# Development of benchmark objective-function-formulation for Derringer's function based model updating method

SHANKAR SEHGAL and HARMESH KUMAR*
Department of Mechanical Engineering
University Institute of Engineering and Technology
Panjab University, Chandigarh, 160014
INDIA
*harmesh@pu.ac.in http://scholar.google.com.sg/citations?user=p7HiX00AAAAJ&hl=en

*Abstract:* - Objective-function-formulation plays very important role in successfully solving any optimization problem in general and model updating problem in particular. This paper presents detailed investigations on three different types of objective-function-formulations for Derringer's function based finite element model updating method. Three case studies are considered wherein during first case study the objective-function-formulations are based upon natural frequencies only; while for second case study, only the modal assurance criterion values are used for formulating the objectives. In third case study, a combination of natural frequencies and modal assurance criterion values are used for formulation of objective-functions. The three objective-function-formulations are then compared against each other in terms of error reduction in prediction of response variables as well as physical input parameters. It is established that error reduction is maximum when both natural frequencies and modal assurance criterion values are collectively used for formulation of objective-functions of model updating problem. This benchmark objective-function-formulation is then used further in refined model updating stage so as to increase the chances of obtaining more accurate and reliable updating results significantly.

*Key-Words:* - Objective-function-formulation; Finite element model updating; Derringer's function; Response surface; Analysis of variance

## Abbreviations

ANOVA analysis of variance
FE finite element
FEMU finite element model updating
MAC modal assurance criterion
PRESS predicted residual error sum of squares
RS response surface
RSM response surface method
SE simulated experimental

## Nomenclature

$A$ coded parameter related to elastic modulus of first finite element
$B$ coded parameter related to elastic modulus of fifth finite element
$C$ coded parameter related to elastic modulus of ninth finite element
$C_i$ coefficient of $i^{th}$ linear term of polynomial model
$D$ coded parameter related to elastic modulus of 13th finite element
$D_i$ coefficient of $i^{th}$ quadratic term of polynomial model
$D_o$ overall desirability function
$E$ coded parameter related to elastic modulus of 17th finite element
$E_i$ elastic modulus of $i^{th}$ finite element
$F$ coded parameter related to elastic modulus of 24th finite element
F' test statistic of F-test
$MAC_{ii}$ modal assurance criterion value for $i^{th}$ finite element mode with $i^{th}$ simulated experimental mode
$\widehat{MAC}_{ii}$ response surface predicted modal assurance criterion value for $i^{th}$ finite element mode with $i^{th}$ simulated experimental mode
$\widehat{MAC}_{iiLL}$ lower limit for response surface predicted modal assurance criterion value for $i^{th}$ finite element mode with $i^{th}$ simulated experimental mode
$\widehat{MAC}_{iiUL}$ upper limit for response surface predicted modal assurance criterion value for $i^{th}$ finite element mode with $i^{th}$

| | |
|---|---|
| | simulated experimental mode |
| $R^2$ | coefficient of determination |
| **V** | variance-covariance matrix |
| **X** | design matrix as being a set of value combinations of coded parameters |
| $X_i$ | $i^{th}$ independent parameter |
| $Y$ | response predicted by response surface method |
| $d_i$ | $i^{th}$ individual desirability function |
| $f$ | response function |
| $i$ | integer |
| $j$ | integer |
| $m$ | number of independent parameters |
| $n$ | number of individual desirability functions / natural frequencies / modes |
| $\varepsilon$ | experimental error |
| $\sigma$ | standard deviation |
| $\omega_i$ | finite element predicted $i^{th}$ natural frequency |
| $\widehat{\omega}_i$ | response surface predicted natural frequency of $i^{th}$ mode |
| $\widehat{\omega}_{iLL}$ | lower limit for response surface predicted natural frequency of $i^{th}$ mode |
| $\widehat{\omega}_{iT}$ | target value for response surface predicted natural frequency of $i^{th}$ mode |
| $\widehat{\omega}_{iUL}$ | upper limit for response surface predicted natural frequency of $i^{th}$ mode |

# 1 Introduction

Use of thin parts made up of low density materials in latest machines and structures is increasing day by day. Thin and light weight products have lot more tendencies to vibrate than their thick and heavy weight counterparts. Excessive vibrations may even cause pre-mature failure of products. Therefore prediction of accurate dynamic behavior is a major step in design of rotors of turbines and many other machines [1]. Dynamic behavior of a structure can be represented by natural frequencies, mode-shapes, damping ratios, frequency response functions etc. Further, to analyze the dynamic behavior of structures, either experimental route or theoretical approach [2–3] can be followed. Theoretical route involves the formation of an analytical model of the system either using a classical method [4] or through Finite Element (FE) method [5]. Application of classical method is generally limited to simple systems only, while FE method is preferred for real life complex systems. However, FE method is not able to predict dynamic responses of structures with complete accuracy due to presence of certain errors in FE model. Thus there is a need to correct an FE model so that its vibration behavior matches with the actual dynamic response obtained experimentally. The procedure used to update the model is called Finite Element Model Updating (FEMU) [6–8].

FEMU methods can be broadly classified into direct and iterative methods. Direct (non-iterative) methods are essentially the one step methods such as those proposed by Baruch and Bar-Itzhack [9], Baruch [10], Berman [11], Berman and Nagy [12]. Updated FE models produced by such methods may not be symmetric and positive definite, hence such methods are not much useful in industry. Industrial applications generally rely upon the use of iterative methods such as those proposed by Collins et al. [13], Lin and Ewins [14], Atalla and Inman [15], Li [16], Lin and Zhu [17], Arora et al. [18–19] and Silva [20].

Recently, Sehgal and Kumar [21] developed a novel technique of FEMU by using Derringer's function method. In this technique, FEMU is treated as multi-objective optimization problem; where number of objectives need to be defined in such a way as to reduce errors in responses predicted by FE model. Major issue behind success or failure of such FEMU technique is proper formulation of objective functions. However, no published work is available to compare the performance of different objective-function-formulations of Derringer's function based FEMU technique. Therefore, purpose of this research work is to compare the performance of three different types of objective-function-formulations of Derringer's function based FEMU technique and hence to find out a benchmark objective-function-formulation for FEMU.

Three case studies are performed involving three different types of objective-function-formulations. During first case study, objective-function-formulations are based upon natural frequencies only. During second case study, objective-functions are formulated by using Modal Assurance Criterion (MAC) values only. While in third case study, a combination of natural frequencies and MAC values are used for objective-function-formulations. Results of the three case studies are then compared against each other to find out the objective-function-formulations with best performance in terms of error reduction in prediction of natural frequencies, MAC values as well as physical input parameters. The best objective-function-formulations are then used further for refined Response Surface (RS) models based FEMU.

Basic theory of Derringer's function based FEMU technique used in this research paper is discussed in section 2. In order to apply Derringer's function based FEMU technique, FE and Simulated Experimental (SE) results are required as explained in section 3. Section 4 discusses about development of RS models. Section 5 presents the current research work related to evaluation of a benchmark objective-function-formulation and its application to refined RS models based FEMU. Section 6 discusses the conclusions drawn out of the present research work.

## 2 Theory

In this paper Derringer' function based FEMU technique [21] has been used for comparing the performance of three different types of objective-function-formulations. The FEMU technique used in this paper is based upon the use of D-optimal design, Response Surface Method (RSM) and desirability function; basic theory of which is presented in subsections 2.1, 2.2 and 2.3 respectively.

### 2.1 D-optimal design

There are several design optimality criterion available in literature such as D-optimality, A-optimality, G-optimality. Among all, D-optimality is the most popular one [22]. It is a type of computer-generated designs, which are an outgrowth of the original work by Kiefer [23–24] and Kiefer and Wolfowitz [25]. In general, modeling accuracy, namely, goodness-of-fit, can be measured by a variance-covariance matrix **V** given by (1).

$$\mathbf{V} = \sigma^2(\mathbf{X}'\mathbf{X})^{-1} \quad (1)$$

where $\sigma$ is the standard deviation and **X** is the design matrix being a set of value combinations of coded parameters. Naturally, it is expected to minimize $(\mathbf{X'X})^{-1}$ in order to obtain an accurate RS model. In statistics, minimizing $(\mathbf{X'X})^{-1}$ is equivalent to maximizing the determinant of $\mathbf{X'X}$. Therefore, the criteria for constructing the design matrix with a maximized $|\mathbf{X'X}|$ from a set of candidate samples can be defined as the D-optimality. The initial 'D' stands for 'determinant'. By using D-optimal designs, the generalized variance of a predefined model is minimized, which means the 'optimality' of a specific D-optimal design is model dependent. Unlike standard designs, D-optimal designs are straight optimization and their matrices are generally not orthogonal with the effect estimates correlated.

### 2.2 Response surface method

RSM is a collection of mathematical and statistical techniques that are useful for modeling and analysis of problems in which a response of interest is influenced by several input variables and the objective is to optimize this response [26–27]. It is a sequential experimentation strategy for empirical model building and optimization as shown in Fig. 1 [28].

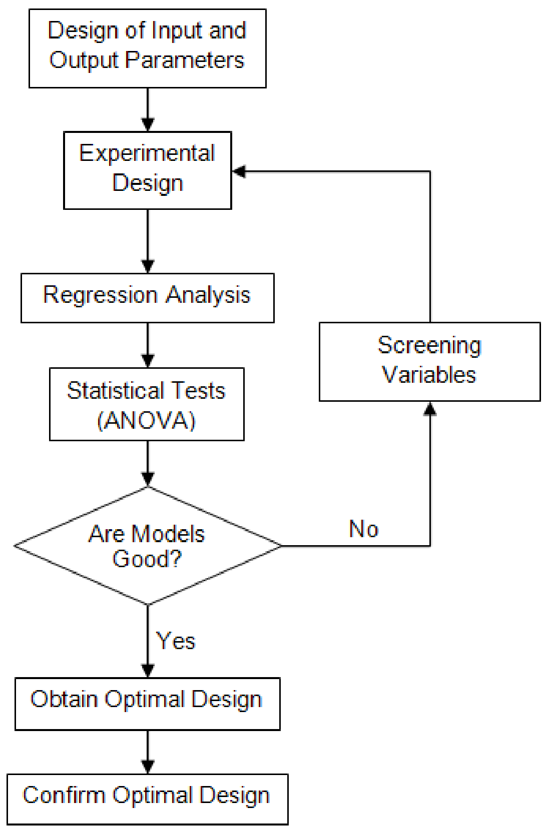


**Fig. 1 Procedure of RSM** [28].

By conducting experiments and applying regression analysis, a model of the response to some independent input variables can be obtained. Based on the model of the response, a near optimal point can then be deduced. RSM is often applied in the characterization and optimization of processes. In RSM, it is possible to represent independent process parameters in quantitative form as written in (2).

$$Y = f(X_1, X_2, X_3, \dots X_m) \pm \varepsilon \quad (2)$$

where $Y$ is the response, $f$ is the response function, $\varepsilon$ is the experimental error, and $X_1, X_2, X_3, \dots X_m$ are independent parameters. By plotting the expected response of $Y$, a surface, known as RS is obtained. The form of $f$ is unknown and may be very complicated. Thus, RSM aims at approximating $f$ by a suitable lower ordered polynomial in some region of the independent process variables. If the response can be well modeled by a linear function of the $m$ independent variables, the function $Y$ can be written as:

$$Y = C_0 + C_1X_1 + C_2X_2 + \dots + C_mX_m \pm \varepsilon \quad (3)$$

However, if a curvature appears in the system, then a higher order polynomial such as the quadratic model as shown in (4) may be used.

$$Y = C_0 + \sum_{i=1}^{m} C_iX_i + \sum_{i=1}^{m} D_iX_i^2 \pm \varepsilon \quad (4)$$

Objective of using RSM is not only to investigate the response over entire factor space, but also to locate the region of interest where the response reaches its optimum or near optimal value. By studying carefully the RS model, the combination of factors, which gives the best response, can then be established.

### 2.3 Desirability function

Derringer and Suich [29] describe a multiple response method called desirability. The method makes use of an objective function, $D_o$, called overall desirability function and transforms an estimated response into a scale free value ($d_i$) called individual desirability. The desirable ranges are from zero to one (least to most desirable, respectively). The factor settings with maximum overall desirability are considered to be the optimal parameter conditions. The simultaneous objective function is a geometric mean of all transformed responses:

$$D_o = (d_1 \times d_2 \times d_3 \times \dots \times d_n)^{1/n} \quad (5)$$

where $n$ is the number of responses in the measure. If any of the responses falls outside the desirability range, the overall function becomes zero. Desirability is an objective function that ranges from zero outside of the limits to one at the goal. The numerical optimization finds a point that maximizes the desirability function. For several responses, all goals get combined into one desirability function. For simultaneous optimization, each response must have a lower and upper limit assigned to each goal. The "Goal" field for responses must be one of five choices: "none", "maximum", "minimum", "target", or "in range". Factors will always be included in the optimization at their design range by default, or as a maximum, minimum of target goal. The meanings of the goal parameters are:

- Maximum:
  - $d_i = 0$ if response < lower limit
  - $0 \le d_i \le 1$ as response varies from lower to upper limit
  - $d_i = 1$ if response > upper limit

- Minimum:
  - $d_i = 1$ if response < lower limit
  - $1 \le d_i \le 0$ as response varies from lower to upper limit
  - $d_i = 0$ if response > upper limit

- Target:
  - $d_i = 0$ if response < lower limit
  - $0 \le d_i \le 1$ as response varies from lower limit to target
  - $1 \ge d_i \ge 0$ as response varies from target to upper limit
  - $d_i = 0$ if response > upper limit

- Range:
  - $d_i = 0$ if response < lower limit
  - $d_i = 1$ as response varies from lower to upper limit
  - $d_i = 0$ if response > upper limit

The $d_i$ for "in range" are included in the product of the desirability function "$D_o$", but are not counted in determining "$n$": $D = (\prod d_i)^{1/n}$. If the goal is none, the response will not be used for the optimization.

## 3 Generation of initial FE and SE results

A cantilever beam structure as drawn in Fig. 2 is considered in present study. This beam structure is taken because of its resemblance with many real life products such as blade of rotor of a turbine, wing of an airplane, wing of a ceiling fan, an integrated chip of a mechatronic product etc. Cantilever beam is having the dimensions 910 x 49 x 7 $mm^3$, density

6728 kg/m$^3$ and Young' modulus of elasticity as 200 GPa. Initial FE model of undamaged beam is constructed using 30 beam elements each having two nodes. The FE model developed in Matlab [30] is then used for producing initial FE natural frequencies and mode-shapes for first five modes of undamaged beam. After that, perturbation is introduced into the FE model of beam structure by reducing the value of modulus of elasticity of six finite elements as per the data provided in Table 1. Six damage locations have been selected in such a way so as to distribute these over entire length of beam. FE model of damaged beam is then processed in Matlab to produce SE natural frequencies and mode-shapes, which are later treated as target results. SE results have been used earlier also by many researchers as target results for model updating related research work [32–35]. In this paper, responses such as natural frequencies of first five modes ($\omega_1$ to $\omega_5$) and first five diagonal elements of MAC matrix ($MAC_{11}$ to $MAC_{55}$) are used during formulation of objective functions of FEMU problem.

Table 2 presents a comparison of SE responses versus FE responses (before FEMU) of beam. It is clear from Table 2 that FE responses do not match with their SE counterparts. So, input parameters of FE model are updated through Derringer's function based FEMU. For this purpose, first the RS models of natural frequencies and MAC values need to be developed as described in section 4.

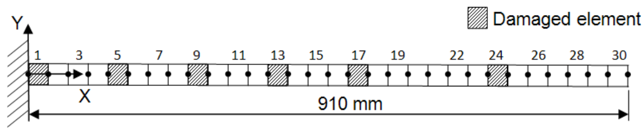


**Fig. 2 FE model of a damaged cantilever beam structure.**

**Table 1: Physical parameters of undamaged and damaged beams.**

| Physical parameter | $E_1$ | $E_5$ | $E_9$ | $E_{13}$ | $E_{17}$ | $E_{24}$ |
|---|---|---|---|---|---|---|
| Stiffness in FE model of undamaged beam (GPa) | 200 | 200 | 200 | 200 | 200 | 200 |
| Stiffness in FE model of damaged beam (GPa) | 60 | 80 | 100 | 60 | 80 | 100 |

**Table 2: Comparison of SE and FE responses (before FEMU).**

| Response variable | Response value | |
|---|---|---|
| | SE | FE (before FEMU) |
| $\omega_1$ (Hz) | 5.98 | 7.45 |
| $\omega_2$ (Hz) | 37.91 | 46.66 |
| $\omega_3$ (Hz) | 111.82 | 130.64 |
| $\omega_4$ (Hz) | 213.02 | 256.01 |
| $\omega_5$ (Hz) | 353.51 | 423.21 |
| $MAC_{11}$ | 1.0000 | 0.9996 |
| $MAC_{22}$ | 1.0000 | 0.9949 |
| $MAC_{33}$ | 1.0000 | 0.9873 |
| $MAC_{44}$ | 1.0000 | 0.9769 |
| $MAC_{55}$ | 1.0000 | 0.9544 |

## 4 Development of RS models of natural frequencies and MAC values

RS models are a basic requirement for Derringer's function based FEMU. In order to develop RS models, firstly an experimental design matrix is generated by using D-optimal design in Design-Expert software [35]. Range of each input physical parameter ($E_1$, $E_5$, $E_9$, $E_{13}$, $E_{17}$ and $E_{24}$) is decided. Lower and upper limits for all the input parameters are taken as 20% and 60% of their corresponding initial FE values respectively. Thus lower limit for each input parameter is taken as 40 GPa and the upper limit of the input parameter is taken as 120 GPa. Six coded parameters ($A$, $B$, $C$, $D$, $E$, and $F$) are defined in such a way that each coded parameter varies linearly between -1 and +1 over complete range of its corresponding physical parameters ($E_1$, $E_5$, $E_9$, $E_{13}$, $E_{17}$ and $E_{24}$). Coordinate exchange method [36] is used for candidate selection, because it does not require a candidate list, which if unchecked grows exponentially as the size of the problem increases [37]. D-optimality criterion is used to develop the design matrix of actual physical variables. Design matrix is consisting of a total of 33 test runs and contains the information about various combinations of different levels of input physical parameters at which different SE runs need to be performed. Design matrix is then imported in

Matlab and used as input to FE model. FE model is then used to produce response variables matrices as its output. The matrices of response variables are then supplemented to the experimental design matrix available in Design-Expert. Relationship between the set of input parameters ($E_1$, $E_5$, $E_9$, $E_{13}$, $E_{17}$ and $E_{24}$) and corresponding set of response variables ($\omega_1$, $\omega_2$, $\omega_3$, $\omega_4$, $\omega_5$, $MAC_{11}$, $MAC_{22}$, $MAC_{33}$, $MAC_{44}$ and $MAC_{55}$) is assumed to be quadratic. A quadratic fit is assumed because it is giving better results than a linear or a cubic model. In order to check adequacy of the RS model, ANalysis Of VAriance (ANOVA) is performed [38]. F-test method is used to carry out the hypothesis testing to check significance of different parameters. Results of ANOVA using F-test for RS predicted first natural frequency ($\hat{\omega}_1$) are presented in Table 3. ANOVA results suggest that the RS model for first natural frequency is statistically significant (p-value < 0.0001). Value of $R^2$ and adjusted $R^2$ is over 99%, which means that RS model provides an excellent explanation of relationship between independent variables ($A$, $B$, $C$, $D$, $E$, and $F$) and response variable ($\hat{\omega}_1$). Further, factor *A* (Elastic modulus of first element), *B* (Elastic modulus of fifth element), *C* (Elastic modulus of ninth element), *D* (Elastic modulus of 13$^{th}$ element), *E* (Elastic modulus of 17$^{th}$ element), *AB* (interaction effect of elastic modulus of first element with elastic modulus of fifth element), *AC* (interaction effect of elastic modulus of first element with elastic modulus of ninth element), *AD* (interaction effect of elastic modulus of first element with elastic modulus of 13$^{th}$ element), *BC* (interaction effect of elastic modulus of fifth element with elastic modulus of ninth element), $A^2$ (square of elastic modulus of first element), $B^2$ (square of elastic modulus of fifth element), $C^2$ (square of elastic modulus of ninth element), $D^2$ (square of elastic modulus of 13$^{th}$ element), $E^2$ (square of elastic modulus of 17$^{th}$ element) have significant effect. These significant effects, in descending order are *A*, *B*, *C*, *D*, $A^2$, $B^2$, *AB*, *AC*, *E*, *BC*, $D^2$, $C^2$, *AD* and $E^2$. The other model terms are said to be non-significant.

To fit the quadratic model for $\hat{\omega}_1$ appropriate, the non-significant terms are eliminated by backward elimination process. Non-significant terms such as *DE*, *F*, *DF*, *EF*, *BE*, *AF*, *CE*, *CD*, *AE*, *BD*, *CF* and *BF* are eliminated during the backward elimination process. Thus the reduced regression model contains very less number of terms compared to the initial regression model of $\hat{\omega}_1$, thereby making the model computationally more efficient without compromising much on accuracy and reliability part. Table 4 shows the ANOVA table for reduced quadratic model for $\hat{\omega}_1$.

**Table 3: ANOVA for $\hat{\omega}_1$ (before backward elimination).**

| Source | Sum of squares | Degrees of freedom | Mean square | F-Value | p-value Prob > F' |
|---|---|---|---|---|---|
| Model | 7.26 | 27 | 0.27 | 775.14 | < 0.0001 |
| *A* | 2.87 | 1 | 2.87 | 8271.61 | < 0.0001 |
| *B* | 1.27 | 1 | 1.27 | 3652.81 | < 0.0001 |
| *C* | 0.46 | 1 | 0.46 | 1334.51 | < 0.0001 |
| *D* | 0.12 | 1 | 0.12 | 342.56 | < 0.0001 |
| *E* | 0.02 | 1 | 0.02 | 46.61 | 0.0010 |
| *F* | 0.00 | 1 | 0.00 | 0.08 | 0.7867 |
| *AB* | 0.04 | 1 | 0.04 | 102.45 | 0.0002 |
| *AC* | 0.02 | 1 | 0.02 | 57.96 | 0.0006 |
| *AD* | 0.01 | 1 | 0.01 | 14.59 | 0.0124 |
| *AE* | 0.00 | 1 | 0.00 | 1.92 | 0.2240 |
| *AF* | 0.00 | 1 | 0.00 | 0.76 | 0.4233 |
| *BC* | 0.01 | 1 | 0.01 | 22.85 | 0.0050 |
| *BD* | 0.00 | 1 | 0.00 | 2.11 | 0.2057 |
| *BE* | 0.00 | 1 | 0.00 | 0.46 | 0.5268 |
| *BF* | 0.00 | 1 | 0.00 | 2.94 | 0.1471 |
| *CD* | 0.00 | 1 | 0.00 | 1.44 | 0.2838 |
| *CE* | 0.00 | 1 | 0.00 | 1.22 | 0.3190 |
| *CF* | 0.00 | 1 | 0.00 | 2.92 | 0.1481 |
| *DE* | 0.00 | 1 | 0.00 | 0.00 | 0.9968 |
| *DF* | 0.00 | 1 | 0.00 | 0.12 | 0.7461 |
| *EF* | 0.00 | 1 | 0.00 | 0.13 | 0.7347 |
| $A^2$ | 0.08 | 1 | 0.08 | 220.46 | < 0.0001 |
| $B^2$ | 0.05 | 1 | 0.05 | 142.51 | < 0.0001 |
| $C^2$ | 0.01 | 1 | 0.01 | 15.01 | 0.0117 |
| $D^2$ | 0.01 | 1 | 0.01 | 20.62 | 0.0062 |
| $E^2$ | 0.00 | 1 | 0.00 | 8.72 | 0.0318 |
| $F^2$ | 0.00 | 1 | 0.00 | 3.58 | 0.1169 |
| Residual | 0.00 | 5 | 0.00 | | |
| Cor Total | 7.26 | 32 | | | |
| Std. Dev. | 0.02 | | | $R^2$ | 0.9998 |
| Mean | 5.85 | | | Adjusted $R^2$ | 0.9985 |
| C.V. % | 0.32 | | | Predicted $R^2$ | 0.9811 |
| PRESS | 0.14 | | | Adequate Precision | 99.0151 |

**Table 4: ANOVA for $\hat{\omega}_1$ (after backward elimination).**

| Source | Sum of squares | Degrees of freedom | Mean square | F-Value | p-value Prob > F' |
|---|---|---|---|---|---|
| Model | 7.25 | 15 | 0.48 | 1273.82 | < 0.0001 |
| *A* | 3.29 | 1 | 3.29 | 8679.56 | < 0.0001 |
| *B* | 1.36 | 1 | 1.36 | 3593.86 | < 0.0001 |
| *C* | 0.53 | 1 | 0.53 | 1406.41 | < 0.0001 |
| *D* | 0.13 | 1 | 0.13 | 332.32 | < 0.0001 |
| *E* | 0.02 | 1 | 0.02 | 51.23 | < 0.0001 |
| *AB* | 0.04 | 1 | 0.04 | 94.91 | < 0.0001 |
| *AC* | 0.02 | 1 | 0.02 | 55.53 | < 0.0001 |
| *AD* | 0.01 | 1 | 0.01 | 13.35 | 0.0020 |
| *BC* | 0.01 | 1 | 0.01 | 22.30 | 0.0002 |
| $A^2$ | 0.08 | 1 | 0.08 | 211.52 | < 0.0001 |
| $B^2$ | 0.05 | 1 | 0.05 | 135.63 | < 0.0001 |
| $C^2$ | 0.01 | 1 | 0.01 | 17.71 | 0.0006 |
| $D^2$ | 0.01 | 1 | 0.01 | 18.70 | 0.0005 |
| $E^2$ | 0.00 | 1 | 0.00 | 7.64 | 0.0133 |
| $F^2$ | 0.00 | 1 | 0.00 | 3.07 | 0.0976 |
| Residual | 0.01 | 17 | 0.00 | | |
| Cor Total | 7.26 | 32 | | | |
| Std. Dev. | 0.02 | | | $R^2$ | 0.9991 |
| Mean | 5.85 | | | Adjusted $R^2$ | 0.9983 |
| C.V. % | 0.33 | | | Predicted $R^2$ | 0.9965 |
| PRESS | 0.03 | | | Adequate Precision | 126.1608 |

The reduced RS model results indicate that the model is significant (p-value < 0.0001). The value of $R^2$ and adjusted $R^2$ is over 99%, which means that the reduced model gives a sufficiently accurate relationship between the input variables and the response variables. Moreover, the "Predicted $R^2$" value is 0.9965, which is in good agreement with the "Adjusted $R^2$" value of 0.9983. The Predicted Residual Error Sum of Squares (PRESS); which is a measure of discrepancy between experimental data and estimated model, is 0.03. Such a low value of PRESS shows that the quadratic model well fits each point in the design.

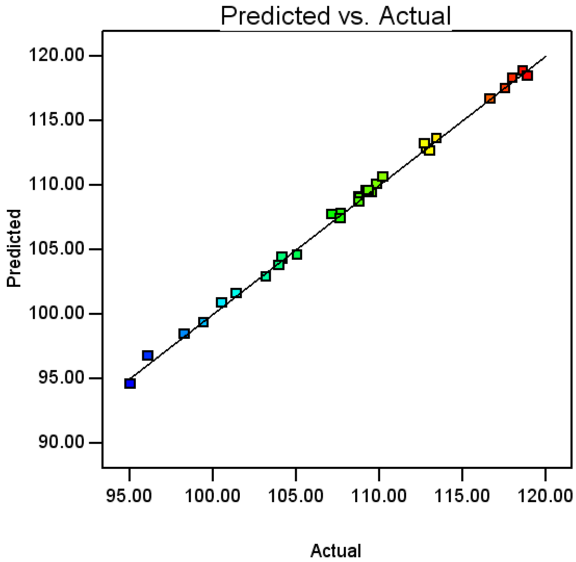


**Fig. 3 Predicted versus actual values of $\hat{\omega}_1$.**

Fig. 3 shows the values of the first natural frequency predicted by the RS model versus the values actually observed by FE analysis. Fig. 3 proves that the regression model is fairly well fitted with the observed values. The polynomial equation for first natural frequency, in coded terms, is written as:-

$$\hat{\omega}_1 = 6.18 + 0.37A + 0.25B + 0.15C + 0.08D + 0.03E + 0.04AB + 0.03AC + 0.02AD + 0.02BC - 0.17A^2 - 0.13B^2 - 0.05C^2 - 0.04D^2 - 0.03E^2 + 0.02F^2 \quad (6)$$

Fig. 4(a) shows the three-dimensional distribution of RS of first natural frequency ($\hat{\omega}_1$) with respect to the set of physical parameters $E_1$ and $E_5$, while keeping rest of the physical parameters ($E_9, E_{13}, E_{17}$ and $E_{24}$) at a constant level of 80 GPa. Three-dimensional distribution of $\hat{\omega}_1$ with respect to a set of input parameters ($E_1$ and $E_9$), ($E_1$ and $E_{13}$), ($E_1$ and $E_{17}$), ($E_1$ and $E_{24}$) and ($E_5$ and $E_9$) have been drawn respectively in part (b), (c), (d), (e) and (f) of Fig. 4. In each figure, two input parameters are varied between their lower and upper limits, while remaining input parameters are held at a constant level of 80 GPa.

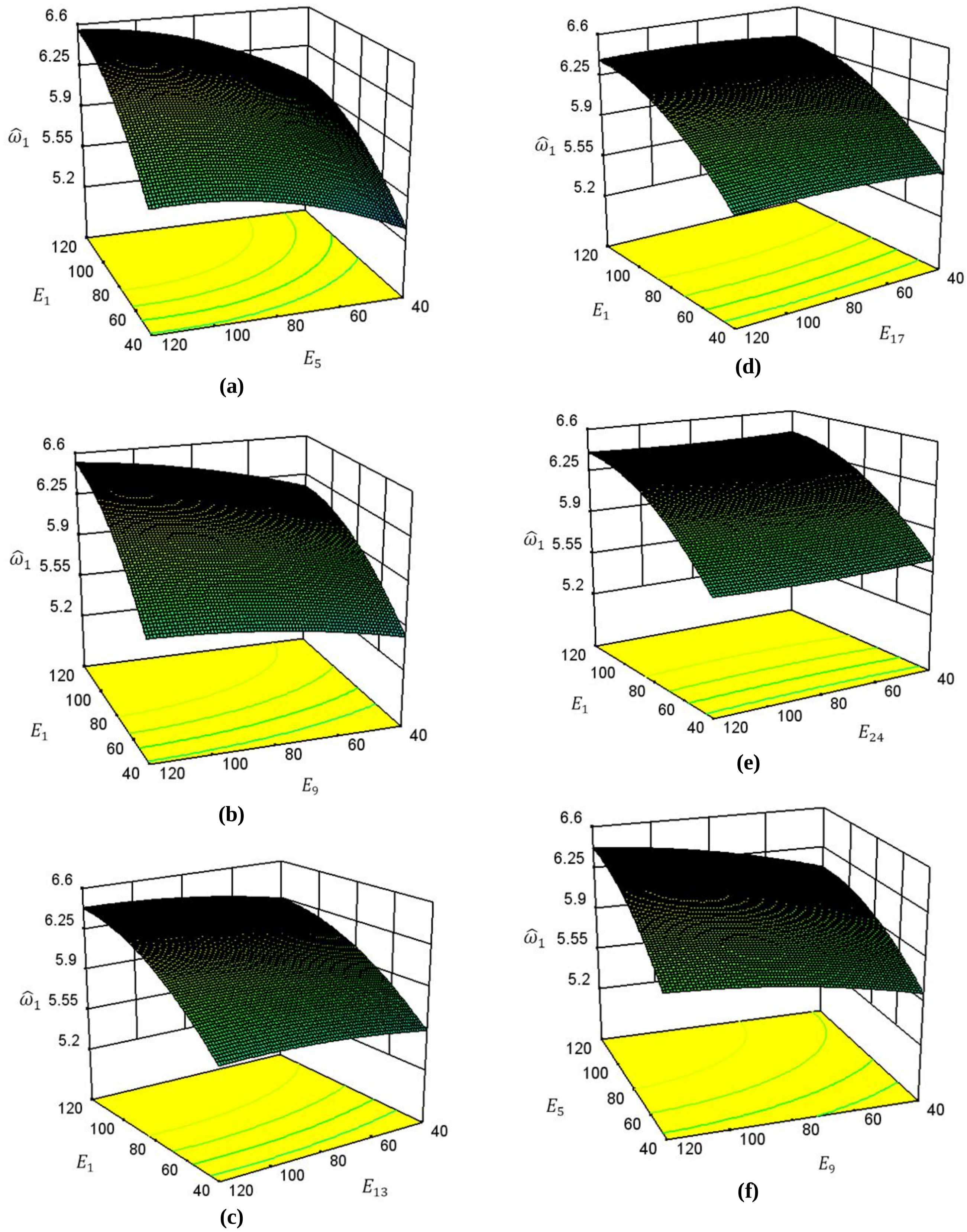


**Fig. 4 Response surface for $\hat{\omega}_1$ with respect to; (a) $E_1$ and $E_5$ (b) $E_1$ and $E_9$ (c) $E_1$ and $E_{13}$ (d) $E_1$ and $E_{17}$ (e) $E_1$ and $E_{24}$ (f) $E_5$ and $E_9$.**

Similar analysis is also performed for next nine RS predicted response variables viz. $\widehat{\omega}_2$, $\widehat{\omega}_3$, $\widehat{\omega}_4$, $\widehat{\omega}_5$, $\widehat{MAC}_{11}$, $\widehat{MAC}_{22}$, $\widehat{MAC}_{33}$, $\widehat{MAC}_{44}$ and $\widehat{MAC}_{55}$. After fitting RS models to actually observed results, the RS models for next nine responses (in coded terms) are given by the regression equations (7) to (15). Corresponding RS plots have been presented in Fig. 5 to 13.

$$\widehat{\omega}_2 = 39.40 + 1.94A + 0.20B + 0.27C + 1.17D + 1.42E + 0.30F + 0.11AB - 0.09AC + 0.14AE - 0.05BD + 0.09CD + 0.09CE + 0.19DE + 0.05EF - 0.63A^2 - 0.12B^2 - 0.18C^2 - 0.57D^2 - 0.68E^2 \quad (7)$$

$$\widehat{\omega}_3 = 112.08 + 4.15A + 0.43B + 3.24C + 0.89D + 1.28E + 3.68F - 0.41AB + 0.26AD - 0.23AE + 0.30AF + 0.36CD + 0.29CF + 0.41EF - 1.10A^2 - 1.39C^2 - 0.44D^2 - 0.59E^2 - 1.88F^2 \quad (8)$$

$$\widehat{\omega}_4 = 214.34 + 6.12A + 3.79B + 2.56C + 3.58D + 4.15E + 9.25F - 0.87AB + 0.59AC - 0.55AD + 0.70AE + 0.73BC - 0.28BD + 0.46BE - 0.47CD + 0.95DE + 0.75DF - 0.94EF - 1.20A^2 - 2.03B^2 - 1.08C^2 - 1.58D^2 - 1.48E^2 - \quad (9)$$

$$\widehat{\omega}_5 = 358.61 + 8.18A + 11.60B + 1.95C + 9.33D + 8.85E + 7.32F - 0.57AB - 0.55AC + 1.42AD - 1.28BC + 2.10BD + 0.68CD - 1.83DE - 0.66DF + 0.64EF - 1.06A^2 - 4.22B^2 - 3.64D^2 - 3.26E^2 - 2.12F^2 \quad (10)$$

$$\widehat{MAC}_{11} = 10^{-4} \times (9998.94 - 2.57A - 0.96B + 0.49C + 1.07D + 0.50E - 1.33AB + 1.70AC + 2.40AD + 1.58AE + 0.66BD - 0.79CD - 0.72DE - 2.37A^2 - 1.87D^2) \quad (11)$$

$$\widehat{MAC}_{22} = 10^{-4} \times (9991.36 - 14.19A + 1.29B - 2.85C - 4.65D + 5.51E + 6.69F - 5.83AC - 1.78AD + 16.12AE + 7.13AF - 2.24CD + 4.50CE + 3.45CF + 3.20DF - 12.30A^2 + 3.95C^2 - 11.13E^2 - 5.60F^2) \quad (12)$$

$$\widehat{MAC}_{33} = 10^{-4} \times (9986.63 - 33.57A - 11.77C + 38.82F - 15.94AB - 19.63AC + 8.83AD + 55.40AF + 16.58BF + 32.80CF + 8.34DE - 10.18DF + 26.83EF - 34.20A^2 - 70.70F^2) \quad (13)$$

$$\widehat{MAC}_{44} = 10^{-4} \times (9925.18 - 69.34A - 40.64B - 30.36D + 40.53E + 104.23F - 56.69AB + 29.06AE + 88.89AF - 21.35BD + 22.89BE + 52.74BF + 26.02CF + 35.00DE + 20.52DF - 51.93E^2 - 122.71F^2) \quad (14)$$

$$\widehat{MAC}_{55} = 10^{-4} \times (9931.21 - 32.15A + 45.96C - 62.02D + 69.05E + 80.61F - 84.45AB - 19.27AC + 59.11AE + 53.46AF - 40.75BC + 30.14BD + 44.71BE + 58.88BF + 35.52CD + 24.57DF - 46.67EF - 43.15A^2 - 57.74C^2 - 58.64E^2 - 76.35F^2) \quad (15)$$

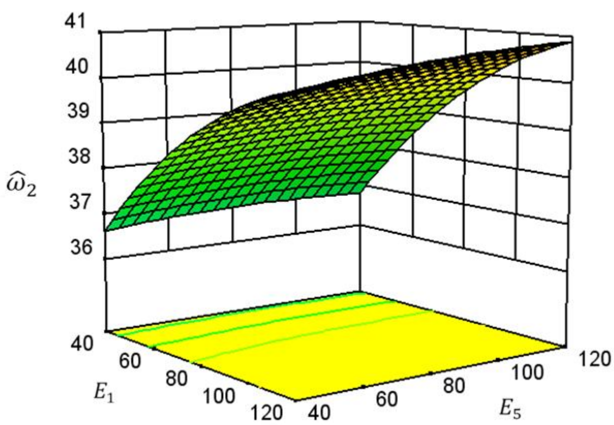


**Fig. 5 Three-dimensional surface plot for $\widehat{\omega}_2$.**

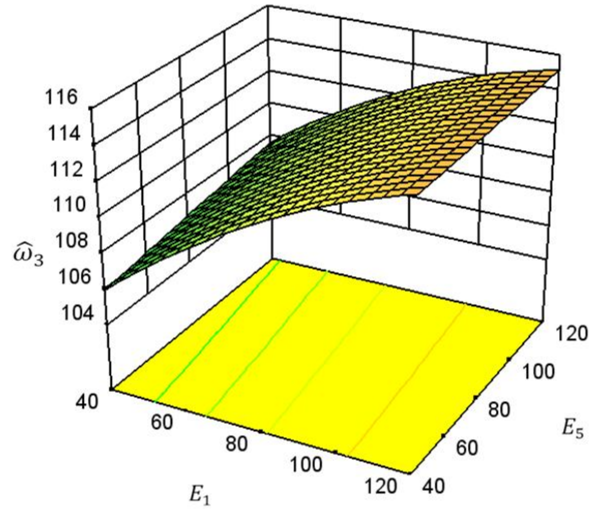


Fig. 6 Three-dimensional surface plot for $\widehat{\omega}_3$.

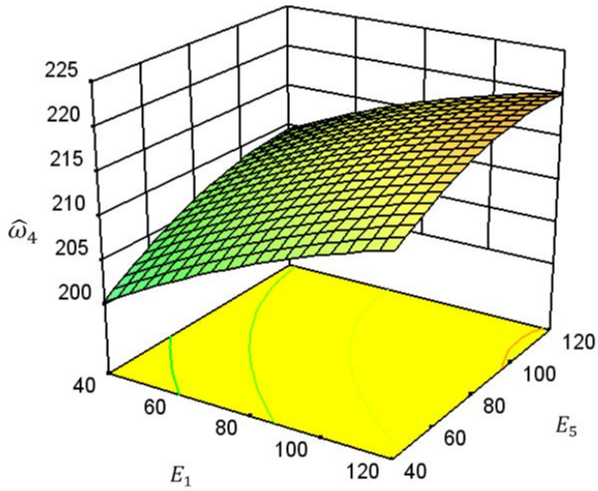


Fig. 7 Three-dimensional surface plot for $\widehat{\omega}_4$.

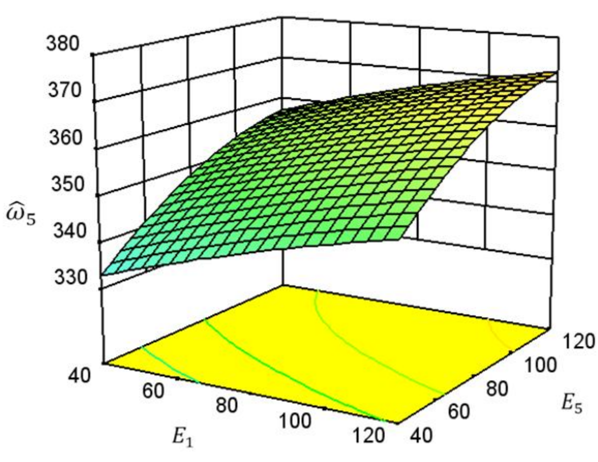


Fig. 8 Three-dimensional surface plot for $\widehat{\omega}_5$.

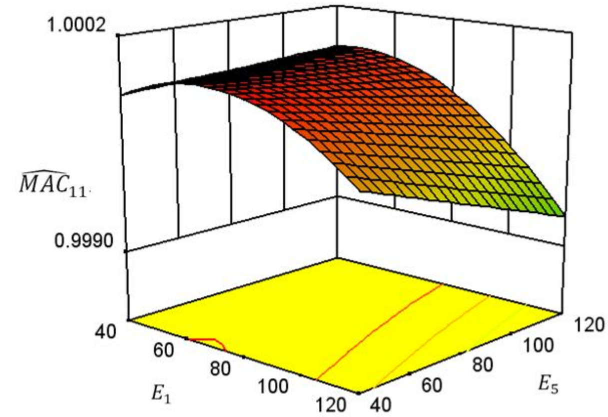


Fig. 9 Three-dimensional surface plot for $\widehat{MAC}_{11}$.

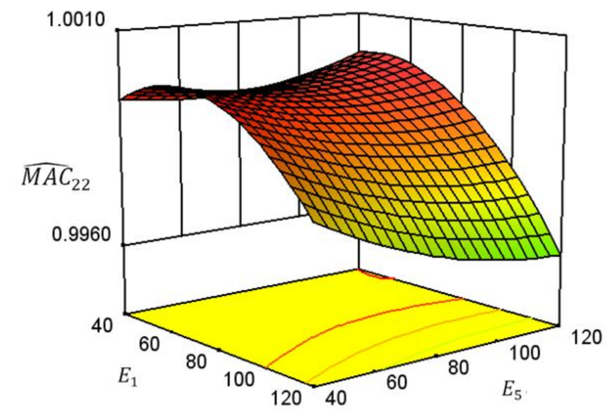


Fig. 10 Three-dimensional surface plot for $\widehat{MAC}_{22}$.

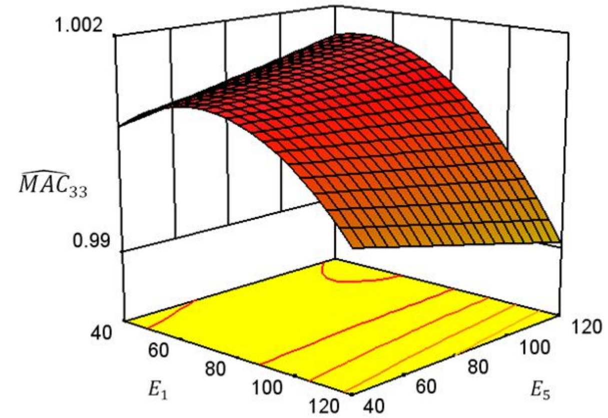


Fig. 11 Three-dimensional surface plot for $\widehat{MAC}_{33}$.

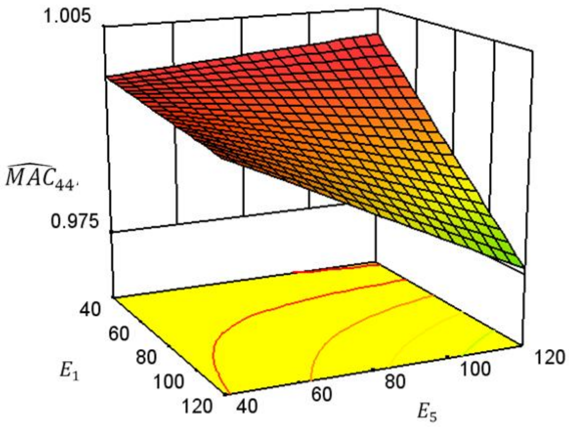


**Fig. 12 Three-dimensional surface plot for $\widehat{MAC}_{44}$.**

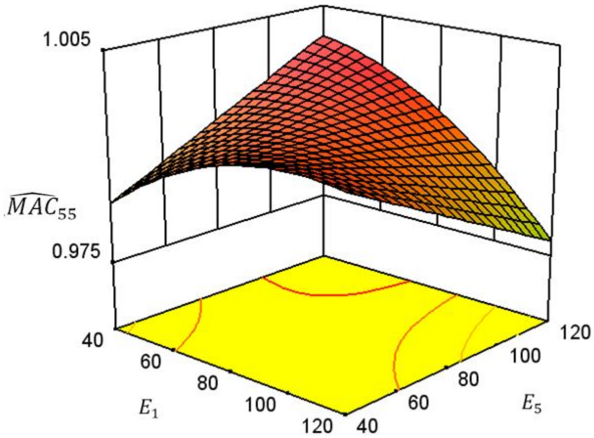


**Fig. 13 Three-dimensional surface plot for $\widehat{MAC}_{55}$.**

After having generated RS models for all response variables, next major step is the objective-function-formulation for Derringer's function based FEMU problem. For this purpose three different case studies have been carried out. Results of the three case studies have been compared in order to find out the best possible formulation of objective functions.

## 5 Evaluation of benchmark objective-function-formulation and its application

Three types of objective-function-formulations are evaluated for their performance and then compared with each other in order to establish a benchmark objective-function-formulation than can be used in further research work related to Derringer's function based FEMU method. For this purpose, three case studies are carried out as explained in sub-sections 5.1, 5.2 and 5.3. Results of the three case studies are then compared in subsection 5.4 in order to find out benchmark objective-function-formulation.

### 5.1 First case study

During first case study, sub-objectives of FEMU are formulated by using natural frequencies only. Five individual desirability functions are defined based on the pattern reflected in (16).

$$d_i = \begin{cases} \left(\dfrac{\widehat{\omega}_i - \widehat{\omega}_{iLL}}{\widehat{\omega}_{iT} - \widehat{\omega}_{iLL}}\right), & \widehat{\omega}_{iLL} \le \widehat{\omega}_i \le \widehat{\omega}_{iT} \\ \left(\dfrac{\widehat{\omega}_{iUL} - \widehat{\omega}_i}{\widehat{\omega}_{iUL} - \widehat{\omega}_{iT}}\right), & \widehat{\omega}_{iT} < \widehat{\omega}_i \le \widehat{\omega}_{iUL} \\ 0 \quad , & \widehat{\omega}_i < \widehat{\omega}_{iLL} \text{ or } \widehat{\omega}_i > \widehat{\omega}_{iUL} \end{cases} \tag{16}$$

where $d_i$ is $i^{\text{th}}$ individual desirability function; $\widehat{\omega}_i$ is $i^{\text{th}}$ natural frequency predicted by RS model; $\widehat{\omega}_{iT}$ is the target value for $i^{\text{th}}$ RS model, which is set as corresponding SE natural frequency; $\widehat{\omega}_{iLL}$ is lower limit for $i^{\text{th}}$ natural frequency predicted by RS model, which is set at a value 0.3% lower than $\widehat{\omega}_{iT}$. $\widehat{\omega}_{iUL}$ is upper limit for $i^{\text{th}}$ natural frequency predicted by RS model, which is set at a value 0.3% higher than $\widehat{\omega}_{iT}$.

Numerical details of formulation of sub-objectives of FEMU are presented in Table 5. Values given in Table 5 are used in conjunction with (16) to formulate the individual desirability functions for first five RS predicted natural frequencies. Fig. 14 shows the individual desirability function for $\widehat{\omega}_1$.

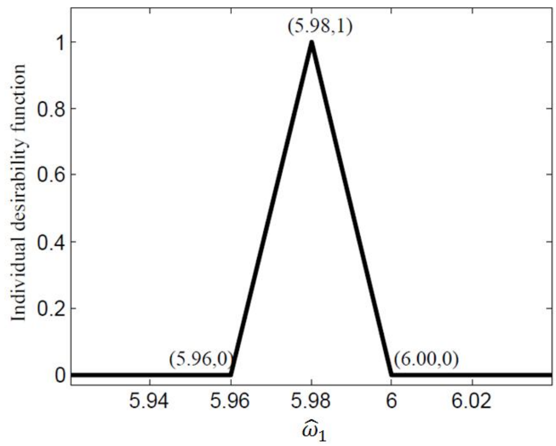


**Fig. 14 Individual desirability function for $\widehat{\omega}_1$.**

Individual desirability functions for $\widehat{\omega}_2$, $\widehat{\omega}_3$, $\widehat{\omega}_4$ and $\widehat{\omega}_5$ also follow the pattern similar to Fig. 14.

Value of any individual desirability function for any particular mode will be unity, only if RS predicted value of natural frequency of that particular mode will be equal to corresponding SE value. Thus FEMU problem gets converted to an optimization problem, where five sub-objectives of the problem are to maximize the scale-free individual desirability functions.

**Table 5: Numerical details of objective-function-formulation during first case study.**

| RS predicted response variable | $\widehat{\omega}_{iT}$ (Hz) | $\widehat{\omega}_{iLL}$ (Hz) | $\widehat{\omega}_{iUL}$ (Hz) |
|---|---|---|---|
| $\widehat{\omega}_1$ | 5.98 | 5.96 | 6.00 |
| $\widehat{\omega}_2$ | 37.91 | 37.80 | 38.02 |
| $\widehat{\omega}_3$ | 111.82 | 111.48 | 112.16 |
| $\widehat{\omega}_4$ | 213.02 | 212.38 | 213.66 |
| $\widehat{\omega}_5$ | 353.51 | 352.45 | 354.57 |

Further, the individual desirability functions are combined together to form a single overall desirability function as per (5), thereby converting the FEMU problem into the maximization of just a single overall desirability function. Overall desirability function will achieve a unit value only if all the individual desirability functions are each equal to unity. Plot of the overall desirability function with respect to $E_1$ and $E_5$ is drawn in Fig. 15.

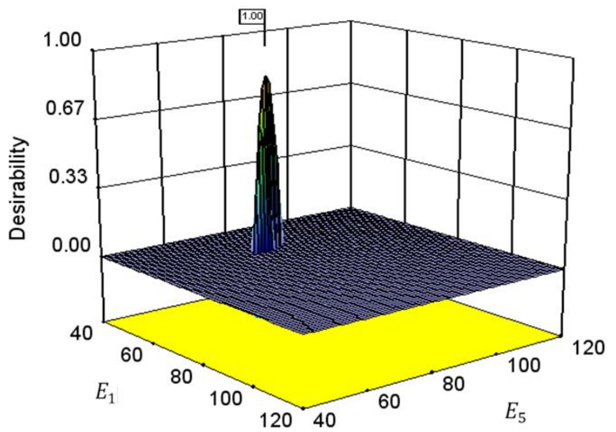


**Fig. 15 Three dimensional plot of overall desirability function for first case study.**

It can be seen from Fig. 15 that high value of overall desirability function is achieved at only a few locations; while in most part of the design space overall desirability function approaches zero. As is highlighted in contour plots drawn in Fig. 16, if $E_1$ takes a value lower than 57 GPa or higher than 62 GPa, the overall desirability function falls to zero; thereby restricting the design space. Moreover, if $E_5$ parameter falls below 71 GPa or above 83 GPa, overall desirability function again falls to zero irrespective of the value of $E_1$. This information is very important in dynamic design applications.

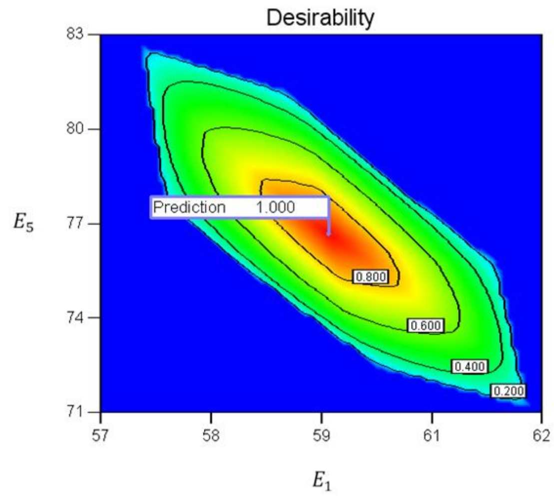


**Fig. 16 Contour plot of overall desirability function for first case study.**

The updating parameters are found in such a way that the overall desirability function approached unity. It is found that optimum value of overall desirability function is 1.0000. This optimum value of overall desirability function is achieved when updating parameters ($E_1$, $E_5$, $E_9$, $E_{13}$, $E_{17}$ and $E_{24}$) are set to (59.64, 74.84, 111.21, 81.82, 65.36 and 94.86) GPa respectively. These settings are then used to produce an updated FE model, which is further executed in Matlab to produce updated natural frequencies and MAC values. Updated FE results are then compared with their SE counterparts. Updating results show that, before FEMU, there is an absolute average error of 20.87% in prediction of natural frequencies (considering first five modes), which gets reduced to just 0.19% after FEMU. Clearly, there is an error reduction of 99.09% in prediction of natural frequencies. An error reduction of 94.94% is observed in prediction of MAC values. But error reduction in prediction of the actual physical parameters is just 91.92%, which is not quite satisfactory and hence motivated the authors to do further work in order to find a better formulation of objective-functions; thereby leading to second and third case studies as explained in next two sub-sections.

## 5.2 Second case study

During second case study, formulation of individual desirability functions is done by using MAC values only. Here formulation is done in such a manner so as to maximize the MAC values as shown in (17).

$$d_i = \begin{cases} 1 & , \widehat{MAC}_{ii} \geq \widehat{MAC}_{iiUL} \\ \left( \dfrac{\widehat{MAC}_{ii} - \widehat{MAC}_{iiLL}}{\widehat{MAC}_{iiUL} - \widehat{MAC}_{iiLL}} \right) & , \widehat{MAC}_{iiLL} < \widehat{MAC}_{ii} < \widehat{MAC}_{iiUL} \\ 0 & , \widehat{MAC}_{ii} \leq \widehat{MAC}_{iiLL} \end{cases} \tag{17}$$

where $d_i$ is $i^{\text{th}}$ individual desirability function; $\widehat{MAC}_{ii}$ is $i^{\text{th}}$ MAC value predicted by RS model; $\widehat{MAC}_{iiUL}$ is upper limit for $i^{\text{th}}$ MAC value predicted by RS model, which is set to be unity, this is because realistically, the maximum possible value for MAC is unity. $\widehat{MAC}_{iiLL}$ is lower limit for $i^{\text{th}}$ MAC value predicted by RS model, which is set at a value 0.997, i.e., 0.3% lower than $\widehat{MAC}_{iiUL}$. Numerical details of formulation of individual desirability functions are also presented in Table 6.

**Table 6: Numerical details of objective-function-formulation during second case study.**

| RS predicted response variable | Type of objective function | $\widehat{MAC}_{iiLL}$ | $\widehat{MAC}_{iiUL}$ |
|---|---|---|---|
| $\widehat{MAC}_{11}$ | Maximize | 0.997 | 1 |
| $\widehat{MAC}_{22}$ | Maximize | 0.997 | 1 |
| $\widehat{MAC}_{33}$ | Maximize | 0.997 | 1 |
| $\widehat{MAC}_{44}$ | Maximize | 0.997 | 1 |
| $\widehat{MAC}_{55}$ | Maximize | 0.997 | 1 |

Values shown in Table 6 are combined as per (17) to develop the individual desirability functions. Graphically, individual desirability functions for $\widehat{MAC}_{11}$ is drawn in Fig. 17. Next four individual desirability functions also have a graphical form as drawn in Fig. 17. Overall desirability function is developed by combining the five individual desirability functions together to produce a single overall desirability function as per (5). Three dimensional plot of overall desirability function is drawn in Fig. 18, while a zoomed view of contour plot of overall desirability function is presented in Fig. 19.

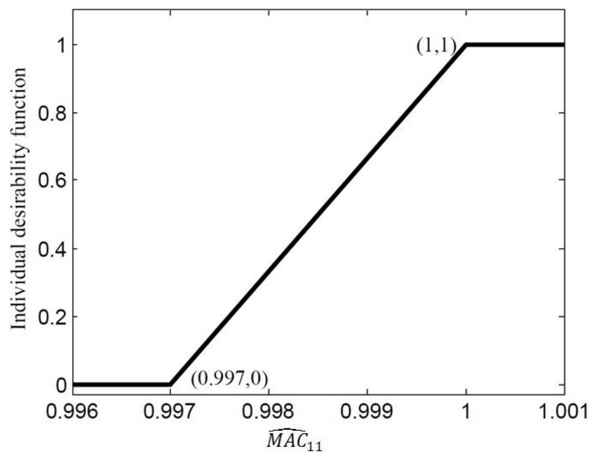


**Fig. 17 Individual desirability function for $\widehat{MAC}_{11}$.**

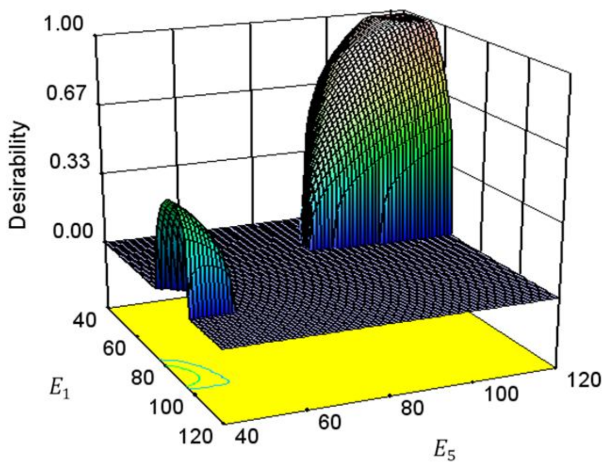


**Fig. 18 Three dimensional plot of overall desirability function for second case study.**

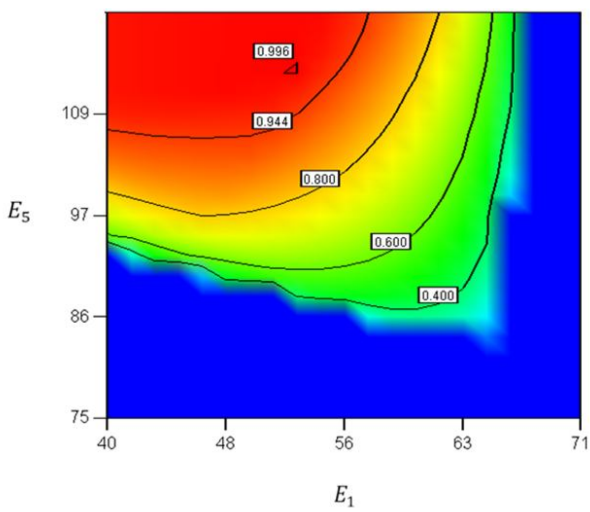


**Fig. 19 Contour plot of overall desirability function for second case study.**

Optimum value of the overall desirability function is found to be 0.996068. Corresponding set of updating parameters ($E_1$, $E_5$, $E_9$, $E_{13}$, $E_{17}$ and $E_{24}$) are found to be (52.49, 113.08, 104.05, 70.59, 79.80 and 95.06) GPa. These updated values of physical parameters are then used to update the FE model of beam structure and later on to produce updated natural frequencies and MAC values. During this case study, updated results show an error reduction of 96.20%, 96.55% and 91.65% in prediction of natural frequencies, MAC values and physical parameters respectively. Since an error reduction of just 91.65% (in case of prediction of physical parameters) is not quite satisfactory, so next case study needs to be carried out, during which objective-functions are formulated by using both natural frequencies as well as MAC values.

### 5.3 Third case study

During third case study, the sub-objectives are formulated by using natural frequencies as well as MAC values Numerical details of formulation of individual desirability functions are shown in Table 7. Values shown in Table 7 are combined as per (18) to develop ten individual desirability functions, which are then processed further to form a single overall desirability function. Three dimensional and contour plots (zoomed) of the overall desirability function have been drawn in Figs. 20 and 21 respectively.

**Table 7: Numerical details of objective-function-formulation during third case study.**

| RS predicted response variable | Type of objective function | Lower limit | Upper limit |
|---|---|---|---|
| $\widehat{\omega}_1$ | Target 5.98 | 5.96 | 6.00 |
| $\widehat{\omega}_2$ | Target 37.91 | 37.80 | 38.02 |
| $\widehat{\omega}_3$ | Target 111.82 | 111.48 | 112.16 |
| $\widehat{\omega}_4$ | Target 213.02 | 212.38 | 213.66 |
| $\widehat{\omega}_5$ | Target 353.51 | 352.45 | 354.57 |
| $\widehat{MAC}_{11}$ | Maximize | 0.997 | 1.000 |
| $\widehat{MAC}_{22}$ | Maximize | 0.997 | 1.000 |
| $\widehat{MAC}_{33}$ | Maximize | 0.997 | 1.000 |
| $\widehat{MAC}_{44}$ | Maximize | 0.997 | 1.000 |
| $\widehat{MAC}_{55}$ | Maximize | 0.997 | 1.000 |

$$d_j = \begin{bmatrix} \left\{ \begin{array}{ll} \left(\frac{\widehat{\omega}_i-\widehat{\omega}_{iLL}}{\widehat{\omega}_{iT}-\widehat{\omega}_{iLL}}\right), & \widehat{\omega}_{iLL} \le \widehat{\omega}_i \le \widehat{\omega}_{iT} \\ \left(\frac{\widehat{\omega}_{iUL}-\widehat{\omega}_i}{\widehat{\omega}_{iUL}-\widehat{\omega}_{iT}}\right), & \widehat{\omega}_{iT} < \widehat{\omega}_i \le \widehat{\omega}_{iUL} \\ 0, & \widehat{\omega}_i < \widehat{\omega}_{iLL} \text{ or } \widehat{\omega}_i > \widehat{\omega}_{iUL} \end{array} \right\} 1 \le j \le 5 \\ \left\{ \begin{array}{ll} 1, & \widehat{MAC}_{ii} \ge \widehat{MAC}_{iiUL} \\ \left(\frac{\widehat{MAC}_{ii}-\widehat{MAC}_{iiLL}}{\widehat{MAC}_{iiUL}-\widehat{MAC}_{iiLL}}\right), & \widehat{MAC}_{iiLL} < \widehat{MAC}_{ii} < \widehat{MAC}_{iiUL} \\ 0, & \widehat{MAC}_{ii} \le \widehat{MAC}_{iiLL} \end{array} \right\} 6 \le j \le 10 \end{bmatrix} \quad (18)$$

where $d_j$ is $j^{th}$ individual desirability function; $i$ is an integer varying from one to five for first five modes; $j$ is an integer varying from one to ten for ten individual desirability functions; other parameters are as defined in (16) and (17).

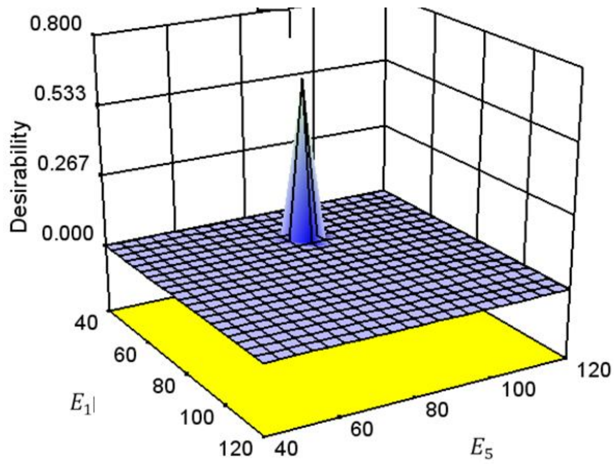


**Fig. 20 Three dimensional plot of overall desirability function for third case study.**

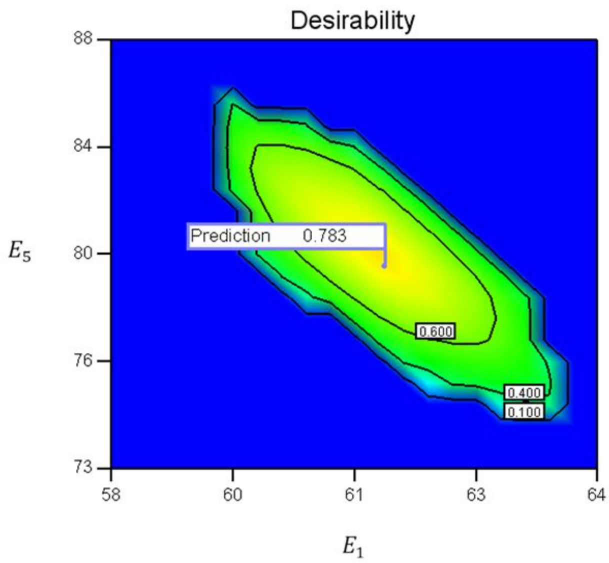


**Fig. 21 Contour plot of overall desirability function for third case study.**

During third case study, the optimum value of the overall desirability function is found to be 0.782667. Corresponding set of updating parameters ($E_1$, $E_5$, $E_9$, $E_{13}$, $E_{17}$ and $E_{24}$) is found to be (61.58, 79.92, 97.71, 57.60, 85.45 and 96.30) GPa respectively. The updated values of the physical parameters are then used to update the FE model of the beam structure and later on to produce updated natural frequencies and MAC values. Here, the updated results show an error reduction of 99.34%, 99.65% and 97.98% in the prediction of natural frequencies, MAC values and physical parameters respectively.

## 5.4 Comparison of results of three case studies

Updated response variables viz., natural frequencies and MAC values, of three case studies have been summarized in Table 8 and 9 respectively. Fig. 22 shows a comparison of response variables based FEMU results of the three case studies. It is clear from Fig. 22 that while considering error reduction in natural frequencies; objective-function-formulations of first case study (where error reduction is 99.1%) perform better than that of second case study (where error reduction is 96.3%). On the other hand, objective-function-formulations of second case study perform (where error reduction is 96.55%) better than that of first case study (where error reduction is 94.94%), if only the error reduction in prediction of MAC values is considered. But, both the first as well as second case study based objective-function-formulations lag behind the performance shown by objective-function-formulations of third case study in reducing the error in prediction of natural frequencies (99.4%) as well as MAC values (99.65%).

Further, performance of any objective-function-formulation should not be judged just by error reduction in response variables, but, error reduction in physical input variables should also be considered particularly when one is relying upon RS type approximation method. This is because in RSM an incorrect set of input variables can also sometimes produce a correct set of response variables. So error reduction in prediction of input physical parameters is also compared in Fig. 23 by using the data available in Table 10 for all three case studies and here also performance of objective-function-formulations of third case study (average error of just 3.26%, along with an error reduction of 97.98%) is far better than that of first (average error of 13.01%, along with an error reduction of 91.92%) or second case study (average error of 13.46%, along with an error reduction of just 91.65%).

**Table 8: Comparison of updated natural frequencies of three case studies.**

| Response variable | SE results (Hz) | FE results before FEMU (Hz) | Initial error (%) | First case study: FE results after FEMU (Hz) | First case study: Final error (%) | Second case study: FE results after FEMU (Hz) | Second case study: Final error (%) | Third case study: FE results after FEMU (Hz) | Third case study: Final error (%) |
|---|---|---|---|---|---|---|---|---|---|
| $\omega_1$ | 5.98 | 7.45 | 24.6 | 6.00 | 0.3 | 6.01 | 0.5 | 5.99 | 0.2 |
| $\omega_2$ | 37.91 | 46.66 | 23.1 | 38.08 | 0.5 | 37.82 | 0.2 | 38.03 | 0.3 |
| $\omega_3$ | 111.82 | 130.64 | 16.8 | 111.93 | 0.1 | 111.35 | 0.4 | 111.81 | 0.0 |
| $\omega_4$ | 213.02 | 256.01 | 20.2 | 212.82 | 0.1 | 214.89 | 0.9 | 212.77 | 0.1 |
| $\omega_5$ | 353.51 | 423.21 | 19.7 | 353.49 | 0.0 | 360.14 | 1.9 | 353.67 | 0.1 |
| Average error (%) | | | 20.9 | | 0.2 | | 0.8 | | 0.1 |
| Error reduction (%) | | | | | 99.1 | | 96.3 | | 99.4 |

**Table 9: Comparison of updated MAC values of three case studies.**

| Response variable | Desired value | FE results before FEMU | Initial error (%) | First case study: FE results after FEMU | First case study: Final error (%) | Second case study: FE results after FEMU | Second case study: Final error (%) | Third case study: FE results after FEMU | Third case study: Final error (%) |
|---|---|---|---|---|---|---|---|---|---|
| $\widehat{MAC}_{11}$ | 1 | 0.9996 | 0.04 | 1.0000 | 0.00 | 1.0000 | 0.00 | 1.0000 | 0.00 |
| $\widehat{MAC}_{22}$ | 1 | 0.9949 | 0.51 | 0.9997 | 0.03 | 0.9999 | 0.01 | 1.0000 | 0.00 |
| $\widehat{MAC}_{33}$ | 1 | 0.9873 | 1.27 | 0.9996 | 0.04 | 0.9999 | 0.01 | 0.9999 | 0.01 |
| $\widehat{MAC}_{44}$ | 1 | 0.9769 | 2.31 | 0.9983 | 0.17 | 0.9988 | 0.12 | 0.9999 | 0.01 |
| $\widehat{MAC}_{55}$ | 1 | 0.9544 | 4.56 | 0.9980 | 0.20 | 0.9984 | 0.16 | 0.9999 | 0.01 |
| Average error (%) | | | 1.74 | | 0.09 | | 0.06 | | 0.01 |
| Error reduction (%) | | | | | 94.94 | | 96.55 | | 99.65 |

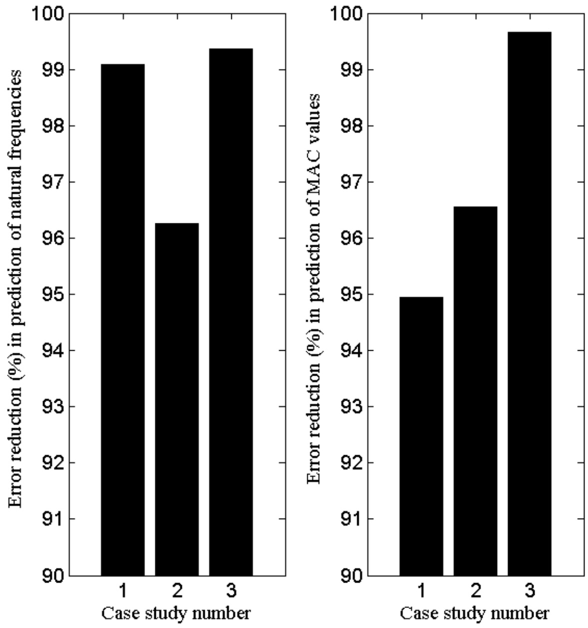


**Fig. 22 Comparison of response variables based FEMU results of three case studies.**

**Table 10: Comparison of updated physical parameters of three case studies.**

| Physical parameter | SE value (GPa) | FE results before FEMU (GPa) | Initial error (%) | First case study | | Second case study | | Third case study | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | FE results after FEMU (GPa) | Final error (%) | FE results after FEMU (GPa) | Final error (%) | FE results after FEMU (GPa) | Final error (%) |
| $E_1$ | 60 | 200 | 233.33 | 59.64 | 0.60 | 52.49 | 12.52 | 61.58 | 2.63 |
| $E_5$ | 80 | 200 | 150.00 | 74.84 | 6.45 | 113.08 | 41.35 | 79.92 | 0.10 |
| $E_9$ | 100 | 200 | 100.00 | 111.21 | 11.21 | 104.05 | 4.05 | 97.71 | 2.29 |
| $E_{13}$ | 60 | 200 | 233.33 | 81.82 | 36.37 | 70.59 | 17.65 | 57.60 | 4.00 |
| $E_{17}$ | 80 | 200 | 150.00 | 65.36 | 18.30 | 79.80 | 0.25 | 85.45 | 6.81 |
| $E_{24}$ | 100 | 200 | 100.00 | 94.86 | 5.14 | 95.06 | 4.94 | 96.30 | 3.70 |
| Average error (%) | | | 161.11 | | 13.01 | | 13.46 | | 3.26 |
| Error reduction (%) | | | | | 91.92 | | 91.65 | | 97.98 |

Thus it is found that performance of objective-function-formulation of third case study is far better than that of first or second case study, thereby prompting its use as a benchmark objective-function-formulation for further research work related to Derringer's function based FEMU. Moreover RSM is an approximation based technique, hence, in order to further improve upon accuracy and reliability of updating results, more refined and reliable RS models need to be created and used for further model updating as explained in sub-section 5.5.

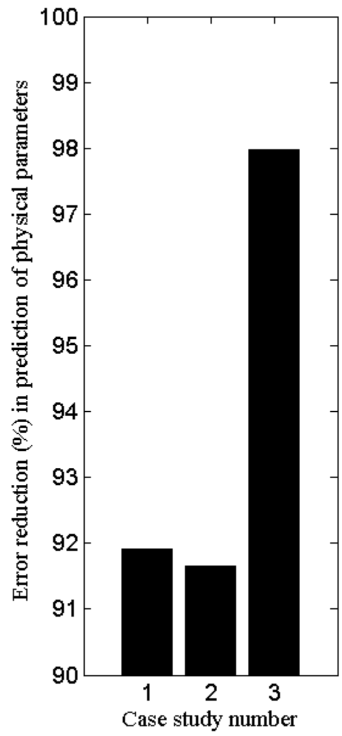


**Fig. 23 Comparison of physical variables based FEMU results of three case studies.**

## 5.5 Application of benchmark objective-function-formulation in Refined RS models based FEMU

During this stage, updating results of third case study are used to generate refined and more reliable RS models of different response variables. Here upper and lower limits of physical variables are set as ±10 GPa of corresponding optimum value obtained during third case study. ANOVA is performed for all ten responses (five natural frequencies and five MAC values). AVOVA results for first natural frequency (after backward elimination) have been presented in Table 11.

ANOVA results indicate that RS model is significant (p-value < 0.0001). The value of $R^2$ and adjusted $R^2$ is 100%, which is also the best one can expect. This means that the refined RS model gives most accurate relationship between input variables and response variables. Moreover, the "Predicted $R^2$" value is also 1.0000, which is in best agreement with "Adjusted $R^2$" value of 1.0000. "PRESS" of 0.00 shows that quadratic model fits in a perfect manner. A comparison of Table 4 with Table 11 shows that the refined RS based ANOVA results are far better than those obtained during the first stage. Fig. 24 shows the normal probability plot of residuals for RS predicted first natural frequency.

**Table 11: ANOVA for $\hat{\omega}_1$ (after backward elimination).**

| Source | Sum of squares | Degrees of freedom | Mean square | F-Value | p-value Prob > F' |
|---|---|---|---|---|---|
| Model | 0.58 | 19 | 0.03 | 222168.20 | < 0.0001 |
| *A* | 0.39 | 1 | 0.39 | 2869935.39 | < 0.0001 |
| *B* | 0.07 | 1 | 0.07 | 527327.28 | < 0.0001 |
| *C* | 0.01 | 1 | 0.01 | 65128.65 | < 0.0001 |
| *D* | 0.02 | 1 | 0.02 | 161714.42 | < 0.0001 |
| *E* | 0.00 | 1 | 0.00 | 4915.92 | < 0.0001 |
| *F* | 0.00 | 1 | 0.00 | 17.83 | 0.0010 |
| *AB* | 0.00 | 1 | 0.00 | 1677.07 | < 0.0001 |
| *AC* | 0.00 | 1 | 0.00 | 255.28 | < 0.0001 |
| *AD* | 0.00 | 1 | 0.00 | 657.10 | < 0.0001 |
| *AE* | 0.00 | 1 | 0.00 | 34.80 | < 0.0001 |
| *BC* | 0.00 | 1 | 0.00 | 44.83 | < 0.0001 |
| *BD* | 0.00 | 1 | 0.00 | 90.66 | < 0.0001 |
| *CD* | 0.00 | 1 | 0.00 | 19.36 | 0.0007 |
| *CE* | 0.00 | 1 | 0.00 | 10.09 | 0.0073 |
| $A^2$ | 0.00 | 1 | 0.00 | 7190.63 | < 0.0001 |
| $B^2$ | 0.00 | 1 | 0.00 | 612.81 | < 0.0001 |
| $C^2$ | 0.00 | 1 | 0.00 | 102.19 | < 0.0001 |
| $D^2$ | 0.00 | 1 | 0.00 | 732.26 | < 0.0001 |
| $E^2$ | 0.00 | 1 | 0.00 | 10.60 | 0.0063 |
| Residual | 0.00 | 13 | 0.00 | | |
| Cor Total | 0.58 | 32 | | | |
| Std. Dev. | 0.00 | | | $R^2$ | 1.0000 |
| Mean | 5.96 | | | Adjusted $R^2$ | 1.0000 |
| C.V. % | 0.01 | | | Predicted $R^2$ | 1.0000 |
| PRESS | 0.00 | | | Adequate precision | 1600.3713 |

In Fig. 24, the residuals are falling along a straight line, which shows that the residuals are normally distributed. Fig. 25 shows the values of the first natural frequency predicted by the refined RS model versus the values actually observed. Fig. 25 proves that the regression model is fairly well fitted with the observed values. The polynomial equation for $\hat{\omega}_1$, in coded terms, is given in (19). Similar analysis is performed for remaining nine response variables also. By using benchmark objective-function-formulation of third case study, ten individual desirability functions are defined by as per data given in Table 12 in conjunction with (18).

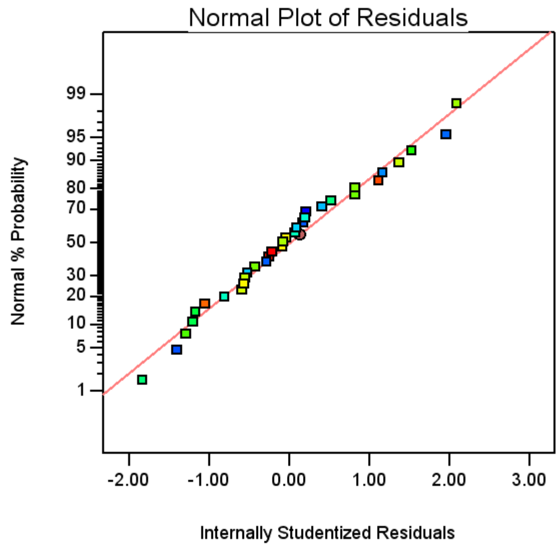


**Fig. 24 Normal probability plot of the residuals for $\hat{\omega}_1$.**

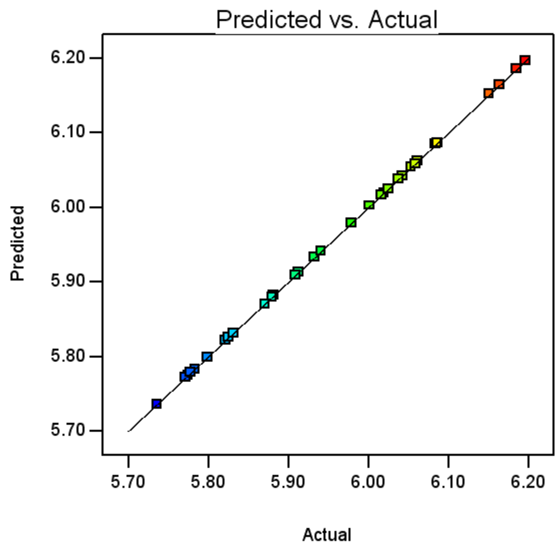


**Fig. 25 Predicted versus actual values of $\hat{\omega}_1$.**

$$\hat{\omega}_1 = 5.99 + 0.13A + 0.05B + 0.02C + 0.03D + 0.01E + 0.0003F + 0.003AB + 0.001AC + 0.002AD + 0.001AE + 0.001BC + 0.001BD + 0.0004CD + 0.0003CE - 0.02A^2 - 0.01B^2 - 0.002C^2 - 0.01D^2 - 0.001E^2 \quad (19)$$

**Table 12: Details of objective-function-formulation during refined RS based FEMU.**

| Response variable | Type of objective function | Lower limit | Upper limit |
|---|---|---|---|
| $\hat{\omega}_1$ | Achieve target of 5.98 | 5.97 | 5.99 |
| $\hat{\omega}_2$ | Achieve target of 37.91 | 37.87 | 37.95 |
| $\hat{\omega}_3$ | Achieve target of 111.82 | 111.71 | 111.93 |
| $\hat{\omega}_4$ | Achieve target of 213.02 | 212.81 | 213.23 |
| $\hat{\omega}_5$ | Achieve target of 353.51 | 353.16 | 353.86 |
| $\widehat{MAC}_{11}$ | Maximize | 0.999 | 1.000 |
| $\widehat{MAC}_{22}$ | Maximize | 0.999 | 1.000 |
| $\widehat{MAC}_{33}$ | Maximize | 0.999 | 1.000 |
| $\widehat{MAC}_{44}$ | Maximize | 0.999 | 1.000 |
| $\widehat{MAC}_{55}$ | Maximize | 0.999 | 1.000 |

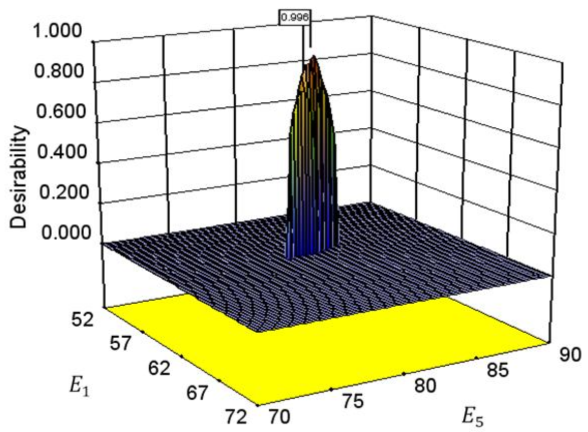


**Fig. 26 Three-dimensional plot of overall desirability function for refined RS based FEMU.**

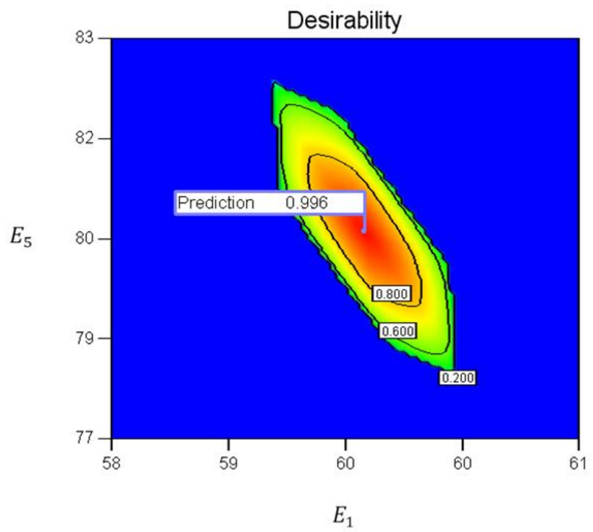


**Fig. 27 Contour plot of overall desirability function for refined RS based FEMU.**

Overall desirability function, as drawn in Fig. 26, is developed by combining the ten individual desirability function as per (5). Contour plot of the overall desirability function is drawn in Fig. 27.Optimum value of overall desirability function is found to be 0.996. Corresponding set of updating parameters ($E_1$, $E_5$, $E_9$, $E_{13}$, $E_{17}$ and $E_{24}$) is found to be (60.21, 80.37, 99.59, 59.25, 80.74 and 99.70) GPa respectively. Updated values of physical parameters are then used to update the FE model of beam structure and later on to produce updated natural frequencies and MAC values as shown in Tables 13 to 15. Updated results show an error reduction of 99.99%, 100.00% and 99.62% in prediction of natural frequencies, MAC values and physical parameters respectively. Thus it is established that the benchmark objective-function-formulation of third case study helps in almost completely removing the errors of FE model, thereby proving the success of the proposed objective-function-formulation.

**Table 13: Updated results of natural frequencies after refined RS based FEMU.**

| Response variable | SE results (Hz) | FE results after refined RS based FEMU (Hz) | Final error (%) |
|---|---|---|---|
| $\omega_1$ | 5.98 | 5.98 | 0.00 |
| $\omega_2$ | 37.91 | 37.91 | 0.00 |
| $\omega_3$ | 111.82 | 111.82 | 0.00 |
| $\omega_4$ | 213.02 | 213.00 | 0.01 |
| $\omega_5$ | 353.51 | 353.53 | 0.01 |
| Average error (%) | | | 0.003 |
| Error reduction (%) | | | 99.99 |

**Table 14: Updated results of MAC values after refined RS based FEMU.**

| Response variable | Desired value | FE results after refined RS based FEMU | Final error (%) |
|---|---|---|---|
| $\widehat{MAC}_{11}$ | 1.000 | 1.0000 | 0.00 |
| $\widehat{MAC}_{22}$ | 1.000 | 1.0000 | 0.00 |
| $\widehat{MAC}_{33}$ | 1.000 | 1.0000 | 0.00 |
| $\widehat{MAC}_{44}$ | 1.000 | 1.0000 | 0.00 |
| $\widehat{MAC}_{55}$ | 1.000 | 1.0000 | 0.00 |
| Average error (%) | | | 0.00 |
| Error reduction (%) | | | 100.00 |

**Table 15: Updated results of physical input parameters after refined RS based FEMU.**

| Physical parameter | SE value (GPa) | FE results after refined RS based FEMU (GPa) | Final error (%) |
|---|---|---|---|
| $E_1$ | 60 | 60.21 | 0.35 |
| $E_5$ | 80 | 80.37 | 0.46 |
| $E_9$ | 100 | 99.59 | 0.41 |
| $E_{13}$ | 60 | 59.25 | 1.25 |
| $E_{17}$ | 80 | 80.74 | 0.92 |
| $E_{24}$ | 100 | 99.70 | 0.30 |
| Average error (%) | | | 0.62 |
| Error reduction (%) | | | 99.62 |

# 6 Conclusions

This paper compares the performance of three different types of objective-function-formulations of Derringer's function based FEMU method and finds out the best formulation that gives maximum error reduction in not only output (response) variables but also input (physical) parameters of FE model. It is established that best updating results are achieved when objectives are formulated by using a combination of natural frequencies and MAC values rather than just natural frequencies or just MAC values. In order to further improve the updating results, refined RS models based FEMU is also performed, wherein best results of earlier case studies are used as starting point. Refined RS models based FEMU helps in increasing accuracy and reliability of RS models and hence updating results also. Updating results show that by using the proposed objective-function-formulation, percentage error in prediction of response variables as well as physical parameters, is almost completely removed, thereby showing the success of the proposed benchmark objective-function-formulation.


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