \end{bmatrix} \qquad \text{(A1)}$$

with $S_{11}^{LT}=\dfrac{1}{E_L}$; $S_{22}^{LT}=\dfrac{1}{E_T}$; $S_{66}^{LT}=\dfrac{1}{G_{LT}}$;

$$S_{12}^{LT}=S_{21}^{LT}=-\frac{v_{LT}}{E_T}$$

Next, eqn. (A1) was recasted with respect to O xyz frame of reference yielding

$$\left[S^{xy}\right]=\begin{bmatrix} S_{11}^{xy} & S_{12}^{xy} & S_{16}^{xy} \\ S_{21}^{xy} & S_{22}^{xy} & S_{26}^{xy} \\ S_{61}^{xy} & S_{62}^{xy} & S_{66}^{xy} \end{bmatrix} \qquad \text{(A2)}$$

with

$$S_{11}^{xy} = \frac{\cos^4(\theta)}{E_L} + \frac{\sin^4(\theta)}{E_T} + \frac{\sin^2(2\theta)}{4}\left(\frac{1}{G_{LT}} - \frac{2v_{LT}}{E_L}\right) \quad \text{(A3)}$$

$$S_{22}^{xy} = \frac{\sin^4(\theta)}{E_L} + \frac{\cos^4(\theta)}{E_T} + \frac{\sin^2(2\theta)}{4}\left(\frac{1}{G_{LT}} - \frac{2v_{LT}}{E_L}\right) \quad \text{(A4)}$$

$$S_{66}^{xy} = \frac{1}{E_L} + \frac{2v_{LT}}{E_L} + \frac{1}{E_T} - \cos^2(2\theta)\left(\frac{1}{E_L} + \frac{2v_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}}\right) \quad \text{(A5)}$$

$$S_{16}^{xy} = \frac{\sin(2\theta)}{E_L}\left[1 + v_{LT} - \frac{E_L}{2G_{LT}} - \sin^2(\theta)\left(1 + 2v_{LT} + \frac{E_L}{E_T} - \frac{E_L}{G_{LT}}\right)\right] \quad \text{(A6)}$$

$$S_{26}^{xy} = \frac{\sin(2\theta)}{E_L}\left[1 + v_{LT} - \frac{E_L}{2G_{LT}} - \cos^2(\theta)\left(1 + 2v_{LT} + \frac{E_L}{E_T} - \frac{E_L}{G_{LT}}\right)\right] \quad \text{(A7)}$$

Here, to introduce the parameter of fiber discontinuity it has been considered that the fundamental elastic constants of the composite are given by the following explicit representations

$$E_L = E_m\left(1 + \frac{\left(2(l/d)+1\right)\left((E_f/E_m)-1\right)U_f}{(E_f/E_m) + 2(l/d) - ((E_f/E_m)-1)U_f}\right) \quad \text{(A8)}$$

$$E_T = E_m\left(1 + \frac{\left((d/l)+2\right)\left((E_f/E_m)-1\right)U_f}{((E_f/E_m) + 1 + (d/l) - ((E_f/E_m)-1))U_f}\right) \quad \text{(A9)}$$

$$G_{LT} = G_m\left(1 + \frac{\left((d/l)+2\right)\left((E_f/E_m)-1\right)U_f}{((G_f/G_m) + 1 + (d/l) - ((G_f/G_m)-1))U_f}\right) \quad \text{(A10)}$$

$$v_{LT} = (1 - U_f)v_m + U_f v_f \quad \text{(A11)}$$

where $\frac{l}{d} = a$ is the ratio between length and diameter of a fiber, whereas the other quantities have been already defined.

On the other hand, Eisenberg method [12] which also focuses on the prediction of the elastic properties of randomly distributed short fiber composites can be synopsized as follows. The stiffness matrix Q for an orthotropic layer of a composite with respect to the principal material axes is defined as

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \quad \text{(A12)}$$

Then a transformation of the matrix consisting of the coefficients Q_{ij} to Q_{ij}' which corresponds to a turn of the rectangular Cartesian coordinate system at angle θ around O $z$axis.

$$[Q] = \begin{bmatrix} Q_{11}' & Q_{12}' & Q_{16}' \\ Q_{21}' & Q_{22}' & Q_{26}' \\ Q_{61}' & Q_{62}' & Q_{66}' \end{bmatrix} \quad \text{(A13)}$$

with

$$Q_{11}' = Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + (2Q_{12} + 4Q_{66})\sin^2\theta\cos^2\theta$$

$$Q_{22}' = Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + (2Q_{12} + 4Q_{66})\sin^2\theta\cos^2\theta$$

$$Q_{66}' = (Q_{11} + Q_{22} - 2Q_{12})\sin^2\theta\cos^2\theta + Q_{66}(\cos^2\theta - \sin^2\theta)^2$$

$$Q_{12}' = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\cos^4\theta + \sin^4\theta)$$

$$Q_{16}' = \sin\theta\cos\theta\left[Q_{11}\cos^2\theta - Q_{22}\sin^2\theta + (Q_{12} + 2Q_{66})(\sin^2\theta - \cos^2\theta)\right]$$

$$Q_{26}' = \sin\theta\cos\theta\left[Q_{11}\sin^2\theta - Q_{22}\cos^2\theta + (Q_{12} + 2Q_{66})(\cos^2\theta - \sin^2\theta)\right]$$

Next, to evaluate the elastic constants of a composite layer with a random distribution of the short fibers, an averaging term of any of the above entries of the matrix Q' is considered and therefore

$$\overline{Q_{ij}} = \frac{1}{\pi}\int_0^{\pi} Q_{ij}' d\theta \quad \text{(A14)}$$

From hence it is evident that,

$$\overline{Q}_{11} = \overline{Q}_{22} = \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}) \quad \text{(A15)}$$

$$\overline{Q}_{12} = \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}) \quad \text{(A16)}$$

$$\overline{Q}_{66} = \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66}) \quad \text{(A17)}$$

$$\overline{Q}_{16} = \overline{Q}_{26} = 0 \quad \text{(A18)}$$

Referring to an isotropic material in a state of plane stress, the matrix Q is simplified as follows

$$Q = \begin{bmatrix} \frac{E}{1-v^2} & \frac{vE}{1-v^2} & 0 \\ \frac{vE}{1-v^2} & \frac{E}{1-v^2} & 0 \\ 0 & 0 & \frac{E}{2(1+v)} \end{bmatrix} \quad \text{(A19)}$$

After all, by equalizing the individual items of the above matrix with the quantities obtained from eqs. (A15) to (A18) one obtains

$$\bar{E} = \frac{\bar{Q}_{11}^2 - \bar{Q}_{12}^2}{\bar{Q}_{11}} = \frac{(Q_{11}+Q_{22}+2Q_{12})(Q_{11}+Q_{22}-2Q_{12}+4Q_{66})}{3Q_{11}+3Q_{22}+2Q_{12}+4Q_{66}} \quad \text{(A20)}$$

$$\bar{v} = \frac{\bar{Q}_{12}}{\bar{Q}_{11}} = \frac{Q_{11}+Q_{22}+6Q_{12}-4Q_{66}}{3Q_{11}+3Q_{22}+2Q_{12}+4Q_{66}} \quad \text{(A21)}$$

$$E_1 = E_f U_f + E_m U_m \quad \text{(A22)}$$

$$E_2 = \frac{E_f U_m}{E_f U_f + E_m U_m} \quad \text{(A23)}$$

$$v_{12} = v_f U_f + v_m U_m \quad \text{(A24)}$$

$$G_f = \frac{E_f}{2(1+v_f)} \quad \text{(A25)}$$

$$G_m = \frac{E_m}{2(1+v_m)} \quad \text{(A26)}$$

$$G_{12} = \frac{G_m G_f}{G_f U_f + G_m U_m} \quad \text{(A27)}$$