Cases of integrability corresponding to the motion of a pendulum on the two-dimensional plane

MAXIM V. SHAMOLIN
Lomonosov Moscow State University
Institute of Mechanics
Michurinskii Ave., 1, 119192 Moscow
RUSSIAN FEDERATION
shamolin@rambler.ru, shamolin@imec.msu.ru

Abstract: We systematize some results on the study of the equations of plane-parallel motion of symmetric fixed rigid bodies–pendulums located in a nonconservative force fields. The form of these equations is taken from the dynamics of real fixed rigid bodies placed in a homogeneous flow of a medium. In parallel, we study the problem of a plane-parallel motion of a free rigid body also located in a similar force fields. Herewith, this free rigid body is influenced by a nonconservative tracing force; under action of this force, either the magnitude of the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint, or the center of mass of the body moves rectilinearly and uniformly; this means that there exists a nonconservative couple of forces in the system.

Key–Words: Rigid body, Pendulum, Resisting Medium, Dynamical Systems With Variable Dissipation, Integrability

1 Model assumptions

Let consider the homogeneous flat plate AB symmetrical relative to the plane which perpendicular to the plane of motion and passing through the holder OD. The plate is rigidly fixed perpendicular to the tool holder OD located on the cylindrical hinge O, and it flows about homogeneous fluid flow. In this case, the body is a physical pendulum, in which the plate AB and the pivot axis perpendicular to the plane of motion. The medium flow moves from infinity with constant velocity $\mathbf{v} = \mathbf{v}_\infty \neq \mathbf{0}$. Assume that the holder does not create a resistance.

We suppose that the total force $\mathbf{S}$ of medium flow interaction is parallel to the holder, and point N of application of this force is determined by at least the angle of attack α, which is made by the velocity vector $\mathbf{v}_D$ of the point D with respect to the flow and the holder, and also the reduced angular velocity

$$\omega \cong \frac{l\Omega}{v_D},\ v_D = |\mathbf{v}_D|$$

(l is the length of the holder, Ω is the algebraic value of a projection of the pendulum angular velocity to the axle hinge). Such conditions arise when one uses the model of streamline flow around plane bodies [1, 2].

Therefore, the force $\mathbf{S}$ is directed along the normal to the plate to its side, which is opposite to the direction of the velocity $\mathbf{v}_D$, and passes through a certain point N of the plate, which is displaced from the point D forward with respect to the flow (see also [1, 3]).

The vector

$$\mathbf{e} = \frac{\mathbf{OD}}{l} \tag{1}$$

determines the orientation of the holder. Then

$$\mathbf{S} = -s(\alpha)v_D^2\mathbf{e}, \tag{2}$$

where

$$s(\alpha) = s_1(\alpha)\mathrm{sign}\cos\alpha, \tag{3}$$

and the resistance coefficient $s_1 \geq 0$ depends only on the angle of attack α. By the plate symmetry properties with respect to the point D, the function $s(\alpha)$ is even.

Let $Dx_1x_2 = Dxy$ be the coordinate system rigidly attached to the body, herewith, the axis $Dx = Dx_1$ has a direction vector $\mathbf{e}$ (see (1)), and the axis $Dx_2 = Dy$ has the same direction with the vector $\mathbf{DA}$.

The space of positions of this physical pendulum is the circle (one-dimensional sphere)

$$\mathbf{S}^1\{\xi \in \mathbf{R}^1 :\ \xi \bmod 2\pi\}, \tag{4}$$

and its phase space is the tangent bundle of a circle

$$T_*\mathbf{S}^1\{(\dot{\xi};\xi) \in \mathbf{R}^2 :\ \xi \bmod 2\pi\}, \tag{5}$$

i.e., two-dimensional cylinder.

To the value Ω, we put in correspondence the skew-symmetric matrix

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}, \ \tilde{\Omega} \in \mathrm{so}(2). \tag{6}$$

The distance from the center D of the plate to the center of pressure (the point N), has the form

$$|\mathbf{r}_N| = r_N = DN\left(\alpha, \frac{l\Omega}{v_D}\right), \tag{7}$$

where

$$\mathbf{r}_N = \{0, x_{2N}\} = \{0, y_N\}$$

in system $Dx_1x_2 = Dxy$.

Immediately, we note that the model used to describe the effects of fluid flow on fixed pendulum is similar to the model constructed for free body and, in further, takes into account of the rotational derivative of the moment of the forces of medium influence with respect to the pendulum angular velocity (see also [3]). An analysis of the problem of the physical pendulum in a flow will allow to find the qualitative analogies in the dynamics of partially fixed bodies and free ones.

2 Set of dynamical equations in Lie algebra so(2)

If I is a central moment of inertia of a rigid body–pendulum then the general equation of motion has the following form:

$$I\dot{\Omega} = DN\left(\alpha, \frac{l\Omega}{v_D}\right) s(\alpha) v_D^2, \tag{8}$$

since the moment of the medium interaction force equals the determinant of the following auxiliary matrix:

$$\begin{pmatrix} 0 & x_{2N} \\ -s(\alpha)v_D^2 & 0 \end{pmatrix}, \tag{9}$$

where

$$\{-s(\alpha)v_D^2, 0\}$$

is the decomposition of the medium interaction force **S** in the coordinate system Dx_1x_2.

Since the dimension of the Lie algebra so(2) is equal to 1, the single equation (8) is a group equations on so(2), and, simply speaking, the motion equation.

We see, that in the right-hand side of Eq. (8), first of all, it includes the angle of attack, therefore, this equation is not closed. In order to obtain a complete system of equations of motion of the pendulum, it is necessary to attach several sets of kinematic equations to the dynamic equation on the Lie algebra so(2).

3 First set of kinematic equations

In order to obtain a complete system of equations of motion, it needs the set of kinematic equations which relate the velocities of the point D (i.e., the formal center of the plate AB) and the over-running medium flow:

$$\mathbf{v}_D = v_D \cdot \mathbf{i}_v(\alpha) = \tilde{\Omega}\begin{pmatrix} l \\ 0 \end{pmatrix} + (-v_\infty)\mathbf{i}_v(-\xi), \tag{10}$$

where

$$\mathbf{i}_v(\alpha) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}. \tag{11}$$

The equation (10) expresses the theorem of addition of velocities in projections on the related coordinate system Dx_1x_2.

Indeed, the left-hand side of Eq. (10) is the velocity of the point D of the pendulum with respect to the flow in the projections on the related with the pendulum coordinate system Dx_1x_2. Herewith, the vector $\mathbf{i}_v(\alpha)$ is the unit vector along the axis of the vector $\mathbf{v}_D$. The vector $\mathbf{i}_v(\alpha)$ is the image of the unit vector along the axis Dx_1, rotated around the vertical (the axis Dx_3) by the angle α and has the decomposition (11).

The right-hand side of the Eq. (10) is the sum of the velocities of the point D when you rotate the pendulum (the first term), and the motion of the flow (the second term). In this case, in the first term, we have the coordinates of the vector $\mathbf{OD} = \{l, 0\}$ in the coordinate system Dx_1x_2.

We explain the second term of the right-hand side of Eq. (10) in more detail. We have in it the coordinates of the vector $(-\mathbf{v}_\infty) = \{-v_\infty, 0\}$ in the immovable space. In order to describe it in the projections on the related coordinate system Dx_1x_2, we need to make a (reverse) rotation of the pendulum at the angle $(-\xi)$ that is algebraically equivalent to multiplying the value $(-v_\infty)$ on the vector $\mathbf{i}_v(-\xi)$.

Thus, the first set of kinematic equations (10) has the following form in our case:

$$\begin{aligned} v_D\cos\alpha &= -v_\infty\cos\xi, \\ v_D\sin\alpha &= l\Omega + v_\infty\sin\xi. \end{aligned} \tag{12}$$

4 Second set of kinematic equations

We also need a set of kinematic equations which relate the angular velocity tensor $\tilde{\Omega}$ and coordinates $\dot{\xi}, \xi$ of the phase space (5) of pendulum studied, i.e., the tangent bundle $T_*\mathbf{S}^1\{\dot{\xi}; \xi\}$.

We draw the reasoning style allowing arbitrary dimension. The desired equations are obtained from the following two sets of relations. Since the motion of the body takes place in a Euclidean space $\mathbf{E}^n, n = 2$ formally, at the beginning, we express the tuple consisting of a phase variable Ω, through new variable z_1 (from the tuple z):

$$\Omega = z_1. \tag{13}$$

Then we substitute the following relationship instead of the variable z:

$$z_1 = \dot{\xi}. \tag{14}$$

Thus, two sets of Eqs. (13) and (14) give the second set of kinematic equations:

$$\Omega = \dot{\xi}. \tag{15}$$

We see that three sets of the relations (8), (12), and (15) form the closed system of equations.

These three sets of equations include the following two functions:

$$r_N = DN\left(\alpha, \frac{l\Omega}{v_D}\right),\ s(\alpha). \tag{16}$$

In this case, the function s is considered to be dependent only on α, and the function $r_N = DN$ may depend on, along with the angle α, generally speaking, the reduced angular velocity $\omega \cong l\Omega/v_D$.

5 Problem on free body motion under assumption of tracing force

Parallel to the present problem of the motion of the fixed body, we study the plane-parallel motion of the free symmetric rigid body with the frontal plane butt-end (one-dimensional plate AB) in the resistance force fields under the quasi-stationarity conditions [4, 5] with the same model of medium interaction.

If (v, α) are the polar coordinates of the velocity vector of the certain characteristic point D of the rigid body (D is the center of the plate AB), Ω is the value of its angular velocity, I, m are characteristics of inertia and mass, then the dynamical part of the equations of motion in which the tangent forces of the interaction of the body with the medium are absent, has the form

$$\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha - \Omega v\sin\alpha + \sigma\Omega^2 = \frac{F_x}{m},$$

$$\dot{v}\sin\alpha + \dot{\alpha}v\cos\alpha + \Omega v\cos\alpha - \sigma\dot{\Omega} = 0, \tag{17}$$

$$I\dot{\Omega} = y_N\left(\alpha, \frac{\Omega}{v}\right)s(\alpha)v^2,$$

where

$$F_x = -S,\ S = s(\alpha)v^2,\ \sigma = CD, \tag{18}$$

in this case

$$\left(0, y_N\left(\alpha, \frac{\Omega}{v}\right)\right) \tag{19}$$

are the coordinates of the point N of application of the force **S** in the coordinate system $Dx_1x_2 = Dxy$ related to the body.

The first two equations of the system (17) describe the motion of the center of a mass in the two-dimensional Euclidean plane $\mathbf{E}^2$ in the projections on the coordinate system Dx_1x_2. In this case, $Dx_1 = Dx$ is the perpendicular to the plate passing through the center of mass C of the symmetric body and $Dx_2 = Dy$ is an axis along the plate. The third equation of the system (17) is obtained from the theorem on the change of the angular moment of a rigid body in the projection on the axis perpendicular to the plane of motion.

Thus, the direct product

$$\mathbf{R}^1 \times \mathbf{S}^1 \times \mathrm{so}(2) \tag{20}$$

of the two-dimensional cylinder and the Lie algebra $\mathrm{so}(2)$ is the phase space of third-order system (17) of the dynamical equations. Herewith, since the medium influence force dos not depend on the position of the body in a plane, the system (17) of the dynamical equations *is separated from the system of kinematic equations* and may be studied independently (see also [2, 6]).

5.1 Nonintegrable constraint

If we consider *a more general problem* on the motion of a body under the action of a certain tracing force **T** passing through the center of mass and providing the fulfillment of the equality

$$v \equiv \text{const}, \tag{21}$$

during the motion (see also [7, 8]), then F_x in system (17) must be replaced by

$$T - s(\alpha)v^2. \tag{22}$$

As a result of an appropriate choice of the magnitude T of the tracing force, we can achieve the fulfillment of Eq. (21) during the motion [9]. Indeed, if we formally express the value T by virtue of system (17), we obtain (for $\cos\alpha \neq 0$):

$$T = T_v(\alpha, \Omega) = m\sigma\Omega^2 +$$

$$+s(\alpha)v^2\left[1-\frac{m\sigma}{I}y_N\left(\alpha,\frac{\Omega}{v}\right)\frac{\sin\alpha}{\cos\alpha}\right]. \quad (23)$$

This procedure can be viewed from two standpoints. First, a transformation of the system has occurred at the presence of the tracing (control) force in the system which provides the corresponding class of motions (21). Second, we can consider this procedure as a procedure that allows one to reduce the order of the system. Indeed, system (17) generates an independent second-order system of the following form:

$$\begin{array}{c}\dot\alpha v\cos\alpha+\Omega v\cos\alpha-\sigma\dot\Omega=0,\\ I\dot\Omega=y_N\left(\alpha,\frac{\Omega}{v}\right)s(\alpha)v^2,\end{array} \quad (24)$$

where the parameter v is supplemented by the constant parameters specified above.

We can see from (24) that the system cannot be solved uniquely with respect to $\dot\alpha$ on the manifold

$$O=\left\{(\alpha,\Omega)\in\mathbf{R}^2:\ \alpha=\frac{\pi}{2}+\pi k,\ k\in\mathbf{Z}\right\} \quad (25)$$

Thus, formally speaking, the uniqueness theorem is violated on manifold (25).

This implies that system (24) outside of the manifold (25) (and only outside it) is equivalent to the following system:

$$\begin{array}{c}\dot\alpha=-\Omega+\frac{\sigma v}{I}\frac{y_N\left(\alpha,\frac{\Omega}{v}\right)s(\alpha)}{\cos\alpha},\\ \dot\Omega=\frac{1}{I}y_N\left(\alpha,\frac{\Omega}{v}\right)s(\alpha)v^2.\end{array} \quad (26)$$

The uniqueness theorem is violated for system (24) on the manifold (25) in the following sense: regular phase trajectories of system (24) pass through almost all points of the manifold (25) and intersect the manifold (25) at a right angle, and also there exists a phase trajectory that completely coincides with the specified point at all time instants. However, these trajectories are different since they correspond to different values of the tracing force. Let us prove this.

As was shown above, to fulfill constraint (21), one must choose the value of T for $\cos\alpha\neq 0$ in the form (23).

Let

$$\lim_{\alpha\to\pi/2}\frac{y_N\left(\alpha,\frac{\Omega}{v}\right)s(\alpha)}{\cos\alpha}=L\left(\frac{\Omega}{v}\right). \quad (27)$$

Note that $|L|<+\infty$ if and only if

$$\lim_{\alpha\to\pi/2}\left|\frac{\partial}{\partial\alpha}\left(y_N\left(\alpha,\frac{\Omega}{v}\right)s(\alpha)\right)\right|<+\infty. \quad (28)$$

For $\alpha=\pi/2$, the necessary magnitude of the tracing force can be found from the equality

$$T=T_v\left(\frac{\pi}{2},\Omega\right)=m\sigma\Omega^2-\frac{m\sigma Lv^2}{I}. \quad (29)$$

where Ω is arbitrary.

On the other hand, if we support the rotation around a certain point W of the Euclidean plane $\mathbf{E}^2$ by means of the tracing force, then the tracing force has the form

$$T=T_v\left(\frac{\pi}{2},\Omega\right)=\frac{mv^2}{R_0}, \quad (30)$$

where R_0 is the distance CW.

Generally speaking, Eqs. (29) and (30) define different values of the tracing force T for almost all points of manifold (25), and the proof is complete.

5.2 Constant velocity of the center of mass

If we consider *a more general problem* on the motion of a body under the action of a certain tracing force $\mathbf{T}$ passing through the center of mass and providing the fulfillment of the equality (see also [10])

$$\mathbf{V}_C\equiv\mathbf{const} \quad (31)$$

during the motion ($\mathbf{V}_C$ is the velocity of the center of mass), then F_x in system (17) must be replaced by zero since the nonconservative couple of the forces acts on the body:

$$T-s(\alpha)v^2\equiv 0. \quad (32)$$

Obviously, we must choose the value of the tracing force T as follows:

$$T=T_v(\alpha,\Omega)=s(\alpha)v^2,\ \mathbf{T}\equiv-\mathbf{S}. \quad (33)$$

The choice (33) of the magnitude of the tracing force T is a particular case of the possibility of separation of an independent second-order subsystem after a certain transformation of the third-order system (17).

Indeed, let the following condition hold for T:

$$\begin{array}{c}T=T_v(\alpha,\Omega)=\tau_1\left(\alpha,\frac{\Omega}{v}\right)v^2+\\ +\tau_2\left(\alpha,\frac{\Omega}{v}\right)\Omega v+\tau_3\left(\alpha,\frac{\Omega}{v}\right)\Omega^2=\\ =T_1\left(\alpha,\frac{\Omega}{v}\right)v^2.\end{array} \quad (34)$$

We can rewrite system (17) as follows:

$$\dot v+\sigma\Omega^2\cos\alpha-\sigma\sin\alpha\left[\frac{v^2}{I}y_N\left(\alpha,\frac{\Omega}{v}\right)s(\alpha)\right]=$$

$$=\frac{T_1\left(\alpha,\frac{\Omega}{v}\right)v^2-s(\alpha)v^2}{m}\cos\alpha,$$

$$\dot{\alpha}v + \Omega v - \sigma\cos\alpha\left[\frac{v^2}{I}y_N\left(\alpha, \frac{\Omega}{v}\right)s(\alpha)\right] -$$
$$-\sigma\Omega^2\sin\alpha =$$
$$= \frac{s(\alpha)v^2 - T_1\left(\alpha, \frac{\Omega}{v}\right)v^2}{m}\sin\alpha, \tag{35}$$
$$\dot{\Omega} = \frac{v^2}{I}y_N\left(\alpha, \frac{\Omega}{v}\right)s(\alpha).$$

If we introduce the new dimensionless phase variable and the differentiation by the formulas

$$\Omega = n_1 v\omega, \;\; <\cdot>= n_1 v <'>, \; n_1 > 0, \tag{36}$$
$$n_1 = \text{const},$$

then system (35) is reduced to the following form:

$$v' = v\Psi(\alpha, \omega), \tag{37}$$

$$\left.\begin{array}{c} \alpha' = -\omega + \sigma n_1\omega^2\sin\alpha + \\ +\left[\frac{\sigma}{In_1}y_N\left(\alpha, n_1\omega\right)s(\alpha)\right]\cos\alpha - \\ -\frac{T_1(\alpha, n_1\omega) - s(\alpha)}{mn_1}\sin\alpha, \\ \omega' = \frac{1}{In_1^2}y_N\left(\alpha, n_1\omega\right)s(\alpha) - \\ -\omega\left[\frac{\sigma}{In_1}y_N\left(\alpha, n_1\omega\right)s(\alpha)\right]\sin\alpha + \\ +\sigma n_1\omega^3\cos\alpha - \omega\frac{T_1(\alpha, n_1\omega) - s(\alpha)}{mn_1}\cos\alpha, \end{array}\right\} \tag{38}$$

$$\Psi(\alpha, \omega) = -\sigma n_1\omega^2\cos\alpha +$$
$$+\left[\frac{\sigma}{In_1}y_N\left(\alpha, n_1\omega\right)s(\alpha)\right]\sin\alpha +$$
$$+\frac{T_1\left(\alpha, n_1\omega\right) - s(\alpha)}{mn_1}\cos\alpha.$$

We see that the independent second-order subsystem (38) can be substituted into the third-order system (37), (38) and can be considered separately on its own two-dimensional phase cylinder.

In particular, if condition (33) holds, then the method of separation of an independent second-order subsystem is also applicable.

6 Case where the moment of nonconservative forces is independent of the angular velocity

We take the function $\mathbf{r}_N$ as follows (the plate AB is given by the equation $x_{1N} \equiv 0$):

$$\mathbf{r}_N = \begin{pmatrix} 0 \\ x_{2N} \end{pmatrix} = R(\alpha)\mathbf{i}_N, \tag{39}$$

where

$$\mathbf{i}_N = \mathbf{i}_v\left(\frac{\pi}{2}\right) \tag{40}$$

(see (11)).

In our case

$$\mathbf{i}_N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{41}$$

Thus, the equality

$$x_{2N} = R(\alpha) \tag{42}$$

holds and shows that for the considered system, the moment of the nonconservative forces is independent of the angular velocity (it depends only on the angle α).

And so, for the construction of the force field, we use the pair of dynamical functions $R(\alpha), s(\alpha)$; the information about them is of a qualitative nature. Similarly to the choice of the Chaplygin analytical functions (see [11]), we take the dynamical functions s and R as follows: s

$$R(\alpha) = A\sin\alpha, \; s(\alpha) = B\cos\alpha, \; A, B > 0. \tag{43}$$

6.1 Reduced systems

Theorem 1 *The simultaneous equations (8), (12), (15) under conditions (39), (43) can be reduced to the dynamical system on the tangent bundle (5) of the one-dimensional sphere (4).*

Indeed, if we introduce the dimensionless parameter and the differentiation by the formulas

$$b_* = ln_0, \; n_0^2 = \frac{AB}{I}, \;\; <\cdot>= n_0 v_\infty <'>, \tag{44}$$

then the obtained equation has the following form:

$$\xi'' + b_*\xi'\cos\xi + \sin\xi\cos\xi = 0. \tag{45}$$

After the transition from the variables z (about the variables z see (14)) to the variables w

$$w_1 = -\frac{1}{n_0 v_\infty}z_1 - b_*\sin\xi, \tag{46}$$

Eq. (45) is equivalent to the system

$$\begin{array}{c} \xi' = -w_1 - b_*\sin\xi, \\ w_1' = \sin\xi\cos\xi, \end{array} \tag{47}$$

on the tangent bundle

$$T_*\mathbf{S}^1\{(w_1;\xi) \in \mathbf{R}^2: \; \xi \bmod 2\pi\} \tag{48}$$

of the one-dimensional sphere $\mathbf{S}^1\{\xi \in \mathbf{R}^1: \; \xi \bmod 2\pi\}$.

6.2 General remarks on integrability of system

In order to integrate the second-order system (47), we have to obtain, generally speaking, one independent first integral.

6.2.1 The system under the absence of a force field

Let study the system (47) on the tangent bundle $T_*\mathbf{S}^1\{w_1;\xi\}$ of the one-dimensional sphere $\mathbf{S}^1\{\xi\}$. At the same time, we get out of this system the conservative one. Furthermore, we assume that the function (7) is identically equal to zero (in particular, $b_* = 0$, and also the coefficient $\sin\xi\cos\xi$ in the second equation of system (47) is absent). The system studied has the form

$$\xi' = -w_1, \tag{49}$$

$$w_1' = 0. \tag{50}$$

The system (49), (50) describes the motion of a rigid body in the absence of an external force field.

Theorem 2 *System (49), (50) has one analytical first integral as follows:*

$$\Phi_1(w_1;\xi) = w_1^2 = C_1 = \textit{const}. \tag{51}$$

This first integral (51) states that as the external force field is not present, it is preserved (in general, nonzero) the component of the angular velocity of a ("two-dimensional") rigid body, precisely

$$\Omega \equiv \Omega^0 = \text{const}. \tag{52}$$

In particular, the existence of the first integral (51) is explained by the equation

$$w_1^2 = \frac{1}{n_0^2 v_\infty^2}\Omega^2 \equiv C_1 = \text{const}. \tag{53}$$

6.2.2 The system under the presence of a conservative force field

Now let us study the system (47) under assumption $b_* = 0$. In this case, we obtain the conservative system. Precisely, the coefficient $\sin\xi\cos\xi$ in the second equation of system (47) (unlike the system (49), (50)) characterizes the presence of the force field. The system studied has the form

$$\alpha' = -w_1, \tag{54}$$

$$w_1' = \sin\xi\cos\xi. \tag{55}$$

Thus, the system (54), (55) describes the motion of a rigid body in a conservative field of external forces.

Theorem 3 *System (54), (55) has one analytical first integral as follows:*

$$\Phi_1(w_1;\xi) = w_1^2 + \sin^2\xi = C_1 = \textit{const}, \tag{56}$$

The first integral (56) is an integral of the total energy.

6.3 Transcendental first integral

We turn now to the integration of the desired second-order system (47) (without any simplifications, i.e., in the presence of all coefficients).

We put in correspondence to system (47) the following nonautonomous first-order equation:

$$\frac{dw_1}{d\xi} = \frac{\sin\xi\cos\xi}{-w_1 - b_*\sin\xi}. \tag{57}$$

Using the substitution $\tau = \sin\xi$, we rewrite Eq. (57) in the algebraic form

$$\frac{dw_1}{d\tau} = \frac{\tau}{-w_1 - b_*\tau}. \tag{58}$$

Further, introducing the homogeneous variable by the formula $w_1 = u\tau$, we reduce Eq. (58) to the following quadrature:

$$\frac{(-b_* - u)du}{1 + b_*u + u^2} = \frac{d\tau}{\tau}. \tag{59}$$

Integration of quadrature (59) leads to the following three cases. Simple calculations yield the following first integrals.

I. $b_*^2 - 4 < 0$.

$$\ln(1 + b_*u + u^2) + \\ + \frac{2b_*}{\sqrt{4 - b_*^2}}\text{arctg}\frac{2u + b_*}{\sqrt{4 - b_*^2}} + \ln\tau^2 = \text{const}. \tag{60}$$

II. $b_*^2 - 4 > 0$.

$$\ln|1 + b_*u + u^2| - \\ - \frac{b_*}{\sqrt{b_*^2 - 4}}\ln\left|\frac{2u + b_* + \sqrt{b_*^2 - 4}}{2u + b_* - \sqrt{b_*^2 - 4}}\right| + \\ + \ln\tau^2 = \text{const}. \tag{61}$$

III. $b_*^2 - 4 = 0$.

$$\ln|u - 1| + \frac{1}{u - 1} + \ln|\tau| = \text{const}. \tag{62}$$

In other words, in the variables (ξ, w_1) the found first integrals have the following forms:

I. $b_*^2 - 4 < 0$.

$$[\sin^2\xi + b_* w_1 \sin\xi + w_1^2]\times$$
$$\times\exp\left\{\frac{2b_*}{\sqrt{4-b_*^2}}\mathrm{arctg}\frac{2w_1 + b_*\sin\xi}{\sqrt{4-b_*^2}\sin\xi}\right\} = \mathrm{const}. \tag{63}$$

II. $b_*^2 - 4 > 0$.

$$[\sin^2\xi + b_* w_1 \sin\xi + w_1^2]\times$$
$$\times\left|\frac{2w_1 + b_*\sin\xi + \sqrt{b_*^2-4}\sin\xi}{2w_1 + b_*\sin\xi - \sqrt{b_*^2-4}\sin\xi}\right|^{-b_*/\sqrt{b_*^2-4}} = \tag{64}$$
$$= \mathrm{const}.$$

III. $b_*^2 - 4 = 0$.

$$(w_1 - \sin\xi)\exp\left\{\frac{\sin\xi}{w_1 - \sin\xi}\right\} = \mathrm{const}. \tag{65}$$

Therefore, in the considered case the system of dynamical equations (47) has the first integral expressed by relations (63)–(65) (or (60)–(62)), which is a transcendental function of its phase variables (in the sense of complex analysis) and is expressed as a finite combination of elementary functions.

Theorem 4 *Three sets of relations (8), (12), (15) under conditions (39), (43) possess the first integral (the complete set), which is a transcendental function (in the sense of complex analysis) and is expressed as a finite combination of elementary functions.*

6.4 Topological analogies

Now we present two groups of analogies related to the system (17), which describes the motion of a free body in the presence of a tracking force.

The first group of analogies deals with the case of the presence the nonintegrable constraint (21) in the system. In this case the dynamical part of the motion equations under certain conditions is reduced to a system (26).

Under onditions (39), (43) the system (26) has the form

$$\begin{aligned}\alpha' &= -\omega + b\sin\alpha,\\ \omega' &= \sin\alpha\cos\alpha,\end{aligned} \tag{66}$$

if we introduce the dimensionless parameter, the variable, and the differentiation analogously to (44):

$$b = \sigma n_0,\ n_0^2 = \frac{AB}{I},\ \Omega = n_0 v\omega, \tag{67}$$
$$< \cdot >= n_0 v <'> .$$

Theorem 5 *System (66) (for the case of a free body) is equivalent to the system (47) (for the case of a fixed pendulum).*

Indeed, it is sufficient to substitute

$$\xi = \alpha,\ w_1 = \omega,\ b_* = -b. \tag{68}$$

Corollary 6 *1. The angle of attack α for a free body is equivalent to the angle of body deviation ξ of a fixed pendulum.*

2. The distance $\sigma = CD$ for a free body corresponds to the length of a holder $l = OD$ of a fixed pendulum.

3. The first integral of a system (66) can be automatically obtained through the Eqs. (60)–(62) (or (63)–(65)) after substitutions (68) (see also [12]):

I. $b^2 - 4 < 0$.

$$[\sin^2\alpha - b\omega\sin\alpha + \omega^2]\times$$
$$\times\exp\left\{-\frac{2b}{\sqrt{4-b^2}}arctg\frac{2\omega - b\sin\alpha}{\sqrt{4-b^2}\sin\alpha}\right\} = \tag{69}$$
$$= const.$$

II. $b^2 - 4 > 0$.

$$[\sin^2\alpha - b\omega\sin\alpha + \omega^2]\times$$
$$\times\left|\frac{2\omega - b\sin\alpha + \sqrt{b^2-4}\sin\alpha}{2\omega - b\sin\alpha - \sqrt{b^2-4}\sin\alpha}\right|^{b/\sqrt{b^2-4}} = \tag{70}$$
$$= const.$$

III. $b^2 - 4 = 0$.

$$(\omega - \sin\alpha)\exp\left\{\frac{\sin\alpha}{\omega - \sin\alpha}\right\} = const. \tag{71}$$

The second group of analogies deals with the case of a motion with the constant velocity of the center of mass of a body, i.e., when the property (31) holds. In this case the dynamical part of the motion equations under certain conditions is reduced to a system (38).

Then, under conditions (31), (39), (43), and (67), the reduced dynamical part of the motion equations (system (38)) has the form of analytical system

$$\begin{aligned}\alpha' &= -\omega + b\sin\alpha\cos^2\alpha + b\omega^2\sin\alpha,\\ \omega' &= \sin\alpha\cos\alpha - b\omega\sin^2\alpha\cos\alpha + b\omega^3\cos\alpha,\end{aligned} \tag{72}$$

in this case, we choose the constant n_1 as follows:

$$n_1 = n_0. \tag{73}$$

If the problem on the first integral of the system (66) is solved using Corollary 6, the same problem for the system (72) can be solved by the following theorem 7.

For this we introduce the following notations and new variables (comp. with [13]):

$$C_1 = 2 - b,\ C_2 = b > 0,\ C_3 = -2 - b < 0,$$

$$u_1 = \omega - \sin\alpha,\ v_1 = \omega + \sin\alpha, \tag{74}$$

$$u_1 = v_1 t_1,\ v_1^2 = \frac{1}{q_1},$$

then the problem on explicit form of the desired first integral reduces to solving of the linear inhomogeneous equation:

$$\frac{dq_1}{dt_1} = a_1(t_1)q_1 + a_2(t_1), \tag{75}$$

where

$$a_1(t_1) = \frac{2(C_3 t_1 + C_2)}{C_3 t_1^2 - C_1}, \tag{76}$$

$$a_2(t_1) = \frac{4C_2 t_1}{C_3 t_1^2 - C_1}.$$

The general solution of Eq. (75) has the following form (see [1, 13]):

I. $b < 2$.

$$q_1(t_1) = k(t_1)(-C_3 t_1^2 + C_1)\times$$

$$\times\exp\left\{-\frac{2b}{\sqrt{4-b^2}}\mathrm{arctg}\sqrt{\frac{2+b}{2-b}}t_1\right\} + \mathrm{const}. \tag{77}$$

II. $b > 2$.

$$q_1(t_1) = k(t_1)(-C_3 t_1^2 + C_1)\times$$

$$\times\left|\frac{\sqrt{-C_1} + \sqrt{-C_3}t_1}{\sqrt{-C_1} - \sqrt{-C_3}t_1}\right|^{C_2/\sqrt{C_1C_3}} + \mathrm{const}. \tag{78}$$

III. $b = 2$.

$$q_1(t_1) = k(t_1)t_1^2\exp\left\{\frac{1}{t_1}\right\} + \mathrm{const}, \tag{79}$$

in this case,

I. $b < 2$.

$$k(t_1) = -\frac{b}{8}\times$$

$$\times\exp\left\{\frac{2b}{\sqrt{4-b^2}}\left[\frac{2b}{\sqrt{4-b^2}}\sin 2\zeta - 2\cos 2\zeta\right]\right\} + \tag{80}$$

$$+\mathrm{const},$$

$$\mathrm{tg}\zeta = \sqrt{\frac{2-b}{2+b}}t_1.$$

II. $b > 2$.

$$k(t_1) = \pm|\zeta|^{b/\sqrt{b^2-4}}\mp$$

$$\mp\frac{b}{b + 2\sqrt{b^2-4}}|\zeta|^{b/\sqrt{b^2-4}+2} + \mathrm{const}, \tag{81}$$

$$t_1 = \sqrt{\frac{b-2}{b+2}}\left(\frac{1-\zeta}{1+\zeta}\right).$$

III. $b = 2$.

$$k(t_1) = -2\frac{t_1+1}{t_1}\exp\left\{-\frac{1}{t_1}\right\}. \tag{82}$$

Thus, Eqs. (77)–(82) allow us to find the desired first integral of system (72) using the notations and substitutions (74).

Theorem 7 *The first integral of the system (72) is a transcendental function of its own phase variables and is expressed as a finite combination of elementary functions.*

Because of cumbersome character of form of the first integral obtained, we represent this form in the case **III** only:

$$\exp\left\{\frac{\sin\alpha + \omega}{\sin\alpha - \omega}\right\}\frac{1 - 4\omega\sin\alpha + 4\omega^2}{(\omega - \sin\alpha)^2} = C_1 = \tag{83}$$

$$= \mathrm{const}.$$

Theorem 8 *The first integral of system (66) is constant on the phase trajectories of the system (72).*

Let us perform the *proof* for the case $b = 2$. Indeed, we rewrite the first integral (83) of system (72) as follows:

$$\exp\left\{\frac{n_0 v\sin\alpha + \Omega}{n_0 v\sin\alpha - \Omega}\right\}\times$$

$$\times\frac{n_0^2v^2 - 2bn_0v\Omega\sin\alpha + b^2\Omega^2}{(\Omega - n_0v\sin\alpha)^2} = \mathrm{const}. \tag{84}$$

We see that the numerator of the second factor is proportional to the square of the velocity of the center of mass $\mathbf{V}_C$ of the body with constant coefficient n_0^2. But, by virtue of (31), this value is constant on trajectories of the system (72). Therefore, the function

$$\exp\left\{\frac{n_0 v\sin\alpha + \Omega}{n_0 v\sin\alpha - \Omega}\right\}\frac{V_C^2}{(\Omega - n_0v\sin\alpha)^2} = \mathrm{const} \tag{85}$$

is also constant on these trajectories.

Further, let us raise the left-hand side of Eq. (85) to the power $(-1/2)$ and conclude that the following function is constant on the trajectories of the system (72):

$$\exp\left\{\frac{\Omega+n_0v\sin\alpha}{2(\Omega-n_0v\sin\alpha)}\right\}(\Omega-n_0v\sin\alpha)=\text{const.} \tag{86}$$

And now, we divide Eq. (86) by $\sqrt{e}$ and obtain the function

$$\exp\left\{\frac{n_0v\sin\alpha}{\Omega-n_0v\sin\alpha}\right\}(\Omega-n_0v\sin\alpha)=\text{const}, \tag{87}$$

which is constant on the phase trajectories of the system (72). But the first integral (87) is completely similar to the first integral (71), as required.

Thus, we have the following topological and mechanical analogies in the sense explained above.

(1) A motion of a fixed physical pendulum on a cylindrical hinge in a flowing medium (nonconservative force fields).

(2) A plane-parallel free motion of a rigid body in a nonconservative force field under a tracing force (in the presence of a nonintegrable constraint).

(3) A plane-parallel composite motion of a rigid body rotating about its center of mass, which moves rectilinearly and uniformly, in a nonconservative force field.

7 Case where the moment of nonconservative forces depends on the angular velocity

7.1 Dependence on the angular velocity

This section is devoted to dynamics of the two-dimensional rigid body on the plane. This subsection is devoted to the study of the case of the motion where the moment of forces depends on the angular velocity. We introduce this dependence in the general case; this will allow us to generalize this dependence both to three-, and multi-dimensional bodies.

Let $x=(x_{1N},x_{2N})$ be the coordinates of the point N of application of a nonconservative force (interaction with a medium) on the one-dimensional plate and $Q=(Q_1,Q_2)$ be the components independent of the angular velocity. We introduce only the linear dependence of the functions $(x_{1N},x_{2N})=(x_N,y_N)$ on the angular velocity since the introduction of this dependence itself is not a priori obvious (see [1, 3]).

Thus, we accept the following dependence:

$$x=Q+R, \tag{88}$$

where $R=(R_1,R_2)$ is a vector-valued function containing the angular velocity. Here, the dependence of the function R on the angular velocity is gyroscopic:

$$R=\begin{pmatrix}R_1\\R_2\end{pmatrix}=$$

$$=-\frac{1}{v_D}\begin{pmatrix}0&-\Omega\\\Omega&0\end{pmatrix}\begin{pmatrix}h_1\\h_2\end{pmatrix}, \tag{89}$$

where (h_1,h_2) are certain positive parameters (comp. with [2, 4]).

Now, for our problem, since $x_{1N}=x_N\equiv 0$, we have

$$x_{2N}=y_N=Q_2-h_1\frac{\Omega}{v_D}. \tag{90}$$

Thus, the function $\mathbf{r}_N$ is selected in the following form (the plate AB is defined by the equation $x_{1N}\equiv 0$):

$$\mathbf{r}_N=\begin{pmatrix}0\\x_{2N}\end{pmatrix}=R(\alpha)\mathbf{i}_N-\frac{1}{v_D}\tilde{\Omega}h, \tag{91}$$

where

$$\mathbf{i}_N=\mathbf{i}_v\left(\frac{\pi}{2}\right),\ h=\begin{pmatrix}h_1\\h_2\end{pmatrix}, \tag{92}$$

$$\tilde{\Omega}=\begin{pmatrix}0&-\Omega\\\Omega&0\end{pmatrix}$$

(see (6), (11)).

In our case

$$\mathbf{i}_N=\begin{pmatrix}0\\1\end{pmatrix}. \tag{93}$$

Thus, the following relation

$$x_{2N}=R(\alpha)-h_1\frac{\Omega}{v_D} \tag{94}$$

holds, which shows that an additional dependence of the damping (or accelerating in some domains of the phase space) moment of the nonconservative forces is also present in the system considered (i.e., the moment depends on the angular velocity).

And so, for the construction of the force field, we use the pair of dynamical functions $R(\alpha), s(\alpha)$; the information about them is of a qualitative nature. Similarly to the choice of the Chaplygin analytical functions (see [1, 5]), we take the dynamical functions s and R as follows:

$$R(\alpha)=A\sin\alpha,\ s(\alpha)=B\cos\alpha,\ A,B>0. \tag{95}$$

7.2 Reduced systems

Theorem 9 *The simultaneous equations (8), (12), (15) under conditions (91), (95) can be reduced to the dynamical system on the tangent bundle (5) of the one-dimensional sphere (4).*

Indeed, if we introduce the dimensionless parameters and the differentiation by the formulas

$$b_* = ln_0,\ n_0^2 = \frac{AB}{I},\ H_{1*} = \frac{h_1 B}{In_0}, \qquad (96)$$

$$< \cdot >= n_0 v_\infty <'>,$$

then the obtained equation has the following form:

$$\xi'' + (b_* - H_{1*})\xi' \cos\xi + \sin\xi\cos\xi = 0. \qquad (97)$$

After the transition from the variables z (about the variables z see (14)) to the variables w

$$w_1 = -\frac{1}{1 + b_* H_{1*}} \left(\frac{1}{n_0 v_\infty} z_1 + b_* \sin\xi \right), \qquad (98)$$

Eq. (97) is equivalent to the system

$$\begin{array}{l} \xi' = -(1 + b_* H_{1*})w_1 - b_* \sin\xi, \\ w_1' = \sin\xi\cos\xi + H_{1*} w_1 \cos\xi. \end{array} \qquad (99)$$

7.3 Transcendental first integral

We put in correspondence to system (99) the following nonautonomous first-order equation:

$$\frac{dw_1}{d\xi} = \frac{\sin\xi\cos\xi + H_{1*} w_1 \cos\xi}{-(1 + b_* H_{1*})w_1 - b_* \sin\xi}. \qquad (100)$$

Using the substitution $\tau = \sin\xi$, we rewrite Eq. (100) in the algebraic form

$$\frac{dw_1}{d\tau} = \frac{\tau + H_{1*} w_1}{-(1 + b_* H_{1*})w_1 - b_* \tau}. \qquad (101)$$

Further, introducing the homogeneous variable by the formula $w_1 = u\tau$, we reduce Eq. (101) to the following quadrature:

$$\frac{(-b_* - (1 + b_* H_{1*})u)du}{1 + (b_* + H_{1*})u + (1 + b_* H_{1*})u^2} = \frac{d\tau}{\tau}. \qquad (102)$$

Integration of quadrature (102) leads to the following three cases. Simple calculations yield the following first integrals.

I. $|b_* - H_{1*}| < 2$.

$$\ln(1 + (b_* + H_{1*})u + (1 + b_* H_{1*})u^2) + \frac{2b_*}{\sqrt{4 - (b_* - H_{1*})^2}} \times$$
$$\times \text{arctg} \frac{2(1 + b_* H_{1*})u + (b_* + H_{1*})}{\sqrt{4 - (b_* - H_{1*})^2}} + \ln\tau^2 = \text{const}. \qquad (103)$$

II. $|b_* - H_{1*}| > 2$.

$$\frac{1}{1 + b_* H_{1*}} \ln|1 + (b_* + H_{1*})u + (1 + b_* H_{1*})u^2| + \ln\tau^2 - \frac{b_* \sqrt{1 + b_* H_{1*}}}{\sqrt{(b_* - H_{1*})^2 - 4}} \times$$
$$\times \ln\left| \frac{A + \sqrt{(b_* - H_{1*})^2 - 4}}{A - \sqrt{(b_* - H_{1*})^2 - 4}} \right| = \text{const}, \qquad (104)$$

$$A = 2(1 + b_* H_{1*})^{3/2} u + (b_* + H_{1*})\sqrt{1 + b_* H_{1*}}.$$

III. $|b_* - H_{1*}| = 2$.

$$\ln\left| u + \frac{b_* + H_{1*}}{2(1 + b_* H_{1*})} \right| - \frac{b_* - H_{1*}}{2(1 + b_* H_{1*})u + (b_* + H_{1*})} + \ln|\tau| = \text{const}. \qquad (105)$$

In the variables (ξ, w_1) the found first integrals have the cumbersome character of their form. Nevertheless, we represent this form in the case **III** in the explicit form:

$$\left(w_1 + \frac{b_* + H_{1*}}{2(1 + b_* H_{1*})} \sin\xi \right) \times \exp\left\{ \frac{(-b_* + H_{1*})\sin\xi}{2(1 + b_* H_{1*})w_1 + (b_* + H_{1*})\sin\xi} \right\} = \text{const}. \qquad (106)$$

Therefore, in the considered case the system of dynamical equations (99) has the first integral expressed by relations (103)–(105) (or, in particular, in the case **III** (106)), which is a transcendental function of its phase variables (in the sense of complex analysis) and is expressed as a finite combination of elementary functions.

Theorem 10 *Three sets of relations (8), (12), (15) under conditions (91), (95) possess the first integral (the complete set), which is a transcendental function (in the sense of complex analysis) and is expressed as a finite combination of elementary functions.*

7.4 Topological analogies

Now we present two groups of analogies related to the system (17), which describes the motion of a free body in the presence of a tracking force.

The first group of analogies deals with the case of the presence the nonintegrable constraint (21) in the system. In this case the dynamical part of the motion equations under certain conditions is reduced to a system (26).

Under conditions (91), (95) the system (26) has the form

$$\begin{aligned} \alpha' &= -(1+bH_1)\omega + b\sin\alpha, \\ \omega' &= \sin\alpha\cos\alpha - H_1\omega\cos\alpha, \end{aligned} \tag{107}$$

if we introduce the dimensionless parameters, the variable, and the differentiation analogously to (44):

$$b = \sigma n_0,\ n_0^2 = \frac{AB}{I},\ H_1 = \frac{h_1 B}{I n_0},\ \Omega = n_0 v\omega, \\ <\cdot> = n_0 v <'> . \tag{108}$$

Theorem 11 *System (107) (for the case of a free body) is equivalent to the system (99) (for the case of a fixed pendulum).*

Indeed, it is sufficient to substitute

$$\xi = \alpha,\ w_1 = \omega,\ b_* = -b,\ H_{1*} = -H_1. \tag{109}$$

Corollary 12 1. *The angle of attack α for a free body is equivalent to the angle of body deviation ξ of a fixed pendulum.*

2. *The distance $\sigma = CD$ for a free body corresponds to the length of a holder $l = OD$ of a fixed pendulum.*

3. *The first integral of a system (107) can be automatically obtained through the Eqs. (100)–(102) (or (103)–(105)) after substitutions (109) (see also [6, 7]).*

In the variables (α, ω) the found first integrals have the cumbersome character of their form. Nevertheless, we represent this form in the case **III** in the explicit form:

$$\left(\omega - \frac{b+H_1}{2(1+bH_1)}\sin\alpha\right)\times \\ \times\exp\left\{\frac{(b-H_1)\sin\alpha}{2(1+bH_1)\omega - (b+H_1)\sin\alpha}\right\} = \\ = \text{const}. \tag{110}$$

The second group of analogies deals with the case of a motion with the constant velocity of the center of mass of a body, i.e., when the property (31) holds. In this case the dynamical part of the motion equations under certain conditions is reduced to a system (38).

Then, under conditions (31), (91), (95), (108) the reduced dynamical part of the motion equations (system (38)) has the form of analytical system

$$\begin{aligned} \alpha' &= -\omega + b\sin\alpha\cos^2\alpha + \\ &+ b\omega^2\sin\alpha - bH_1\omega\cos^2\alpha, \\ \omega' &= \sin\alpha\cos\alpha - b\omega\sin^2\alpha\cos\alpha + \\ &+ b\omega^3\cos\alpha + \\ &+ bH_1\omega^2\sin\alpha\cos\alpha - H_1\omega\cos\alpha, \end{aligned} \tag{111}$$

in this case, we choose the constant n_1 as follows:

$$n_1 = n_0. \tag{112}$$

If the problem on the first integral of the system (107) is solved using Corollary 12, the same problem for the system (111) can be solved by the following Theorem 13.

For this we introduce the following notations and new variables (comp. with [8, 9]):

$$\begin{gathered} A_1 = \tfrac{b}{2} - \tfrac{bH_1}{2} - \tfrac{H_1}{2}, \\ A_2 = 1 + \tfrac{b}{2} + \tfrac{bH_1}{2} + \tfrac{H_1}{2} > 0, \\ A_3 = 1 - \tfrac{b}{2} + \tfrac{bH_1}{2} - \tfrac{H_1}{2}, \\ u_1 = \omega - \sin\alpha,\ v_1 = \omega + \sin\alpha, \\ u_1 = v_1 t_1,\ v_1^2 = \tfrac{1}{q_1}, \end{gathered} \tag{113}$$

then the problem on explicit form of the desired first integral reduces to solving of the linear inhomogeneous equation:

$$\frac{dq_1}{dt_1} = a_1(t_1)q_1 + a_2(t_1), \tag{114}$$

where

$$a_1(t_1) = \frac{2(A_2 t_1 - A_1)}{A_2 t_1^2 + bH_1 t_1 + A_3}, \\ a_2(t_1) = \frac{2b(-t_1 + H_1(t_1^2 - 1)/4))}{A_2 t_1^2 + bH_1 t_1 + A_3}. \tag{115}$$

The general solution of Eq. (114) has the following form [10]:

I. $|b - H_1| < 2$.

$$q_1(t_1) = k(t_1)(A_2 t_1^2 + bH_1 t_1 + A_3)\times \\ \times\exp\left\{-\frac{2(b - bH_1 - H_1)}{\sqrt{4-(b-H_1)^2}}B\right\}, \tag{116}$$

$$B = \text{arctg}\left\{\frac{2+b+bH_1+H_1}{\sqrt{4-(b-H_1)^2}}t_1 + \right.$$

$$+\frac{bH_1}{\sqrt{4-(b-H_1)^2}}\Bigg\}.$$

II. $|b-H_1|>2.$

$$q_1(t_1)=k(t_1)(A_2t_1^2+bH_1t_1+A_3)\times$$

$$\times\left|\frac{\sqrt{4-(b-H_1)^2}+C}{\sqrt{4-(b-H_1)^2}-C}\right|^{(b-bH_1-H_1)/\sqrt{4-(b-H_1)^2}},$$

$$C=(2+b+bH_1+H_1)t_1+bH_1. \quad (117)$$

III. $|b-H_1|=2.$

$$q_1(t_1)=k(t_1)\left(t_1+\frac{bH_1}{2A_2}\right)^2\times$$

$$\times\exp\left\{\frac{2(b-H_1)}{(2+b+bH_1+H_1)t_1+bH_1}\right\}. \quad (118)$$

To find a solution of the nonhomogeneous equations (114), (115), we must express the value of k as a function of t_1, which is expressed as a finite combination of elementary functions.

Thus, Eqs. (116)–(118) allow to obtain the desired first integral of the system (111) using the notations and substitutions (113).

Theorem 13 *The first integral of the system (111) is a transcendental function of its own phase variables and is expressed as a finite combination of elementary functions.*

Because of cumbersome character of form of the first integral obtained, we represent this form in the case **III** only:

$$\exp\left\{\frac{-2(b-H_1)\sin\alpha}{2(1+bH_1)\omega-(b+H_1)\sin\alpha}\right\}\times$$

$$\times\frac{1-4\omega\sin\alpha+4\omega^2}{(\omega-2\sin\alpha/(b+H_1))^2}=C_1=\text{const.} \quad (119)$$

Theorem 14 *The first integral of system (107) is constant on the phase trajectories of the system (111).*

Let us perform the *proof* for the case $|b-H_1|=2$. Indeed, we rewrite the first integral (119) as follows:

$$\exp\left\{\frac{-2n_0v(b-H_1)\sin\alpha}{2(1+bH_1)\Omega-n_0v(b+H_1)\sin\alpha}\right\}\times$$

$$\times\frac{n_0^2v^2-4n_0v\Omega\sin\alpha+4\Omega^2}{(\Omega-2n_0v\sin\alpha/(b+H_1))^2}=\text{const.} \quad (120)$$

We see that the numerator of the second factor is proportional to the square of the velocity of the center of mass $\mathbf{V}_C$ of the body with constant coefficient n_0^2. But, by virtue of (31), this value is constant on trajectories of the system (111). Therefore, the function

$$\exp\left\{\frac{-2n_0v(b-H_1)\sin\alpha}{2(1+bH_1)\Omega-n_0v(b+H_1)\sin\alpha}\right\}\times$$

$$\times\frac{V_C^2}{(\Omega-2n_0v\sin\alpha/(b+H_1))^2}=\text{const.} \quad (121)$$

Further, let us raise the left-hand side of Eq. (121) to the power $(-1/2)$ and conclude that the following function is constant on the trajectories of the system (111):

$$\exp\left\{\frac{n_0v(b-H_1)\sin\alpha}{2(1+bH_1)\Omega-n_0v(b+H_1)\sin\alpha}\right\}\times$$

$$\times(\Omega-2n_0v\sin\alpha/(b+H_1))=\text{const.} \quad (122)$$

And now, it is clear that the function (122) is equivalent to the function (110), since in the case **III** the following relation

$$(b+H_1)^2=4(1+bH_1) \quad (123)$$

holds.

Thus, we have the following topological and mechanical analogies in the sense explained above.

(1) A motion of a fixed physical pendulum on a cylindrical hinge in a flowing medium (nonconservative force fields under assumption of additional dependence of the moment of the forces on the angular velocity).

(2) A plane-parallel free motion of a rigid body in a nonconservative force field under a tracing force (in the presence of a nonintegrable constraint under assumption of additional dependence of the moment of the forces on the angular velocity).

(3) A plane-parallel composite motion of a rigid body rotating about its center of mass, which moves rectilinearly and uniformly, in a nonconservative force field under assumption of additional dependence of the moment of the forces on the angular velocity.

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