Analysis of Vorticity Transport in Magnetic Couple-Stress Fluid

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Abstract:- Results of investigations on the transport of vorticity in couple-stress fluid in the presence of suspended magnetic particles is presented here. Equations governing the transport of vorticity in couple-stress fluid in the presence of suspended magnetic particles are obtained from the equations of magnetic fluid flow proposed by Wagh and Jawandhia [1] in their study on the transport of vorticity in magnetic fluid. It follows from the analysis of these equations that the transport of solid vorticity is coupled with the transport of fluid vorticity. Also it is found that due to thermo-kinetic process, fluid vorticity may exist in the absence of solid vorticity, but when fluid vorticity is zero, then solid vorticity is necessarily zero. A two-dimensional case is also studied and found that the fluid vorticity is indirectly influenced by the temperature and the magnetic field gradient.

Key-Words:- Couple-stress fluid, suspended magnetic particles, vorticity

1 Introduction

The theory of couple-stress fluid has been formulated by Stokes [2]. One of the applications of couple-stress fluid is its use to the study of the mechanisms of lubrications of synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing which has articular cartilage as the bearing and synovial fluid as the lubricant. When a fluid is generated, squeeze-film action is capable of providing considerable protection to the cartilage surface. The shoulder, ankle, knee and hip joints are the loaded-bearing synovial joints of the human body and these joints have a low friction coefficient and negligible wear. Normal synovial fluid is a viscous, non-Newtonian fluid and is generally clear or yellowish. According to the theory of Stokes [2], couple-stresses appear in noticeable magnitudes in fluids with very large molecules.

Many of the flow problems in fluids with couple-stresses, discussed by Stokes, indicate some possible experiments, which could be used for determining the material constants, and the results are found to differ from those of Newtonian fluid. Couple-stresses are found to appear in noticeable magnitudes in polymer solutions for force and couple-stresses. This theory is developed in an effort to examine the simplest generalization of the classical theory, which would allow polar effects. The constitutive equations proposed by Stokes [2] are:

$$T_{(ij)} = \left(-p + \lambda D_{kk}\right)\delta_{ij} + 2\mu D_{ij} \ ,$$

$$T_{[ij]} = -2\eta \vec{W}_{ij.kk} - \frac{\rho}{2}\vec{\varepsilon}_{ijs} G_s \ ,$$

and

$$M_{ij} = 4\eta \vec{\omega}_{j,i} + 4\eta' \vec{\omega}_{i,j} \ ,$$

where

$$D_{ij} = \frac{1}{2}\left(V_{i,j} + V_{j,i}\right),\ \vec{W}_{ij} = -\frac{1}{2}\left(V_{i,j} - V_{j,i}\right)$$

and $$\vec{\omega}_i = \frac{1}{2}\vec{\varepsilon}_{ijk} V_{k,j} \ .$$

Here T_{ij}, $T_{(ij)}$, $T_{[ij]}$, M_{ij}, D_{ij}, $\vec{W}_{i,j}$, $\vec{\omega}_i$, G_s, $\vec{\varepsilon}_{ijk}$, V, ρ and λ, μ, η, η', are stress tensor, symmetric part of T_{ij}, anti-symmetric part of T_{ij}, the couple-stress tensor, deformation tensor, the vorticity tensor, the vorticity vector, body couple, the alternating unit tensor, velocity field, the density and material constants respectively. The dimensions of λ and μ are those of viscosity whereas the dimensions of η and η' are those of momentum.

Since the long chain hyaluronic acid molecules are found as additives in synovial fluids, Walicki and Walicka [3] modeled the synovial fluid as a couple-stress fluid. The synovial fluid is the natural lubricant of joints of the vertebrates. The detailed description of the joint lubrication has very important practical implications. Practically all diseases of joints are caused by or connected with a malfunction of the lubrication. The efficiency of the physiological joint lubrication is caused by several mechanisms. The synovial fluid is, due to its content of the hyaluronic acid, a fluid of high viscosity, near to a gel. Goel et al. [4] have studied the hydromagnetic stability of an unbounded couple-stress binary fluid mixture under rotation with vertical temperature and concentration gradients. Sharma et al. [5] have considered a couple-stress fluid with suspended particles heated from below. They have found that for stationary convection, couple-stress has a stabilizing effect whereas suspended particles have a d estabilizing effect. Kumar et al. [6] have considered the thermal instability of a layer of a couple-stress fluid acted on by a uniform rotation, and have found that for stationary convection, the rotation has a stabilizing effect whereas couple-stress has both stabilizing and destabilizing effects. Thermosolutal convection in a couple-stress fluid in presence of magnetic field and rotation, separately, has been investigated by Kumar and Singh [7, 8].

Further, magnetic fluids are suspensions of small magnetic particles in a liquid carrier. Under normal conditions, the material behaves like a v iscous fluid. When it is exposed to a magnetic field, the particles inside align and it responds to the field, exhibiting magnetized behaviour. There are a n umber of uses for magnetic fluids, ranging from medicine to industrial manufacturing. It is, therefore, a two-phase system consisting of solid and liquid phases. We assume that the liquid phase is non-magnetic in nature and the magnetic force acts only on the magnetic particles. Thus, the magnetic force changes the velocity of the magnetic particles. Consequently, the dragging force acting on the carrier liquid is changed and thus the flow of carrier liquid is also influenced by the magnetic force. Because of the relative velocity between the solid and liquid particles, the net effect of the particles suspended in the fluid is extra dragging force acting on t he system. Taking this force into consideration, Saffman [9] proposed the equations of the flow of suspensions of non-magnetic particles. These equations were modified by Wagh [10] to describe the flow of magnetic fluid, by including the magnetic body force $\mu_0 M\nabla H$.

The transport of vorticity in a magnetic fluid was studied by Wagh and Jawandhia [1]. Transport and sedimentation of suspended particles in inertial pressure-driven flow has been considered by Yan and Koplik [11].

Studying magnetic fluids arouses considerable interest in recent years; and keeping in mind the importance of couple-stress fluid, the present paper attempts to study thetransport of vorticity in magnetic couple-stress fluid-particles mixtures by using the equations proposed by Wagh and Jawandhia [1].

2 Basic Assumptions and Magnetic Body Force

Particles of magnetic material are much larger than the size of the molecules of the carrier liquid. Accordingly considering the limit of a microscopic volume element in which the fluid can be assumed to be continuous medium, magnetic particles must be treated as discrete entities. Now if we consider a cell of magnetic fluid containing a larger number of magnetic particles, then one must consider the microrotation of the cell in addition to its translations as a point mass. Thus, one has to assign average velocity $\vec{q}_d$ and the average angular velocity $\vec{\omega}$ of the cell. But, here as an approximation, we neglect the effect of microrotation. We shall also make the following assumptions:

(i) The free current density $\vec{J}$ is negligible, and $\vec{J} \times \vec{B}$ is insignificant.

(ii) The magnetic field is curl free i.e. $\nabla \times \vec{H} = 0.$

(iii) In many practical situations liquid compressibility is not important. Hence, the contribution due to magnetic friction can be neglected. The remaining force of the magnetic field is referred as magnetization force.

(iv) All time-dependent magnetization effects in the fluid such as hysteresis are negligible, and the magnetization $\vec{M}$ is collinear with $\vec{H}$.

From electromagnetic theory, the force per unit volume (in MKS units) on a piece of magnetized material of magnetization $\vec{M}$ (i.e. dipole moment per unit volume) in the field of magnetic intensity $\vec{H}$ is $\mu_0\left(\vec{M}.\nabla\right)\vec{H}$, where μ_0 is the free space permeability.

Using assumption (iv), we obtain

$$\mu_0\left(\vec{M}.\nabla\right)\vec{H} = \frac{\mu_0 M}{H}\left(\vec{H}.\nabla\right)\vec{H}, \text{ where}$$

$$M = \left|\vec{M}\right| \text{ and } H = \left|\vec{H}\right| . \tag{1}$$

But by assumption (ii), we have

$$\left(\vec{H}.\nabla\right)\vec{H} = \frac{1}{2}\nabla\left(\vec{H}.\vec{H}\right) - \vec{H} \times \left(\nabla \times \vec{H}\right) = \frac{1}{2}\nabla\left(\vec{H}.\vec{H}\right) \tag{2}$$

Hence

$$\mu_0\left(\vec{M}.\nabla\right)\vec{H} = \left(\frac{\mu_0 M}{H}\right)\frac{1}{2}\nabla\left(\vec{H}.\vec{H}\right) = \mu_0 M\nabla H.$$

Thus, the magnetic body force assumes the form (Rosensweig [12])

$$\vec{f}_m = \mu_0 M\nabla H. \tag{3}$$

3 Derivation of Equations Governing Transport of Vorticity in a Magnetic Couple-Stress Fluid

To describe the flow of magnetic fluid by including the magnetic body force $\mu_0 M\nabla H$ acting on the suspended magnetic particles, Wagh [10] modified the Saffman′s equations for flow of suspension. Then the equations expressing the flow of suspended magnetic particles and the flow of couple-stress fluid in which magnetic particles are suspended are written as

$$mN\left[\frac{\partial\vec{V}}{\partial t} + \left(\vec{V}.\nabla\right)\vec{V}\right] = mN\vec{g} + \mu_0 M\nabla H + KN\left(\vec{u} - \vec{V}\right), \tag{4}$$

$$\rho\left[\frac{\partial\vec{u}}{\partial t} + \left(\vec{u}.\nabla\right)\vec{u}\right] = -\nabla P + \rho\vec{g} + \left(\mu - \mu'\nabla^2\right)\nabla^2\vec{u} + KN\left(\vec{V} - \vec{u}\right), \tag{5}$$

where

$P,\ \rho,\ \vec{u}\left(u_x, u_y, u_z\right),\ \vec{g}\left(0, 0, -g\right), \vec{V}\left(l, r, s\right), m, \text{ and } N\left(\bar{x}, t\right)$

respectively denote the pressure minus hydrostatic pressure, density, velocity of fluid particles, gravity force, velocity of solid particles, particle mass and particle number density. Moreover, $\bar{x} = (x, y, z)$ and $K = 6\pi\mu\eta$, where η is the particle radius, is the Stokes' drag coefficient.

In the equations of motion for the fluid, the presence of suspended particles adds an additional force term, proportional to the velocity difference between suspended particles and fluid. Since the force exerted by the fluid on the suspended particles is equal and opposite to that exerted by the particles on t he fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion for the suspended particles.

By making use of the Lagrange's vector identities

$$\left(\vec{q}_d.\nabla\right)\vec{q}_d = \frac{1}{2}\nabla q_d^2 - \vec{q}_d \times \vec{\Omega}, \left(\vec{q}.\nabla\right)\vec{q} = \frac{1}{2}\nabla q^2 - \vec{q} \times \vec{\Omega}_1, \tag{6}$$

equations (4) and (5) become

$$mN\left[\frac{\partial\vec{V}}{\partial t} - \left(\vec{V} \times \vec{\Omega}\right)\right] = -\nabla mNgz - \frac{1}{2}mN\nabla\vec{V}^2 + \mu_0 M\nabla H + KN\left(\vec{u} - \vec{V}\right), \tag{7}$$

$$\rho\left[\frac{\partial \vec{u}}{\partial t}-\left(\vec{u}\times\vec{\Omega}_1\right)\right]=-\nabla P-\nabla\rho g z-\frac{1}{2}\rho\nabla\vec{u}^2+\left(\mu-\mu'\nabla^2\right)\nabla^2\vec{u}+KN\left(\vec{V}-\vec{u}\right), \quad (8)$$

where $\vec{\Omega}=\nabla\times\vec{V}$ and $\vec{\Omega}_1=\nabla\times\vec{u}$ are solid vorticity and fluid vorticity.

Taking the curl of these equations and keeping in view that the curl of a gradient is identically zero, we get

$$mN\left[\frac{\partial\vec{\Omega}}{\partial t}-\left(\nabla\times\vec{V}\times\vec{\Omega}\right)\right]=\mu_0\nabla\times M\nabla H+KN\left(\vec{\Omega}_1-\vec{\Omega}\right), \quad (9)$$

$$\rho\left[\frac{\partial\vec{\Omega}_1}{\partial t}-\left(\nabla\times\vec{u}\times\vec{\Omega}_1\right)\right]=\left(\mu-\mu'\nabla^2\right)\nabla^2\vec{\Omega}_1+KN\left(\vec{\Omega}-\vec{\Omega}_1\right). \quad (10)$$

By making use of the vector identities

$$\nabla\times\left(\vec{V}\times\vec{\Omega}\right)=\left(\vec{\Omega}.\nabla\right)\vec{V}-\left(\vec{V}.\nabla\right)\vec{\Omega}+\vec{V}\nabla.\vec{\Omega}-\vec{\Omega}\nabla.\vec{V}=\left(\vec{\Omega}.\nabla\right)\vec{V}-\left(\vec{V}.\nabla\right)\vec{\Omega}, \quad (11)$$

$$\nabla\times\left(\vec{u}\times\vec{\Omega}_1\right)=\left(\vec{\Omega}_1.\nabla\right)\vec{u}-\left(\vec{u}.\nabla\right)\vec{\Omega}_1+\vec{u}\nabla.\vec{\Omega}_1-\vec{\Omega}_1\nabla.\vec{u}=\left(\vec{\Omega}_1.\nabla\right)\vec{u}-\left(\vec{u}.\nabla\right)\vec{\Omega}_1, \quad (12)$$

equations (9) and (10) become

$$mN\frac{D\vec{\Omega}}{Dt}=\mu_0\nabla\times M\nabla H+mN\left(\vec{\Omega}.\nabla\right)\vec{V}+KN\left(\vec{\Omega}_1-\vec{\Omega}\right), \quad (13)$$

$$\frac{D\vec{\Omega}_1}{Dt}=\left(\nu-\frac{\mu'}{\rho}\nabla^2\right)\nabla^2\vec{\Omega}_1+\left(\vec{\Omega}_1.\nabla\right)\vec{u}+\frac{KN}{\rho}\left(\vec{\Omega}-\vec{\Omega}_1\right), \quad (14)$$

where ν is kinematic viscosity and $\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+\left(\vec{V}.\nabla\right)$ is the convective derivative.

In equation (13),

$$\nabla\times\left(M\nabla H\right)=\left(\nabla M\times\nabla H\right)+\left(M\nabla\times\nabla H\right). \quad (15)$$

Since the curl of the gradient is zero, the last term in equation (15) is zero. Also since $M=M\left(H,T\right)$.

Therefore, $\nabla M=\left(\frac{\partial M}{\partial H}\right)\nabla H+\left(\frac{\partial M}{\partial T}\right)\nabla T.$ (16)

By making use of (16), equation (15) becomes

$$\nabla\times\left(M\nabla H\right)=\left(\frac{\partial M}{\partial H}\right)\nabla H\times\nabla H+\left(\frac{\partial M}{\partial T}\right)\nabla T\times\nabla H. \quad (17)$$

The first term on the right hand side of this equation is clearly zero, hence we get

$$\nabla\times\left(M\nabla H\right)=\left(\frac{\partial M}{\partial T}\right)\nabla T\times\nabla H. \quad (18)$$

Putting this expression in equation (13), we obtain

$$mN\frac{D\vec{\Omega}}{Dt}=\mu_0\left(\frac{\partial M}{\partial T}\right)\nabla T\times\nabla H+mN\left(\vec{\Omega}.\nabla\right)\vec{V}+KN\left(\vec{\Omega}_1-\vec{\Omega}\right). \quad (19)$$

Here (14) and (19) are the equations governing the transport of vorticity in magnetic couple-stress fluid-particle mixtures.

4 Discussion

In equation (19), the first term in the right-hand side i.e. $\mu_0\left(\frac{\partial M}{\partial T}\right)\nabla T\times\nabla H$ describes the production of vorticity due to thermo-kinetic processes. The last term $KN\left(\vec{\Omega}_1-\vec{\Omega}\right)$ gives the change in solid vorticity on account of the exchange of vorticity between the liquid and solid.

From equations (14) and (19), it follows that the transport of solid vorticity $\vec{\Omega}$ is coupled with the transport of fluid vorticity $\vec{\Omega}_1$.

From equation (19), we see that if solid vorticity $\vec{\Omega}$ is zero, then the fluid vorticity $\vec{\Omega}_1$ is not-zero, but is given by

$$\vec{\Omega}_1=-\frac{\mu_0}{KN}\left(\frac{\partial M}{\partial T}\right)\nabla T\times\nabla H. \quad (20)$$

This implies that due to thermo-kinetic processes, fluid vorticity can exist in the absence of solid vorticity.

From equation (14), we find that if $\vec{\Omega}_1$ is zero, then $\vec{\Omega}$ is also zero. This implies that when fluid vorticity is zero, then solid vorticity is necessarily zero**.**

In the absence of suspended magnetic particles, N is zero and magnetization M is also zero. Then, equation (19) is identically satisfied and equation (14) reduces to

$$\frac{D\vec{\Omega}_1}{Dt} = \left(\nu - \frac{\mu'}{\rho}\nabla^2\right)\nabla^2\vec{\Omega}_1 + \left(\vec{\Omega}_1.\nabla\right)\vec{u}. \qquad (21)$$

This equation is the vorticity transport equation. The last term on the right hand side of equation (21) represents the rate at which $\vec{\Omega}_1$ varies for a given particle, when the vortex lines move with the fluid, the strengths of the vortices remaining constant. The first term represents the rate of dissipation of vorticity through friction (resistance) and rate of change of vorticity due to fluid viscoelasticity.

Two-Dimensional Case

Here we consider the two-dimensional case:

Let $$\vec{V} = v_x(x,y)\hat{i} + v_y(x,y)\hat{j}\ , \quad \vec{u} = u_x(x,y)\hat{i} + u_y(x,y)\hat{j}\ , \qquad (22)$$

where components v_x, v_y *and* u_x, u_y are functions of x, y *and* t, then

$$\vec{\Omega} = \Omega_z\hat{k}\ ,\ \vec{\Omega}_1 = \Omega_{1z}\hat{k}\ . \qquad (23)$$

In two-dimensional case, equation (19) becomes

$$\frac{D\Omega_z}{Dt} = \frac{\mu_0}{mN}\left(\frac{\partial M}{\partial T}\right)\left(\frac{\partial T}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial x}\frac{\partial T}{\partial y}\right) + \frac{K}{m}\left(\Omega_{1z} - \Omega_z\right), \qquad (24)$$

and equation (14) similarly becomes

$$\frac{D\Omega_{1z}}{Dt} = \nu\nabla^2\left(\Omega_{1z}\right) - \frac{\mu'}{\rho}\nabla^2\nabla^2\left(\Omega_{1z}\right) + \frac{KN}{\rho}\left(\Omega_z - \Omega_{1z}\right), \qquad (25)$$

since it can be easily verified that

$$\left(\vec{\Omega}.\nabla\right)\vec{V} = 0 \text{ and } \left(\vec{\Omega}_1.\nabla\right)\vec{u} = 0. \qquad (26)$$

The first term on the right hand side of equation (25) is the change of fluid vorticity due to internal friction (resistance). The second term is the gradient of fluid vorticity due to couple-stress viscosity and the third term is change in fluid vorticity due to the exchange of vorticity between solid and liquid. Equation (25) does not explicitly involve the term representing change of vorticity due to magnetic field gradient and/or temperature gradient. But equation (24) shows that solid vorticity Ω_z depends on these factors. Hence, it follows that the fluid vorticity is indirectly influenced by the temperature and the magnetic field gradient.

In the absence of magnetic particles, N is zero and magnetization M is also zero. Equation (24) is therefore satisfied identically, and equation (25) reduces to the classical equation for the transport of fluid vorticity. If we consider a suspension of non-magnetic particles instead of a magnetic fluid, then the corresponding equation for the transport of vorticity may be obtained by setting M equal to zero in the equations governing the transport of vorticity in magnetic fluids. If magnetization M of the magnetic particles is independent of temperature, then the first term of equations (19) and (24) vanishes, and the equations governing the transport of vorticity in a magnetic fluid become the same as those governing the transport of vorticity in non-magnetic suspensions.

If the temperature gradient ∇T vanishes or if the magnetic field gradient ∇H vanishes or if ∇T is parallel to ∇H, then also the first term in (19) and (24) vanishes. We thus see that the transport of vorticity in a magnetic fluid is also the same as the transport of vorticity in non-magnetic suspension.

5 Conclusions

Magnetic fluids are suspensions of small magnetic particles in a liquid carrier. Saffman [9] proposed the equations of the flow of suspension of non-magnetic particles. These equations were modified by Wagh [10] to describe the flow of magnetic fluid, by including the magnetic body force, and Wagh and Jawandhia [1] have studied the transport of vorticity in a magnetic fluid. The fluid is considered as Newtonian in all these studies. Theory of couple-stress fluid has been formulated by Stokes [2]. One of the applications of couple-stress fluid is its use to the study of the mechanism of lubrications of synovial joints, which has become the object of scientific research; therefore, in the present study we discuss the transport of vorticity in magnetic couple-stress fluid. We obtain the equations governing such a transport of vorticity from the equations of magnetic fluid flow proposed by Wagh and Jawandhia [1] and by taking Stokes [2] constitutive relations into consideration. The main conclusions from the analysis of this paper are as follows:

Transport of solid vorticity is coupled with the transport of fluid vorticity.

Due to thermo-kinetic process, fluid vorticity may exist in the absence of solid vorticity, but when fluid vorticity is zero, then solid vorticity is necessary zero.

In two-dimensional case, it is found that fluid vorticity is indirectly influenced by the temperature and the magnetic field gradient.

The effect of couple-stress viscosity is seen in equation (25) where the second term in the right hand side gives the gradient of fluid vorticity due to couple-stress viscocity.

It is also found that if magnetization of the magnetic particles is independent of temperature, then equations governing the transport of vorticity in a magnetic fluid become the same as those governing the transport of vorticity in non-magnetic suspensions.

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