Significant New Types of P-Contraction with Application

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Abstract: This paper develops two new classes of contraction mappings by combining the notion of P-contraction with Ω_b-distance mappings and altering distance functions in the setting of G_b-metric spaces. First, we introduce the (Φ, Ω_b) P-contraction and prove that every such mapping on a complete G_b-metric space has a unique fixed point, for an arbitrary altering distance function and for every contraction constant $\alpha \in [0, 1)$. The fixed point theorem for P-contractions in complete metric spaces and Banach-type Ω_b-contractions are obtained as special cases, and an example shows that the new class is strictly larger than the class of Banach-type Ω_b-contractions. Second, we introduce the $\mathcal{H}(\Phi, \Omega_b)$ P-contraction, which incorporates $\mathcal{H}$-simulation functions. We prove that every $\mathcal{H}(\Phi, \Omega_b)$ P-contraction is a (Φ, Ω_b) P-contraction and that the two notions coincide for the $\mathcal{H}$-simulation function $h(t_1, t_2) = t_2/t_1$; hence the $\mathcal{H}$-formulation provides a unified way of generating concrete contractive conditions of P-type. Illustrative examples are given, together with an application to the existence and uniqueness of a solution in [0,1] of a class of polynomial equations.

Key-Words: P-contraction; G_b-metric space; Ω_b-distance mapping; altering distance function; $\mathcal{H}$-simulation function; fixed point.

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1 Introduction

Fixed point theory, originating from Banach's contraction principle [1], has evolved substantially through the generalizations of Kannan [2], Chatterjea [3], Geraghty [4], and others. Khan, Swaleh, and Sessa [5] introduced altering distance functions, Popescu [6] studied P-contractions, and Jleli and Samet [7] provided a new generalization of the Banach contraction principle see also [8]. These classical developments form the foundation of modern metric fixed point theory. Another major direction is the extension of these results to generalized metric structures. Kamran, Samreen, and Ul Ain [9] introduced extended b-metric spaces, while Aghajani, Abbas, and Roshan [10] introduced G_b-metric spaces, which combine the structures of G-metric and b-metric spaces. Related results were obtained in bipolar b-metric spaces [11] and in neutrosophic double controlled metric spaces [12]. In MR-metric spaces, Malkawi and co-authors studied the existence and uniqueness of fixed points [13], fundamental theorems with applications to integral equations and neutron transport [14], and the convergence of self-mappings [15]. Abodayeh, Bataihah, and Shatanawi [16] introduced generalized Ω-distance mappings (Ω_b-distance mappings) on G_b-metric spaces, extending the Ω-distance concept known in G-metric spaces. Distance mappings of this kind have been developed in several directions: Abu-Irwaq, Nuseir, and Bataihah [17] obtained common fixed point theorems in G-metric spaces, while Shatanawi and collaborators established Suzuki-type results [18], results for cyclic mappings [19],[20], and general remarks on G-metric spaces [21]. Other recent directions include tripled coincidence point theorems for weak Φ-contractions [22], multivalued contractions in b-gauge spaces [23], Suzuki mappings in modular spaces [24], Chatterjea-type cyclic contractions [25], new types of distance spaces [26], Gamma-k and Gamma distance mappings [27],[28], mixed-power and integral type contractions [29],[30], Geraghty type contractions [31],[32],[33], and fixed point results in fuzzy, neutrosophic and MR-type settings [34],[35],[36],[37],[38],[39]. More recent studies have extended fixed point theory to generalized distance spaces, neutrosophic and fuzzy metric settings, and new contraction frameworks [40],[41],[42],[43],[44]. Other developments include applications to parameter estimation and fractional differential equations [45, 46], as well as minimizing-sequence and displacement-functional approaches [47, 48, 49]. The works closest to the present paper are the paper of Abodayeh et al. [16], where Ω_b-distance mappings were introduced and Ω_b-Banach contractions were studied; the papers of Qawasmeh [50], where $\mathcal{H}$-simulation functions were combined with Ω_b-distance mappings in G_b-metric spaces; the paper of Shatnawi and Bataihah [30], where integral type contractions via Ω_b-distance mappings were considered; and the work of Popescu [6] on

P-contractions in metric spaces. The present paper differs from these works in the following respects.

(i) The contractive conditions (5) and (31) contain the P-type term

$$|\Omega(s_1, fs_1, fs_1) - \Omega(s_2, fs_2, fs_2)|.$$

To the best of our knowledge, this term has not previously been combined with Ω_b-distance mappings and altering distance functions in G_b-metric spaces; the contractive conditions in [16, 50, 30] do not involve it.

(ii) The main result (Theorem 3.7) holds for an arbitrary altering distance function ϕ and for every $\alpha \in [0, 1)$, without any additional restriction linking α and the parameter s of the space.

(iii) The fixed point theorem for P-contractions in complete metric spaces is recovered as a special case (Corollary 3.10), Banach-type Ω_b-contractions are covered (Corollary 3.9), and Example 3.13 shows that the new class is strictly larger than the class of Banach-type Ω_b-contractions.

(iv) The relation between the (Φ, Ω_b) and the $\mathcal{H}(\Phi, \Omega_b)$ formulations is described precisely (Proposition 4.2).

The paper is organized as follows. Section 2 collects the preliminaries. In Section 3, we introduce the (Φ, Ω_b) P-contraction, prove the main fixed point theorem, derive its consequences, and give examples. Section 4 deals with the $\mathcal{H}(\Phi, \Omega_b)$ P-contraction. Section 5 contains an application to a class of polynomial equations, and Section 6 concludes the paper.

2 Preliminaries

The notion of a G_b-metric space was introduced by Aghajani et al. [10] as a generalization of a G-metric space, and they used it to establish several fixed point theorems. The definition of a G_b-metric space reads as follows.

Definition 2.1. *[10] Let $\mathcal{D}$ be a nonempty set and let $s \in [1, +\infty)$. A function $G : \mathcal{D} \times \mathcal{D} \times \mathcal{D} \to [0, +\infty)$ is called a generalized b-metric (or G_b-metric) on $\mathcal{D}$ with parameter s if it satisfies the following conditions for all $s_1, s_2, s_3, a \in \mathcal{D}$:*

(G_b1) $G(s_1, s_2, s_3) = 0$ *if and only if* $s_1 = s_2 = s_3$;

(G_b2) $0 < G(s_1, s_1, s_2)$ *whenever* $s_1 \neq s_2$;

(G_b3) $G(s_1, s_1, s_2) \leq G(s_1, s_2, s_3)$ *whenever* $s_2 \neq s_3$;

(G_b4) $G(s_1, s_2, s_3) = G(s_1, s_3, s_2) = G(s_2, s_1, s_3) = \cdots$ *(symmetry in all three variables);*

(G_b5) $G(s_1, s_2, s_3) \leq s\,[G(s_1, a, a) + G(a, s_2, s_3)]$.

In this case, the pair $(\mathcal{D}, G)$ is called a G_b-metric space.

From now on, $(\mathcal{D}, G)$ denotes a G_b-metric space with parameter $s \geq 1$, and, for a mapping $f : \mathcal{D} \to \mathcal{D}$, Θ_f denotes the set of all fixed points of f. The letter s is reserved for the parameter of the space; points of $\mathcal{D}$ are denoted by s_1, s_2, s_3, a, t, u, and so on.

The concepts of convergence, Cauchy sequence and completeness in G_b-metric spaces are stated as follows.

Definition 2.2. *[10] Let $\{s_n\}$ be a sequence in $(\mathcal{D}, G)$.*

1. *$\{s_n\}$ is called a G_b-Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(s_n, s_m, s_l) < \epsilon$ for all $n, m, l \geq N$;*
2. *$\{s_n\}$ is said to be G_b-convergent to $t \in \mathcal{D}$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(t, s_n, s_m) < \epsilon$ for all $n, m \geq N$; we then write $s_n \to t$;*
3. *$(\mathcal{D}, G)$ is called complete if every G_b-Cauchy sequence in $\mathcal{D}$ is G_b-convergent in $\mathcal{D}$.*

A mapping $f : \mathcal{D} \to \mathcal{D}$ is called G_b-continuous if $fs_n \to ft$ whenever $s_n \to t$.

We shall use the following elementary facts, which follow directly from (G_b4) and (G_b5).

Remark 2.3. *Let $(\mathcal{D}, G)$ be a G_b-metric space with parameter $s \geq 1$.*

(a) $G(s_1, s_2, s_2) \leq 2s\,G(s_1, s_1, s_2)$ for all $s_1, s_2 \in \mathcal{D}$. Indeed, by (G_b4) and (G_b5) (with $a = s_1$),

$$\begin{aligned} G(s_1, s_2, s_2) &= G(s_2, s_2, s_1) \\ &\leq s[G(s_2, s_1, s_1) + G(s_1, s_2, s_1)] \\ &= 2sG(s_1, s_1, s_2) \end{aligned} \tag{1}$$

(b) $G(s_1, s_2, s_3) \leq s\,[G(s_1, s_2, s_2) + G(s_3, s_2, s_2)]$ for all $s_1, s_2, s_3 \in \mathcal{D}$; this is (G_b5) with $a = s_2$ combined with (G_b4).

(c) A sequence $\{s_n\}$ is G_b-convergent to t if and only if $G(s_n, s_n, t) \to 0$, if and only if $G(s_n, t, t) \to 0$, as $n \to \infty$. Indeed, $G(s_n, s_n, t) = G(t, s_n, s_n)$, $G(s_n, t, t) \leq 2sG(s_n, s_n, t)$ by (a), and $G(t, s_n, s_m) \leq s[G(s_n, t, t) + G(s_m, t, t)]$ by (b).

(d) The limit of a G_b-convergent sequence is unique: if $s_n \to t$ and $s_n \to t'$, then $G(t, t', t') \leq s[G(t, s_n, s_n) + G(s_n, t', t')] \to 0$, so $t = t'$.

Abodayeh et al. [16] introduced generalized Ω-distance mappings, or Ω_b-distance mappings, and used this notion to establish a number of fixed point results. The definition of an Ω_b-distance mapping is as follows.

Definition 2.4. *[16] An Ω_b-distance mapping on $(\mathcal{D}, G)$ (shortly, an Ω_b-distance) is a function $\Omega : \mathcal{D} \times \mathcal{D} \times \mathcal{D} \to [0, \infty)$ satisfying the following conditions:*

(Ω_b1) $\Omega(s_1, s_2, s_3) \leq s\,[\Omega(s_1, a, a) + \Omega(a, s_2, s_3)]$ *for all $s_1, s_2, s_3, a \in \mathcal{D}$, where $s \geq 1$ is the parameter of $(\mathcal{D}, G)$;*

($\Omega_b 2$) *for all* $s_1, s_2 \in \mathcal{D}$, *the functions* $\Omega(s_1, s_2, \cdot), \Omega(s_1, \cdot, s_2) : \mathcal{D} \to [0, \infty)$ *are lower semi-continuous;*

($\Omega_b 3$) *for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $\Omega(s_1, a, a) \leq \delta$ *and* $\Omega(a, s_2, s_3) \leq \delta$ *imply* $G(s_1, s_2, s_3) \leq \epsilon$, *for all* $s_1, s_2, s_3, a \in \mathcal{D}$.

Abodayeh et al. [16] introduced the notion of an Ω_b-Banach contraction as follows.

Definition 2.5. *[16] Let* Ω *be an* Ω_b*-distance on* $(\mathcal{D}, G)$. *A mapping* $f : \mathcal{D} \to \mathcal{D}$ *is called an* Ω_b*-Banach contraction if there exists* $\beta \in [0, 1)$ *with* $\beta s < 1$ *such that*

$$\Omega(fs_1, f^2 s_1, fs_2) \leq \beta\, \Omega(s_1, fs_1, s_2), \qquad (2)$$

for all $s_1, s_2 \in \mathcal{D}$.

According to Abodayeh et al. [16], every Ω_b-Banach contraction on a complete G_b-metric space has a unique fixed point. Several authors have since established further contractions by means of Ω_b-distance mappings; see, for instance, Abu-Irwaq et al. [49], Qawasmeh [50], and Shatnawi and Bataihah [30].

Definition 2.6. *[5] A function* $\phi : [0, \infty) \to [0, \infty)$ *is called an altering distance function if it satisfies the following conditions:*

1. ϕ *is continuous and non-decreasing;*
2. $\phi(t) = 0$ *if and only if* $t = 0$.

From now on, Φ denotes the set of all altering distance functions.

The notion of P-contraction is due to Popescu [6]. We recall its definition.

Definition 2.7. *[6] Let* $(\mathcal{D}, d)$ *be a metric space. A mapping* $f : \mathcal{D} \to \mathcal{D}$ *is called a* P*-contraction if there exists* $\alpha \in [0, 1)$ *such that for all* $s_1, s_2 \in \mathcal{D}$,

$$\begin{aligned} d(fs_1, fs_2) \quad &\leq \alpha\big[d(s_1, s_2) \\ &\quad + |d(s_1, fs_1) - d(s_2, fs_2)|\big]. \end{aligned} \qquad (3)$$

The notion of $\mathcal{H}$-simulation functions was introduced by Bataihah et al. [51], who applied it to obtain fixed point results. Recent studies have used this concept to establish various fixed point theorems; see, for instance, [50].

Definition 2.8. *[51] A function* $h : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$ *is called an* $\mathcal{H}$*-simulation function if* $h(t_1, t_2) \leq \frac{t_2}{t_1}$ *for all* $t_1, t_2 \in [1, +\infty)$. *The set of all* $\mathcal{H}$*-simulation functions is denoted by* $\mathcal{H}$.

Remark 2.9. *[51] Let* $h \in \mathcal{H}$. *If* (t_n) *and* (t'_n) *are sequences in* $[1, +\infty)$ *with* $1 \leq \lim_{n\to\infty} t'_n < \lim_{n\to\infty} t_n$, *then* $\limsup_{n\to\infty} h(t_n, t'_n) < 1$.

Next, we provide several examples of $\mathcal{H}$-simulation functions.

Example 2.10. *[51] The following functions belong to* $\mathcal{H}$:

1. $h_1(t_1, t_2) = \dfrac{k\, t_2^r}{t_1}$, *where* $k, r \in (0, 1]$;
2. $h_2(t_1, t_2) = \dfrac{\min\{t_1, t_2\}}{\max\{t_1, t_2\}}$;
3. $h_3(t_1, t_2) = \dfrac{t_2}{t_1 + \ln t_1}$;
4. $h_4(t_1, t_2) = \dfrac{t_2}{t_1 + \sqrt{t_2}}$;
5. $h_5(t_1, t_2) = \dfrac{t_2}{1 + t_1 t_2}$;
6. $h_6(t_1, t_2) = \dfrac{f_1(t_2)}{f_2(t_1)}$, *where* $f_1, f_2 : (0, +\infty) \to (0, +\infty)$ *are continuous functions such that* $f_1(t) < t$ *and* $f_2(t) \geq t$ *for each* $t \in (0, +\infty)$.

3 Fixed point results for (Φ, Ω_b) P-contraction

Throughout this section, Ω is an Ω_b-distance on a G_b-metric space $(\mathcal{D}, G)$ with parameter $s \geq 1$. For a mapping $f : \mathcal{D} \to \mathcal{D}$ and $s_1, s_2 \in \mathcal{D}$ we write

$$\begin{aligned} \mathcal{M}_f(s_1, s_2) \quad &= \Omega(s_1, s_2, s_2) \\ &\quad + |\Omega(s_1, fs_1, fs_1) - \Omega(s_2, fs_2, fs_2)|. \end{aligned} \qquad (4)$$

Definition 3.1. *A mapping* $f : \mathcal{D} \to \mathcal{D}$ *is called a* (Φ, Ω_b) P*-contraction if there exist* $\phi \in \Phi$ *and* $\alpha \in [0, 1)$ *such that*

$$\phi\left(s\, \Omega(fs_1, fs_2, fs_2)\right) \leq \alpha\, \phi\left(\mathcal{M}_f(s_1, s_2)\right), \qquad (5)$$

for all $s_1, s_2 \in \mathcal{D}$.

Remark 3.2. *(a) Condition* (5) *involves the values of* Ω *with repeated second and third arguments only, in accordance with the two-point nature of the* P*-contraction* (3). *If* f *satisfies the three-variable inequality:*

$$\begin{aligned} \phi\left(s\, \Omega(fs_1, fs_2, fs_3)\right) \quad &\leq \alpha\, \phi\big(\Omega(s_1, s_2, s_3) \\ &\quad + \big|\Omega(s_1, fs_1, fs_1) \\ &\quad - \Omega(s_2, fs_2, fs_2)\big|\big) \end{aligned} \qquad (6)$$

for all $s_1, s_2, s_3 \in \mathcal{D}$, *then, choosing* $s_3 = s_2$, *we see that* f *is a* (Φ, Ω_b) P*-contraction. The converse is false (see Example 3.13).*

(b) The parameter s *appears inside* ϕ *in* (5). *For* $\phi(t) = t$, *condition* (5) *reads* $s\, \Omega(fs_1, fs_2, fs_2) \leq \alpha\, \mathcal{M}_f(s_1, s_2)$. *More generally, if* ϕ *satisfies* $\phi(st) \leq s\phi(t)$ *for all* $t \geq 0$ *(for instance, if* ϕ *is concave), then the inequality* $s\, \phi\left(\Omega(fs_1, fs_2, fs_2)\right) \leq \alpha\, \phi\left(\mathcal{M}_f(s_1, s_2)\right)$ *implies* (5).

The following lemma gives the basic estimate that can be derived from (5) for a general $\phi \in \Phi$.

Lemma 3.3. *Let $f : \mathcal{D} \to \mathcal{D}$ be a (Φ, Ω_b) P-contraction. Then for all $s_1, s_2 \in \mathcal{D}$,*

$$s\,\Omega(fs_1, fs_2, fs_2) \le \mathcal{M}_f(s_1, s_2), \tag{7}$$

and the inequality is strict whenever $\mathcal{M}_f(s_1, s_2) > 0$. If, in addition, $\phi(t) = t$, then $s\,\Omega(fs_1, fs_2, fs_2) \le \alpha\,\mathcal{M}_f(s_1, s_2)$.

Proof. If $\mathcal{M}_f(s_1, s_2) = 0$, then (5) gives $\phi\left(s\,\Omega(fs_1, fs_2, fs_2)\right) \le \alpha\phi(0) = 0$, hence $\Omega(fs_1, fs_2, fs_2) = 0$. If $\mathcal{M}_f(s_1, s_2) > 0$, then $\phi(\mathcal{M}_f(s_1, s_2)) > 0$ and, since $\alpha < 1$,

$$\begin{aligned}\phi\left(s\,\Omega(fs_1, fs_2, fs_2)\right) &\le \alpha\,\phi(\mathcal{M}_f(s_1, s_2)) \\ &< \phi(\mathcal{M}_f(s_1, s_2)).\end{aligned} \tag{8}$$

If we had $s\,\Omega(fs_1, fs_2, fs_2) \ge \mathcal{M}_f(s_1, s_2)$, then the monotonicity of ϕ would give the reverse inequality; hence $s\,\Omega(fs_1, fs_2, fs_2) < \mathcal{M}_f(s_1, s_2)$. The last assertion is (5) with $\phi(t) = t$. □

Remark 3.4. *For a general $\phi \in \Phi$, an estimate of the form $\Omega(fs_1, fs_2, fs_2) \le \frac{\alpha}{s}\mathcal{M}_f(s_1, s_2)$ cannot be expected. For example, if $\phi(t) = t^2$, then* (5) *only gives $s\,\Omega(fs_1, fs_2, fs_2) \le \sqrt{\alpha}\,\mathcal{M}_f(s_1, s_2)$. For this reason, the proofs below rely only on* (5) *and* (7)*, and not on a linear estimate.*

Lemma 3.5. *If $f : \mathcal{D} \to \mathcal{D}$ is a (Φ, Ω_b) P-contraction, then $\Omega(t, t, t) = 0$ for every $t \in \Theta_f$, and Θ_f contains at most one element.*

Proof. Let $t \in \Theta_f$. Applying (5) with $s_1 = s_2 = t$, and using $ft = t$, we get $\mathcal{M}_f(t, t) = \Omega(t, t, t)$ and

$$\phi\left(\Omega(t, t, t)\right) \le \phi\left(s\,\Omega(t, t, t)\right) \le \alpha\,\phi\left(\Omega(t, t, t)\right),$$

where the first inequality holds because $s \ge 1$ and ϕ is non-decreasing. Since $\alpha < 1$, we obtain $\phi(\Omega(t, t, t)) = 0$, and hence

$$\Omega(t, t, t) = 0 \quad \text{for all } t \in \Theta_f. \tag{9}$$

Now, let $t_1, t_2 \in \Theta_f$. By (9),

$$\begin{aligned}\mathcal{M}_f(t_1, t_2) &= \Omega(t_1, t_2, t_2) + \big|\Omega(t_1, t_1, t_1) \\ &\quad -\Omega(t_2, t_2, t_2)\big| \\ &= \Omega(t_1, t_2, t_2)\end{aligned} \tag{10}$$

so (5) gives

$$\begin{aligned}\phi\left(\Omega(t_1, t_2, t_2)\right) &\le \phi\left(s\,\Omega(t_1, t_2, t_2)\right) \\ &\le \alpha\,\phi\left(\Omega(t_1, t_2, t_2)\right).\end{aligned} \tag{11}$$

Hence $\Omega(t_1, t_2, t_2) = 0$. Let $\epsilon > 0$ and let $\delta > 0$ be as in $(\Omega_b 3)$. Since $\Omega(t_1, t_1, t_1) = 0 \le \delta$ and $\Omega(t_1, t_2, t_2) = 0 \le \delta$, condition $(\Omega_b 3)$ (with $s_1 = a = t_1$ and $s_2 = s_3 = t_2$) gives $G(t_1, t_2, t_2) \le \epsilon$. As $\epsilon > 0$ is arbitrary, $G(t_1, t_2, t_2) = 0$, and therefore $t_1 = t_2$ by $(G_b 1)$. □

The next lemma provides a criterion for the G_b-Cauchy property in terms of Ω; it will be used in the proof of the main theorem.

Lemma 3.6. *Let $\{s_n\}$ be a sequence in $\mathcal{D}$ such that*

(a) $\lim_{n\to\infty} \Omega(s_n, s_n, s_n) = 0$, and

(b) for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\Omega(s_n, s_m, s_m) < \epsilon$ for all $m > n \ge N$.

Then $\{s_n\}$ is a G_b-Cauchy sequence.

Proof. Let $\epsilon > 0$ and let $\delta > 0$ be as in $(\Omega_b 3)$. By (a) and (b), there exists $N \in \mathbb{N}$ such that $\Omega(s_n, s_n, s_n) < \delta$ for all $n \ge N$ and $\Omega(s_n, s_m, s_m) < \delta$ for all $m > n \ge N$. Let $m > n \ge N$. Applying $(\Omega_b 3)$ with $s_1 = s_n$, $a = s_m$ and $s_2 = s_3 = s_m$, we obtain

$$G(s_n, s_m, s_m) \le \epsilon \qquad (m > n \ge N).$$

If $n > m \ge N$, then, by Remark 2.3(a), $(G_b 4)$ and the previous inequality, $G(s_n, s_m, s_m) \le 2s\,G(s_n, s_n, s_m) = 2s\,G(s_m, s_n, s_n) \le 2s\epsilon$. Thus $G(s_n, s_m, s_m) \le 2s\epsilon$ for all $n, m \ge N$. Finally, for $n, m, l \ge N$, Remark 2.3(b) gives

$$\begin{aligned}G(s_n, s_m, s_l) &\le s\left[G(s_n, s_m, s_m)\right. \\ &\quad \left.+G(s_l, s_m, s_m)\right] \\ &\le 4s^2\epsilon.\end{aligned} \tag{12}$$

Since $\epsilon > 0$ is arbitrary, $\{s_n\}$ is a G_b-Cauchy sequence. □

Theorem 3.7. *Let Ω be an Ω_b-distance on a complete G_b-metric space $(\mathcal{D}, G)$ with parameter $s \ge 1$, and let $f : \mathcal{D} \to \mathcal{D}$ be a (Φ, Ω_b) P-contraction. Assume that one of the following conditions is satisfied:*

(C1) f is G_b-continuous;

(C2) for every $t \in \mathcal{D}$ with $ft \ne t$,

$$\inf\{G(u, t, t) + \Omega(u, fu, fu) : u \in \mathcal{D}\} > 0.$$

Then Θ_f consists of exactly one element t. Moreover, $\Omega(t, t, t) = 0$, and for every $s_0 \in \mathcal{D}$ the Picard sequence $\{f^n s_0\}$ is G_b-convergent to t.

Proof. Let $s_0 \in \mathcal{D}$ be arbitrary, and let $\{s_n\}$ be the Picard sequence defined by $s_{n+1} = fs_n$ for all $n \ge 0$. Put

$$\begin{aligned}\vartheta_n &= \Omega(s_n, s_{n+1}, s_{n+1}), \\ \gamma_n &= \Omega(s_n, s_n, s_n), \qquad n \ge 0.\end{aligned}$$

For $n \ge 1$ we have $\mathcal{M}_f(s_{n-1}, s_n) = \vartheta_{n-1} + |\vartheta_{n-1} - \vartheta_n|$, so (5) with $s_1 = s_{n-1}$ and $s_2 = s_n$ gives

$$\phi(s\,\vartheta_n) \le \alpha\,\phi\left(\vartheta_{n-1} + |\vartheta_{n-1} - \vartheta_n|\right),\ n \ge 1. \tag{13}$$

Step 1: the case $\vartheta_{n_0} \ge \vartheta_{n_0-1}$ for some $n_0 \ge 1$. In this case, the argument of ϕ on the right-hand side of (13) (with $n = n_0$) equals ϑ_{n_0}, and so

$$\phi(s\,\vartheta_{n_0}) \le \alpha\,\phi(\vartheta_{n_0}) \le \alpha\,\phi(s\,\vartheta_{n_0}).$$

Since $\alpha < 1$, we get $\phi(s\,\vartheta_{n_0}) = 0$, hence $\vartheta_{n_0} = 0$, and therefore $\vartheta_{n_0-1} = 0$ as well. That is,

$$\Omega(s_{n_0-1}, s_{n_0}, s_{n_0}) = 0 \text{ and } \Omega(s_{n_0}, s_{n_0+1}, s_{n_0+1}) = 0. \quad (14)$$

Let $\epsilon > 0$ and let $\delta > 0$ be as in $(\Omega_b 3)$. By (14) and $(\Omega_b 3)$ (with $s_1 = s_{n_0-1}$, $a = s_{n_0}$, $s_2 = s_3 = s_{n_0+1}$), we get $G(s_{n_0-1}, s_{n_0+1}, s_{n_0+1}) \leq \epsilon$. Since ϵ is arbitrary, $G(s_{n_0-1}, s_{n_0+1}, s_{n_0+1}) = 0$, and so $s_{n_0-1} = s_{n_0+1}$. Replacing s_{n_0-1} by s_{n_0+1} in the first equality of (14), we obtain

$$\Omega(s_{n_0+1}, s_{n_0}, s_{n_0}) = 0. \quad (15)$$

By $(\Omega_b 1)$, (15) and (14),

$$\begin{aligned}\Omega(s_{n_0+1}, s_{n_0+1}, s_{n_0+1}) &\leq s\,[\Omega(s_{n_0+1}, s_{n_0}, s_{n_0}) \\ &\quad +\Omega(s_{n_0}, s_{n_0+1}, s_{n_0+1})] \\ &= 0.\end{aligned} \quad (16)$$

Applying $(\Omega_b 3)$ once more, now with $s_1 = s_{n_0}$ and $a = s_2 = s_3 = s_{n_0+1}$, and using (14) and (16), we obtain $G(s_{n_0}, s_{n_0+1}, s_{n_0+1}) = 0$. Hence $fs_{n_0} = s_{n_0+1} = s_{n_0}$, that is, $s_{n_0} \in \Theta_f$. Moreover, $s_n = s_{n_0}$ for all $n \geq n_0$, so the Picard sequence converges to s_{n_0}. By Lemma 3.5, $\Theta_f = \{s_{n_0}\}$ and $\Omega(s_{n_0}, s_{n_0}, s_{n_0}) = 0$, and the proof is complete in this case.

Step 2: $\lim_{n\to\infty} \vartheta_n = 0$**.** By Step 1, we may assume from now on that $\vartheta_n < \vartheta_{n-1}$ for all $n \geq 1$. Then $\{\vartheta_n\}$ is a strictly decreasing sequence of non-negative real numbers, so it converges to some $r \geq 0$. In this case, (13) reads

$$\phi(s\,\vartheta_n) \leq \alpha\,\phi\,(2\vartheta_{n-1} - \vartheta_n)\,, \qquad n \geq 1.$$

Letting $n \to \infty$ and using the continuity of ϕ, we get $\phi(sr) \leq \alpha\,\phi(r) \leq \alpha\,\phi(sr)$. Hence $\phi(sr) = 0$, so $r = 0$, that is,

$$\lim_{n\to\infty} \Omega(s_n, s_{n+1}, s_{n+1}) = 0. \quad (17)$$

Step 3: $\lim_{n\to\infty} \gamma_n = 0$**.** Applying (5) with $s_1 = s_2 = s_{n-1}$, we have $\mathcal{M}_f(s_{n-1}, s_{n-1}) = \gamma_{n-1}$ and

$$\phi(s\,\gamma_n) \leq \alpha\,\phi(\gamma_{n-1}), \qquad n \geq 1. \quad (18)$$

By Lemma 3.3, $\gamma_n \leq s\gamma_n \leq \gamma_{n-1}$, so $\{\gamma_n\}$ is non-increasing and converges to some $\rho \geq 0$. Letting $n \to \infty$ in (18), we get $\phi(s\rho) \leq \alpha\,\phi(\rho) \leq \alpha\,\phi(s\rho)$, hence $\rho = 0$.

Step 4: for every $\epsilon > 0$ **there exists** $N \in \mathbb{N}$ **such that** $\Omega(s_n, s_m, s_m) < \epsilon$ **for all** $m > n \geq N$**.** Suppose, to the contrary, that there exist $\epsilon > 0$ and, for every $k \in \mathbb{N}$, integers $m > n_k \geq k$ with $\Omega(s_{n_k}, s_m, s_m) \geq \epsilon$. By (17), there is $k_0 \in \mathbb{N}$ such that $\vartheta_n < \epsilon/s$ for all $n \geq k_0$; we consider only $k \geq k_0$. For each such k, let m_k be the smallest integer $m > n_k$ with $\Omega(s_{n_k}, s_m, s_m) \geq \epsilon$. Since $\Omega(s_{n_k}, s_{n_k+1}, s_{n_k+1}) = \vartheta_{n_k} < \epsilon$, we have $m_k \geq n_k + 2$, and by the minimality of m_k,

$$\Omega(s_{n_k}, s_{m_k}, s_{m_k}) \geq \epsilon \text{ and } \Omega(s_{n_k}, s_{m_k-1}, s_{m_k-1}) < \epsilon. \quad (19)$$

By $(\Omega_b 1)$ (with $a = s_{n_k+1}$) and (19),

$$\epsilon \leq \Omega(s_{n_k}, s_{m_k}, s_{m_k}) \leq s\,[\vartheta_{n_k} + \Omega(s_{n_k+1}, s_{m_k}, s_{m_k})],$$

so that

$$s\,\Omega(s_{n_k+1}, s_{m_k}, s_{m_k}) \geq \epsilon - s\vartheta_{n_k} > 0. \quad (20)$$

On the other hand, applying (5) with $s_1 = s_{n_k}$ and $s_2 = s_{m_k-1}$, and using (19), we get

$$\begin{aligned}\phi\,(s\,\Omega(s_{n_k+1}, s_{m_k}, s_{m_k})) &\leq \alpha\,\phi\big(\Omega(s_{n_k}, s_{m_k-1} \\ &\quad , s_{m_k-1}) \\ &\quad +|\vartheta_{n_k} - \vartheta_{m_k-1}|\big) \\ &\leq \alpha\,\phi\big(\epsilon+ \\ &\quad \vartheta_{n_k} + \vartheta_{m_k-1}\big).\end{aligned} \quad (21)$$

Combining this with (20) and the monotonicity of ϕ, we obtain

$$\phi\,(\epsilon - s\vartheta_{n_k}) \leq \alpha\,\phi\,(\epsilon + \vartheta_{n_k} + \vartheta_{m_k-1})\,, \quad (22)$$

where $k \geq k_0$. Since $n_k \to \infty$ and $m_k \to \infty$, (17) and the continuity of ϕ give $\phi(\epsilon) \leq \alpha\,\phi(\epsilon)$ as $k \to \infty$. Hence $\phi(\epsilon) = 0$, i.e., $\epsilon = 0$, a contradiction. This proves Step 4.

Step 5: $\{s_n\}$ **converges.** By Steps 3 and 4, the sequence $\{s_n\}$ satisfies conditions (a) and (b) of Lemma 3.6; hence it is a G_b-Cauchy sequence. Since $(\mathcal{D}, G)$ is complete, there exists $t \in \mathcal{D}$ such that $s_n \to t$.

Step 6: $t \in \Theta_f$**.** Assume first that (C1) holds. Then $s_{n+1} = fs_n \to ft$. Since also $s_{n+1} \to t$, Remark 2.3(d) gives $ft = t$.

Assume now that (C2) holds and, to the contrary, that $ft \neq t$. By Remark 2.3(c), $G(s_n, t, t) \to 0$, and by (17), $\Omega(s_n, fs_n, fs_n) = \vartheta_n \to 0$. Therefore

$$\begin{aligned}0 &< \inf\{G(u,t,t) + \Omega(u, fu, fu) : u \in \mathcal{D}\} \\ &\leq G(s_n, t, t) + \vartheta_n \longrightarrow 0 \quad (n \to \infty).\end{aligned} \quad (23)$$

a contradiction. Consequently, $ft = t$. **Step 7: uniqueness.** By Lemma 3.5, $\Theta_f = \{t\}$ and $\Omega(t, t, t) = 0$. Finally, since s_0 was arbitrary, every Picard sequence converges to t. □

Now we derive several consequences of Theorem 3.7. Taking $\phi(t) = t$ for all $t \in [0, \infty)$, we obtain the following corollary.

Corollary 3.8. *Let Ω be an Ω_b-distance on a complete G_b-metric space $(\mathcal{D}, G)$ with parameter $s \geq 1$, and*

let $f : \mathcal{D} \to \mathcal{D}$ be a mapping for which there exists $\alpha \in [0,1)$ such that

$$\begin{aligned} s\,\Omega(fs_1, fs_2, fs_2) \quad &\le \alpha\big[\Omega(s_1, s_2, s_2) \\ &\quad + \big|\Omega(s_1, fs_1, fs_1) \\ &\quad - \Omega(s_2, fs_2, fs_2)\big|\big]. \end{aligned} \tag{24}$$

for all $s_1, s_2 \in \mathcal{D}$. If (C1) or (C2) of Theorem 3.7 is satisfied, then Θ_f consists of exactly one element.

Since the term $|\Omega(s_1, fs_1, fs_1) - \Omega(s_2, fs_2, fs_2)|$ is non-negative and ϕ is non-decreasing, every mapping satisfying the following Banach-type condition is a (Φ, Ω_b) P-contraction. Hence we obtain the next corollary.

Corollary 3.9. *Let Ω be an Ω_b-distance on a complete G_b-metric space $(\mathcal{D}, G)$ with parameter $s \ge 1$, and let $f : \mathcal{D} \to \mathcal{D}$ be a mapping for which there exist $\phi \in \Phi$ and $\alpha \in [0,1)$ such that*

$$\phi\left(s\,\Omega(fs_1, fs_2, fs_2)\right) \le \alpha\,\phi\left(\Omega(s_1, s_2, s_2)\right), \tag{25}$$

for all $s_1, s_2 \in \mathcal{D}$. If (C1) or (C2) of Theorem 3.7 is satisfied, then Θ_f consists of exactly one element. In particular, this holds if $s\,\Omega(fs_1, fs_2, fs_2) \le \alpha\,\Omega(s_1, s_2, s_2)$ for all $s_1, s_2 \in \mathcal{D}$.

The next corollary shows that Theorem 3.7 contains the fixed point theorem for P-contractions in complete metric spaces.

Corollary 3.10. *Let $(\mathcal{D}, d)$ be a complete metric space and let $f : \mathcal{D} \to \mathcal{D}$ be a P-contraction, i.e., f satisfies* (6) *for some $\alpha \in [0,1)$. Then f has a unique fixed point.*

Proof. Define $G_d(s_1, s_2, s_3) = d(s_1, s_2) + d(s_2, s_3) + d(s_1, s_3)$. It is easy to check that $(\mathcal{D}, G_d)$ is a complete G_b-metric space with parameter $s = 1$ (that is, a complete G-metric space). Indeed, (G_b1)–(G_b4) are easily verified by using the triangle inequality of d, and the triangle inequality also gives

$$\begin{aligned} G_d(s_1, s_2, s_3) \quad &\le 2d(s_1, a) + d(a, s_2) + \\ &\quad d(s_2, s_3) + d(a, s_3) \\ &= G_d(s_1, a, a) + G_d(a, s_2, s_3). \end{aligned} \tag{26}$$

which is (G_b5) with $s = 1$; moreover, $G_d(s_n, s_m, s_l) \ge d(s_n, s_m)$ and $G_d(s_n, t, t) = 2d(s_n, t)$, so completeness of $(\mathcal{D}, d)$ implies completeness of $(\mathcal{D}, G_d)$. Let $\Omega = G_d$. Then (Ω_b1) is the inequality just proved, (Ω_b2) holds since G_d is continuous, and (Ω_b3) holds with $\delta = \epsilon/2$, because $G_d(s_1, s_2, s_3) \le G_d(s_1, a, a) + G_d(a, s_2, s_3)$. Since $\Omega(s_1, s_2, s_2) = 2d(s_1, s_2)$, multiplying (6) by 2 shows that f satisfies (24) with $s = 1$; that is, f is a (Φ, Ω_b) P-contraction with $\phi(t) = t$.

It remains to verify (C2). Let $t \in \mathcal{D}$ with $ft \ne t$ and suppose that $\inf\{G_d(u, t, t) + \Omega(u, fu, fu) : u \in \mathcal{D}\} = \inf\{2d(u, t) + 2d(u, fu) : u \in \mathcal{D}\} = 0$. Then there is a sequence $\{u_k\}$ with $d(u_k, t) \to 0$ and $d(u_k, fu_k) \to 0$, so that $d(fu_k, t) \le d(fu_k, u_k) + d(u_k, t) \to 0$. By (6),

$$d(fu_k, ft) \le \alpha\left[d(u_k, t) + |d(u_k, fu_k) - d(t, ft)|\right],$$

and

$$\alpha\left[d(u_k, t) + |d(u_k, fu_k) - d(t, ft)|\right] \longrightarrow \alpha\, d(t, ft),$$

while $d(fu_k, ft) \to d(t, ft)$. Hence $d(t, ft) \le \alpha\, d(t, ft)$, so $d(t, ft) = 0$, a contradiction. Therefore (C2) holds, and Theorem 3.7 completes the proof. □

Remark 3.11. *We summarize the relation of Theorem 3.7 to known results.*

(a) For $s = 1$, $\Omega = G_d$ and $\phi(t) = t$, Theorem 3.7 reduces to the fixed point theorem for P-contractions in complete metric spaces (Corollary 3.10).

(b) Banach-type Ω_b-contractions (25) *are (Φ, Ω_b) P-contractions (Corollary 3.9). Example 3.13 below shows that the converse is false, even when ϕ is allowed to vary over Φ; thus the P-type term in* (5) *genuinely enlarges the class of mappings covered.*

(c) No restriction on α other than $\alpha \in [0,1)$ is needed; the parameter s of the space is absorbed by placing it inside ϕ in (5)*, and the Cauchy property is established in Step 4 without any geometric series argument.*

(d) The function ϕ allows nonlinear contractive conditions. For instance, for $\phi(t) = e^t - 1$, condition (5) *reads $s\,\Omega(fs_1, fs_2, fs_2) \le \ln\left(1 + \alpha\left(e^{\mathcal{M}_f(s_1, s_2)} - 1\right)\right)$; the ratio of the right-hand side to $\mathcal{M}_f(s_1, s_2)$ tends to 1 as $\mathcal{M}_f(s_1, s_2) \to \infty$, so this condition does not force a linear estimate $s\,\Omega(fs_1, fs_2, fs_2) \le k\,\mathcal{M}_f(s_1, s_2)$ with a fixed $k < 1$.*

Setting for the examples. In Examples 3.12–3.14, $\mathcal{D}$ is a closed subset of $\mathbb{R}$, and

$$G(s_1, s_2, s_3) = \left(|s_1 - s_2| + |s_2 - s_3| + |s_1 - s_3|\right)^2,$$

$$\Omega(s_1, s_2, s_3) = \left(|s_1 - s_2| + |s_1 - s_3|\right)^2.$$

Since G is the square of the G-metric $G_0(s_1, s_2, s_3) = |s_1 - s_2| + |s_2 - s_3| + |s_1 - s_3|$ and $(u + v)^2 \le 2(u^2 + v^2)$, G is a G_b-metric with $s = 2$. Moreover, $|s_1 - s_2| + |s_1 - s_3| \le 2|s_1 - a| + (|a - s_2| + |a - s_3|)$, which gives (Ω_b1) with $s = 2$; (Ω_b2) holds because Ω is continuous; and if $\Omega(s_1, a, a) = 4|s_1 - a|^2 \le \delta$ and $\Omega(a, s_2, s_3) \le \delta$, then $G(s_1, s_2, s_3) \le (3\sqrt{\delta})^2 = 9\delta$, so (Ω_b3) holds with $\delta = \epsilon/9$. Hence Ω is an Ω_b-distance on $(\mathcal{D}, G)$. Since G-convergent and G_b-Cauchy sequences coincide with convergent and Cauchy sequences in the usual sense, $(\mathcal{D}, G)$ is

complete. Finally, $\Omega(s_1,s_2,s_2)=4(s_1-s_2)^2$, so with $\phi(t)=t$ and $s=2$, condition (5) is equivalent to

$$2(fs_1-fs_2)^2 \quad \le \alpha\big[(s_1-s_2)^2+ \big|(s_1-fs_1)^2-(s_2-fs_2)^2\big|\big]. \tag{27}$$

Example 3.12. *Let $\mathcal{D}=\{1,2,3,5,6,7,9,10\}$, and define $f:\mathcal{D}\to\mathcal{D}$ by*

$$fs=\begin{cases}1, & s\in\{1,2,3\},\\ 2, & s\in\{5,6,7\},\\ 3, & s\in\{9,10\}.\end{cases}$$

We show that (27) *holds with $\alpha=\frac{1}{2}$, that is,*

$$(fs_1-fs_2)^2 \quad \le \tfrac{1}{4}\big[(s_1-s_2)^2 +\big|(s_1-fs_1)^2-(s_2-fs_2)^2\big|\big], \tag{28}$$

for all $s_1,s_2\in\mathcal{D}$. Both sides of (28) *are symmetric in s_1 and s_2, so it suffices to consider the following cases.*

Case 1: *s_1 and s_2 belong to the same set among $\{1,2,3\}$, $\{5,6,7\}$, $\{9,10\}$. Then $fs_1=fs_2$, and the left-hand side of* (28) *is* 0.

Case 2: *one of s_1,s_2 belongs to $\{1,2,3\}$ and the other to $\{5,6,7\}$, or one belongs to $\{5,6,7\}$ and the other to $\{9,10\}$. Then $(fs_1-fs_2)^2=1$, while $|s_1-s_2|\ge 2$. Hence the right-hand side of* (28) *is at least $\frac{1}{4}\cdot 4=1$.*

Case 3: *one of s_1,s_2 belongs to $\{1,2,3\}$ and the other to $\{9,10\}$. Then $(fs_1-fs_2)^2=4$, while $|s_1-s_2|\ge 6$. Hence the right-hand side of* (28) *is at least $\frac{1}{4}\cdot 36=9>4$.*

Thus f is a (Φ,Ω_b) P-contraction with $\phi(t)=t$ and $\alpha=\frac{1}{2}$. Since $\mathcal{D}$ is finite, every G_b-convergent sequence in $\mathcal{D}$ is eventually constant, so f is G_b-continuous and (C1) holds. Hence f satisfies all the hypotheses of Theorem 3.7, and Theorem 3.7 shows that Θ_f consists of exactly one element; indeed, $\Theta_f=\{1\}$.

The following example shows that the P-type term in (5) is essential: the mapping below is a (Φ,Ω_b) P-contraction, but it is not a Banach-type Ω_b-contraction (25) for any $\phi\in\Phi$ and $\alpha\in[0,1)$.

Example 3.13. *Let $\mathcal{D}=\{0,1,2\}$ and define $f:\mathcal{D}\to\mathcal{D}$ by $f0=1$, $f1=1$ and $f2=0$. Then $(s-fs)^2$ equals* 1, 0 *and* 4 *for $s=0,1,2$, respectively. We check* (27) *with $\alpha=\frac{1}{2}$, i.e.,* (28)*. Both sides of* (28) *are symmetric in s_1,s_2, and the left-hand side vanishes if $fs_1=fs_2$. The remaining pairs are:*

- $\{s_1,s_2\}=\{0,2\}$: $(fs_1-fs_2)^2=1\le\frac{1}{4}[4+|1-4|]=\frac{7}{4}$;
- $\{s_1,s_2\}=\{1,2\}$: $(fs_1-fs_2)^2=1\le\frac{1}{4}[1+|0-4|]=\frac{5}{4}$.

Hence f is a (Φ,Ω_b) P-contraction with $\phi(t)=t$ and $\alpha=\frac{1}{2}$, and (C1) holds since $\mathcal{D}$ is finite. By Theorem 3.7, Θ_f is a singleton; indeed, $\Theta_f=\{1\}$.

On the other hand, for $s_1=1$ and $s_2=2$ we have $s\,\Omega(f1,f2,f2)=2\,\Omega(1,0,0)=8$ and $\Omega(1,2,2)=4$. Since ϕ is non-decreasing and $\phi(4)>0$, we get $\phi(8)\ge\phi(4)>\alpha\,\phi(4)$ for every $\phi\in\Phi$ and $\alpha\in[0,1)$, so (25) *fails. Similarly, for $s_1=s_2=1$ and $s_3=2$, we have $s\,\Omega(f1,f1,f2)=2$, while $\Omega(1,1,2)+|\Omega(1,f1,f1)-\Omega(1,f1,f1)|=1$; hence the three-variable inequality of Remark 3.2(a) fails for every $\phi\in\Phi$ and $\alpha\in[0,1)$. Note also that $|f1-f2|=|1-2|$, so f is not a Banach contraction with respect to the usual metric.*

Example 3.14. *Let $\mathcal{D}=[0,1]$, let $m\in\mathbb{N}$ and $L>2\sqrt{2}\,m$, and define $f:[0,1]\to[0,1]$ by*

$$fs=\frac{L-s^m}{L+s^m}. \tag{29}$$

Since $L>1$ and $0\le s^m\le 1$ for $s\in[0,1]$, we have $fs\in\left[\frac{L-1}{L+1},1\right]\subset[0,1]$, so f is well defined, and f is continuous. For $s_1,s_2\in[0,1]$,

$$fs_1-fs_2=\frac{2L\,(s_2^m-s_1^m)}{(L+s_1^m)(L+s_2^m)},$$

hence,

$$|fs_1-fs_2|\le\frac{2}{L}\,|s_1^m-s_2^m|\le\frac{2m}{L}\,|s_1-s_2|,$$

where the last inequality follows from the mean value theorem, since $|\frac{d}{ds}s^m|=ms^{m-1}\le m$ on $[0,1]$ for $m\in\mathbb{N}$. Therefore, with $\alpha=\frac{8m^2}{L^2}\in(0,1)$,

$$\begin{aligned} s\,\phi\left(\Omega(fs_1,fs_2,fs_2)\right) &= 2\,\Omega(fs_1,fs_2,fs_2)\\ &= 8(fs_1-fs_2)^2\\ &\le \frac{32m^2}{L^2}(s_1-s_2)^2\\ &= \frac{8m^2}{L^2}\,\Omega(s_1,s_2,s_2)\\ &\le \alpha\big[\Omega(s_1,s_2,s_2)+\\ &\quad \big|\Omega(s_1,fs_1,fs_1)-\\ &\quad \Omega(s_2,fs_2,fs_2)\big|\big]. \end{aligned} \tag{30}$$

Thus f is a (Φ,Ω_b) P-contraction with $\phi(t)=t$ (for which $s\phi(\cdot)=\phi(s\,\cdot)$), and (C1) holds. Consequently, f satisfies all the hypotheses of Theorem 3.7, and Theorem 3.7 shows that Θ_f is a singleton.

Remark 3.15. *The restriction $m\in\mathbb{N}$ (that is, $m\ge 1$) in Example 3.14 cannot simply be dropped: for $0<m<1$, the map $s\mapsto s^m$ is not Lipschitz near* 0*. For instance, for $m=\frac{1}{2}$ and $L=2$ (so that $8m^2<L^2$ and $\alpha=\frac{1}{2}$), taking $s_1=1$ and $s_2=0$ gives $fs_1=\frac{1}{3}$, $fs_2=1$, and the left-hand side of* (27) *equals $\frac{8}{9}$, while its right-hand side equals $\frac{1}{2}\left[1+\left|\frac{4}{9}-1\right|\right]=\frac{7}{9}$.*

4 Fixed point results for $\mathcal{H}(\Phi,\Omega_b)$ P-contraction

As in Section 3, Ω is an Ω_b-distance on a G_b-metric space $(\mathcal{D},G)$ with parameter $s\geq 1$, and $\mathcal{M}_f$ is given by (4).

Definition 4.1. *Let $h\in\mathcal{H}$. A mapping $f:\mathcal{D}\to\mathcal{D}$ is called an $\mathcal{H}(\Phi,\Omega_b)$ P-contraction with respect to h if there exist $\phi\in\Phi$ and $\alpha\in[0,1)$ such that*

$$1 \leq h\Big(1+\phi\big(s\,\Omega(fs_1,fs_2,fs_2)\big),\ 1+\alpha\,\phi\left(\mathcal{M}_f(s_1,s_2)\right)\Big) \tag{31}$$

for all $s_1,s_2\in\mathcal{D}$.

The constant 1 is added to both arguments of h in (31) because h is defined on $[1,+\infty)\times[1,+\infty)$, whereas the values of ϕ may lie in $[0,1)$.

Proposition 4.2. *Let $f:\mathcal{D}\to\mathcal{D}$.*

(a) If f is an $\mathcal{H}(\Phi,\Omega_b)$ P-contraction with respect to some $h\in\mathcal{H}$, with $\phi\in\Phi$ and $\alpha\in[0,1)$, then f is a (Φ,Ω_b) P-contraction with the same ϕ and α.

(b) f is a (Φ,Ω_b) P-contraction if and only if it is an $\mathcal{H}(\Phi,\Omega_b)$ P-contraction with respect to $h_*(t_1,t_2)=t_2/t_1$.

Proof. (a) Fix $s_1,s_2\in\mathcal{D}$ and put $u=1+\phi\left(s\,\Omega(fs_1,fs_2,fs_2)\right)\geq 1$ and $v=1+\alpha\,\phi\left(\mathcal{M}_f(s_1,s_2)\right)\geq 1$. By (31) and the definition of $\mathcal{H}$-simulation functions,

$$1\leq h(u,v)\leq \frac{v}{u}.$$

Hence $u\leq v$, that is, $\phi\left(s\,\Omega(fs_1,fs_2,fs_2)\right)\leq \alpha\,\phi\left(\mathcal{M}_f(s_1,s_2)\right)$. Since $s_1,s_2\in\mathcal{D}$ are arbitrary, f satisfies (5).

(b) Clearly $h_*\in\mathcal{H}$. With u,v as in (a), we have $h_*(u,v)\geq 1$ if and only if $u\leq v$, which is exactly (5) for the pair s_1,s_2. □

Theorem 4.3. *Let Ω be an Ω_b-distance on a complete G_b-metric space $(\mathcal{D},G)$ with parameter $s\geq 1$, let $h\in\mathcal{H}$, and let $f:\mathcal{D}\to\mathcal{D}$ be an $\mathcal{H}(\Phi,\Omega_b)$ P-contraction with respect to h. Assume that one of the following conditions is satisfied:*

(C1) f is G_b-continuous;

(C2) for every $t\in\mathcal{D}$ with $ft\neq t$, $\inf\{G(u,t,t)+\Omega(u,fu,fu):u\in\mathcal{D}\}>0$.

Then Θ_f consists of exactly one element t. Moreover, $\Omega(t,t,t)=0$, and for every $s_0\in\mathcal{D}$ the Picard sequence $\{f^n s_0\}$ is G_b-convergent to t.

Proof. Let $\phi\in\Phi$ and $\alpha\in[0,1)$ be such that (31) holds. By Proposition 4.2(a), for all $s_1,s_2\in\mathcal{D}$,

$$\phi\left(s\,\Omega(fs_1,fs_2,fs_2)\right)\leq\alpha\,\phi\left(\mathcal{M}_f(s_1,s_2)\right);$$

that is, f is a (Φ,Ω_b) P-contraction with the same ϕ and α. In particular, all the estimates used in the proof of Theorem 3.7, namely (13), (18), and the inequality obtained in Step 4 by applying (5) to the pair (s_{n_k},s_{m_k-1}), hold for f. Conditions (C1) and (C2) concern only f, G and Ω, and they are exactly the conditions of Theorem 3.7. Hence all the hypotheses of Theorem 3.7 are satisfied, and Theorem 3.7 yields the conclusion. □

Remark 4.4. *By Proposition 4.2, a mapping is an $\mathcal{H}(\Phi,\Omega_b)$ P-contraction with respect to some $h\in\mathcal{H}$ if and only if it is a (Φ,Ω_b) P-contraction. Therefore, the $\mathcal{H}$-formulation does not enlarge the class of mappings covered by Theorem 3.7; its role is to provide a unified way of writing a family of concrete sufficient conditions, one for each $h\in\mathcal{H}$. The case $h=h_*$ gives back Theorem 3.7, and other choices of h give conditions which may be easier to verify in particular situations. For instance, the function $h(t_1,t_2)=t_2^r/t_1$ with $r\in(0,1]$ (that is, h_1 with $k=1$) yields the following corollary.*

Corollary 4.5. *Let Ω be an Ω_b-distance on a complete G_b-metric space $(\mathcal{D},G)$ with parameter $s\geq 1$, and let $f:\mathcal{D}\to\mathcal{D}$ be a mapping for which there exist $\phi\in\Phi$, $\alpha\in[0,1)$ and $r\in(0,1]$ such that*

$$1+\phi\left(s\,\Omega(fs_1,fs_2,fs_2)\right)\leq\left(1+\alpha\,\phi\left(\mathcal{M}_f(s_1,s_2)\right)\right)^r,$$

for all $s_1,s_2\in\mathcal{D}$. If (C1) or (C2) of Theorem 4.3 is satisfied, then Θ_f consists of exactly one element.

Proof. The function $h(t_1,t_2)=t_2^r/t_1$ belongs to $\mathcal{H}$, since $t_2^r\leq t_2$ for $t_2\geq 1$ and $r\in(0,1]$. The assumed inequality is exactly (31) for this h, so the result follows from Theorem 4.3. □

5 Application

In this section, we use Example 3.14 to study the polynomial equation

$$x^{m+1}+x^m+Lx-L=0, \tag{32}$$

where $m\in\mathbb{N}$ and $L>2\sqrt{2}\,m$.

Theorem 5.1. *Let $m\in\mathbb{N}$ and $L>2\sqrt{2}\,m$. Then Equation (32) has exactly one solution x^* in the interval $I=[0,1]$. Moreover, for every $x_0\in[0,1]$, the sequence $x_{n+1}=\dfrac{L-x_n^m}{L+x_n^m}$ converges to x^*.*

Proof. Let $f:[0,1]\to[0,1]$ be the mapping defined in (29). Since $L+x^m>0$, for $x\in[0,1]$ we have

$$\begin{aligned} fx=x &\iff L-x^m=Lx+x^{m+1}\\ &\iff x^{m+1}+x^m+Lx-L=0. \end{aligned} \tag{33}$$

Hence the solutions of (32) in $[0,1]$ are exactly the fixed points of f. By Example 3.14, f satisfies

all the hypotheses of Theorem 3.7; therefore Θ_f consists of exactly one element x^*, and the Picard sequence $x_{n+1} = fx_n$ converges to x^* for every $x_0 \in [0,1]$. Since G-convergence in this setting is the usual convergence, the result follows. □

Example 5.2. *Consider the equation*

$$x^{51} + x^{50} + 200x - 200 = 0. \tag{34}$$

Here $m = 50$ *and* $L = 200 > 2\sqrt{2} \cdot 50 \approx 141.42$, *and* $\alpha = \dfrac{8m^2}{L^2} = \dfrac{8(50)^2}{200^2} = 0.5$. *By Theorem 5.1, Equation* (34) *has exactly one solution* x^* *in* $[0,1]$, *and it is the limit of the iteration* $x_{n+1} = (200 - x_n^{50})/(200 + x_n^{50})$. *Starting from* $x_0 = 0$, *we obtain the iterates in Table 1, which give*

$$x^* \approx 0.9929899899 \approx 0.992990.$$

This agrees with a direct numerical solution of (34) *by the bisection method. We also note that, by Descartes' rule of signs, the polynomial in* (34) *has exactly one positive root, which is consistent with Theorem 5.1.*

Table 1. Picard iterates
$x_{n+1} = (200 - x_n^{50})/(200 + x_n^{50})$ with $x_0 = 0$.

n	x_n	$\lvert x_n - x^*\rvert$
0	0.0000000000	9.93×10^{-1}
1	1.0000000000	7.01×10^{-3}
2	0.9900497512	2.94×10^{-3}
3	0.9939530569	9.63×10^{-4}
4	0.9926431277	3.47×10^{-4}
5	0.9931109634	1.21×10^{-4}
10	0.9929893371	6.53×10^{-7}
15	0.9929899934	3.51×10^{-9}
20	0.9929899898	1.89×10^{-11}
25	0.9929899899	1.02×10^{-13}

Source: created by the authors.

6 Conclusion

In this work, we derived new types of contraction mappings by integrating several existing notions within the framework of P-contractions. Specifically, we introduced the (Φ, Ω_b) P-contraction by combining Ω_b-distance mappings and altering distance functions with the P-contraction, and we proved that such mappings have a unique fixed point in complete G_b-metric spaces (Theorem 3.7), for an arbitrary altering distance function and every $\alpha \in [0,1)$. This result contains the fixed point theorem for P-contractions in complete metric spaces (Corollary 3.10) and Banach-type Ω_b-contractions (Corollary 3.9), and Example 3.13 shows that the inclusion of the latter class is strict. We then incorporated $\mathcal{H}$-simulation functions and introduced the $\mathcal{H}(\Phi, \Omega_b)$ P-contraction (Theorem 4.3). We showed that every such mapping is a (Φ, Ω_b) P-contraction and that the two notions coincide for $h(t_1, t_2) = t_2/t_1$ (Proposition 4.2); thus the $\mathcal{H}$-formulation provides a unified way of generating concrete contractive conditions of P-type (Corollary 4.5).

To demonstrate the applicability of our results, we provided numerical examples together with an application showing that the polynomial equation $x^{m+1} + x^m + Lx - L = 0$, with $m \in \mathbb{N}$ and $L > 2\sqrt{2}\, m$, has a unique solution in $[0,1]$, which can be approximated by Picard iteration.

As directions for future research, it would be of interest to extend the notions introduced in this paper to the setting of fuzzy and neutrosophic metric spaces, in the spirit of [37, 38], to multivalued mappings, and to further applications involving integral and fractional differential equations, as considered in [26].

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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