Structural Properties of the Simple-Intersection Graph of Graded Rings

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Abstract: Let Q be a group with identity e, and let $R = \bigoplus_{q\in Q} R_q$ be a Q-graded ring with unity. We introduce the *graded-simple-intersection graph* $GS_Q(R)$, whose vertices are the nonzero graded ideals of R, where two distinct vertices are adjacent precisely when their intersection is a nonzero graded simple ideal. This graph naturally is a subgraph of the graded-ideal intersection graph by requiring intersections to be graded simple rather than merely nonzero.
We investigate how the algebraic structure of a graded ring R influences the graph-theoretic properties of $GS_Q(R)$. In particular, we characterize the connectedness of $GS_Q(R)$, determine its possible girth and diameter, prove that its diameter is never equal to 3, and establish necessary and sufficient conditions for $GS_Q(R)$ to be bipartite. We also characterize when $GS_Q(R)$ is complete and when it is complete bipartite, revealing further connections between the graded ring structure and the associated graph.

Key-Words: graded simple-intersection graph; graded ring; graded ideal; girth; diameter; bipartite graph.

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1 Introduction

Intersection graphs have become an important tool for studying algebraic structures by translating algebraic relationships into graph-theoretic properties. In the setting of ring theory, Chakrabarty *et al.* introduced the intersection graph of ideals of a commutative ring, where vertices correspond to nonzero proper ideals and adjacency is determined by nonzero intersection. Since then, intersection graphs of ideals have attracted considerable attention, and various graph-theoretic properties, including connectedness, diameter, clique number, chromatic number, planarity, and genus, have been investigated in relation to the algebraic structure of the underlying rings. A comprehensive overview of these developments can be found in the survey of Chakrabarty and Kureethara [1]. Further investigations on intersection graphs of ideals of rings and their structural properties appear in [2], [3], [4], [5], and [6].

The study of intersection graphs has also been extended to modules and other algebraic structures. Several related graph constructions have been introduced to describe module-theoretic properties through graph-theoretic methods, including intersection graphs of modules, co-intersection graphs, and sum-essential graphs; see, for example, [7], [8], [9], and [10]. These developments demonstrate the effectiveness of graph constructions in capturing algebraic properties of rings and modules through their substructures.

More recently, graph constructions associated with graded algebraic structures have received increasing attention. Alraqad *et al.* introduced the intersection graph of graded ideals of graded rings and investigated several of its structural properties [11, 12]. Other graphs arising from graded rings and modules have also been studied, including graded annihilating submodule graphs and graphs related to homogeneous elements; see [13, 14].

Motivated by these developments and by the simple-intersection graph of rings introduced by Ahmed and Moh'd [15], we study a graded version of this construction. In their work, Ahmed and Moh'd introduced the simple-intersection graph of a ring and investigated several of its graph-theoretic properties, including coloring and structural aspects. In the present paper, we extend this idea to graded rings by incorporating the role of graded ideals and graded simple ideals in determining adjacency. Informally, the graph considered in this paper associates a graph to a graded ring whose vertices are graded ideals, with adjacency determined by the presence of a nonzero graded simple ideal in their intersection. This provides a refinement of the usual intersection graph of graded ideals by taking into account the internal structure of their intersections.

The main objective of this paper is to investigate the relationship between the algebraic structure of a graded ring and the graph-theoretic properties of its graded-simple-intersection graph. We organize the paper as follows. In Section 2, we recall the necessary definitions and preliminary results concerning graded rings, graded ideals, and graded simple ideals, and introduce the graded-simple-intersection graph of a graded ring. In Section 3, we characterize when the graph is null and investigate its cycle structure, determining its possible girth values. Section 4 studies paths, connectedness, and the diameter of the graph, where we prove that the diameter cannot be equal to 3. In Section 5, we characterize the bipartite

cases and provide necessary and sufficient conditions for the graph to be bipartite.

2 Preliminaries

Let Q be a group and let

$$R = \bigoplus_{q \in Q} R_q$$

be a Q-graded ring with $1 \neq 0$. Throughout this paper, graded rings are not necessarily assumed to be commutative, and all ideals are considered to be left graded ideals unless otherwise stated.

Definition 1. Let R be a graded ring. A nonzero graded ideal I of R is called *graded simple* if the only graded ideals of R contained in I are 0 and I.

Definition 2. A graded ideal I of a graded ring R is called *graded semisimple* if it can be written as a direct sum of graded simple ideals. The graded simple ideals appearing in such a decomposition are called the components of I.

Clearly, every graded simple ideal is a graded semisimple ideal with a single component. We recall the following standard fact about semisimple ideals.

Lemma 3. [16] Every graded ideal contained in a graded semisimple ideal is itself graded semisimple.

Proof. This follows from the standard result that every submodule of a semisimple module is semisimple, applied within the graded category; see, for example, [17]. □

Definition 4. The graded socle of R, denoted by gr-Soc(R), is the direct sum of all nonzero graded simple ideals of R. If

$$R = \text{gr-Soc}(R),$$

then R is called a *graded semisimple ring*.

Remark 5. Throughout the paper, whenever we write

$$R = L_1 \oplus \cdots \oplus L_n$$

for a graded semisimple ring, the ideals $L_1, \ldots, L_n$ are understood to be the graded simple components of the chosen decomposition. In particular, whenever the number of components is used, we assume that this decomposition contains all nonzero graded simple ideals of R.

Definition 6. A graded ring R is called *graded Artinian* if every descending chain of graded ideals of R eventually stabilizes.

Definition 7. A graded ideal M of a graded ring R is called graded essential if it intersects every nonzero graded ideal of R nontrivially, that is, $M \cap J \neq 0$ for all such J.

The following statements are straightforward graded analogues of well-known module-theoretic results and will be used throughout the paper.

Lemma 8. Let R be a graded ring.

1. If R is graded Artinian, then every nonzero graded ideal of R contains a nonzero graded simple ideal.
2. Every nonzero graded ideal of R contains a nonzero graded simple ideal if and only if gr-Soc(R) is graded essential in R.

The proof of the next corollary follows from Lemma 8 (2), and noticing that $R = grsoc(R)$ when R is graded semisimple.

Corollary 9. If R is graded semisimple, then every nonzero graded ideal of R contains a nonzero graded simple ideal.

The reader should be careful when dealing with the terminology of graded rings. When the word "graded" precedes a math terminology this means that we are working inside the category of graded rings. For example, when we say graded simple ideal, we mean a graded ideal that contains no nonzero proper graded ideals. The word "non-graded" when precedes a math terminology means that the terminology is not satisfied in the category of graded rings. For example, "non-graded simple ideal" means a graded ideal that is not graded simple. For structures without gradation we declare this explicitly by saying "the structure, which is not graded.". For example "A simple ideal which is not graded" means a simple ideal in the category of rings without gradation equipped.

Next, we recall some basic terminology from graph theory. Let G be a simple graph with vertex set $V(G)$. Two vertices u and v are said to be adjacent, denoted by $u \leftrightarrow v$, if they are connected by an edge.

Definition 10. For a vertex $u \in V(G)$, the *neighborhood* of u, denoted by $N(u)$, is the set of all vertices adjacent to u.

A vertex whose neighborhood is empty is called an isolated vertex.

Definition 11. The degree of a vertex v in a graph G is defined by

$$\deg(v) = |N(v)|.$$

In particular, v is an isolated vertex if and only if $\deg(v) = 0$.

Definition 12. For vertices $u, v \in V(G)$, the *distance* $d(u, v)$ is the length of a shortest path joining u and v. The *diameter*of a connected graph G, denoted by diam(G), is defined by

$$\text{diam}(G) = \max\{d(u, v) : u, v \in V(G)\}.$$

If G is disconnected, we write diam$(G) = \infty$.

Definition 13. Let Q be a group and let

$$R = \bigoplus_{q \in Q} R_q$$

be a Q-graded ring with $1 \neq 0$. The *graded simple-intersection graph* of R, denoted by $GS_Q(R)$, is the simple graph whose vertices are the nonzero graded ideals of R. Two distinct vertices K and L are adjacent, denoted by $K \leftrightarrow L$, if and only if

$$K \cap L$$

is a nonzero graded simple ideal of R.

Definition 14. A graph G is called *connected* if every pair of vertices in G can be joined by a path.

Definition 15. A simple graph is called *complete* if every pair of distinct vertices is adjacent. The complete graph with n vertices is denoted by K_n.

Definition 16. A *cycle* in a graph G is a path

$$v_1 \leftrightarrow v_2 \leftrightarrow \cdots \leftrightarrow v_n \leftrightarrow v_1,$$

where $n \geq 3$ and the vertices $v_1, v_2, \ldots, v_n$ are distinct. The number of edges in the cycle is called its length.

Definition 17. The *girth* of a graph G, denoted by $g(G)$, is the length of its shortest cycle. If G contains no cycles, then

$$g(G) = \infty.$$

3 Girth and Cycle Structure of $GS_Q(R)$

This section focuses on analyzing the cycles and girth of $GS_Q(R)$. While some findings align with previous studies on similar graphs, other parts present novel techniques.

Recall that a null graph is one with no edges (i.e., its vertices are all isolated).

Theorem 18. If $GS_Q(R)$ is null, then either R is a graded simple ring or R lacks any nonzero graded simple ideals, and conversely.

Proof. Assume that $GS_Q(R)$ is a null graph. If R is not graded simple and contains a nonzero graded simple ideal I, then

$$R \cap I = I,$$

which is a nonzero graded simple ideal. Hence R and I are adjacent in $GS_Q(R)$, contradicting the assumption that $GS_Q(R)$ is null. Therefore, either R is graded simple or R has no nonzero graded simple ideals.

Conversely, if R is graded simple, then $GS_Q(R)$ has no vertices other than R, and hence no edges. If R contains no nonzero graded simple ideals, then no pair of vertices can satisfy the adjacency condition. Therefore, in both cases, $GS_Q(R)$ is a null graph. □

Remark 19. In $GS_Q(R)$, a vertex J (distinct from R) is adjacent to R if and only if J is a nonzero graded simple ideal of R. Indeed,

$$J \cap R = J,$$

which is a nonzero graded simple ideal precisely when J is graded simple. Therefore, the subgraph induced by R together with all nonzero graded simple ideals of R is a star centered at R.

In particular, $\deg(R)$ equals the number of distinct nonzero graded simple ideals of R. If R is graded simple, then $\deg(R) = 0$, since R is the unique vertex of $GS_Q(R)$ and a vertex is not adjacent to itself. If R is graded semisimple with

$$R = L_1 \oplus \cdots \oplus L_n \qquad (n \geq 2),$$

where $L_1, \ldots, L_n$ exhaust all nonzero graded simple ideals of R (see Remark 5), then

$$\deg(R) = n.$$

Now let I be a nonzero graded simple ideal of R. For any graded ideal $J \neq I$,

$$I \leftrightarrow J \quad \Longleftrightarrow \quad I \subseteq J.$$

Indeed, if $I \subseteq J$, then

$$I \cap J = I,$$

which is a nonzero graded simple ideal, so $I \leftrightarrow J$. Conversely, if $I \leftrightarrow J$, then $I \cap J$ is a nonzero graded simple ideal contained in I. Since I is graded simple,

$$I \cap J \in \{0, I\},$$

and the intersection is nonzero. Hence

$$I \cap J = I,$$

which implies that

$$I \subseteq J.$$

Finally, two distinct nonzero graded simple ideals are never adjacent. Indeed, let I and I' be distinct graded simple ideals. Then $I \cap I'$ is a graded ideal contained in I. Since I is graded simple,

$$I \cap I' \in \{0, I\}.$$

Similarly,

$$I \cap I' \in \{0, I'\}.$$

Because $I \neq I'$, the only common possibility is

$$I \cap I' = 0.$$

Hence the subgraph induced by all nonzero graded simple ideals of R is a null graph.

We now show that all cycles of length at least 4 fall into only two specific types. Then, we establish that the girth of $GS_Q(R)$ can only be one of the values ∞, 3, or 4.

Theorem 20. A cycle of length 3 exists in $GS_Q(R)$ if and only if $GS_Q(R)$ has at least two adjacent ideals that are not graded simple.

Proof. Suppose that $GS_Q(R)$ contains a cycle of length 3, say

$$L \leftrightarrow M \leftrightarrow N \leftrightarrow L.$$

By Remark 19, two distinct nonzero graded simple ideals cannot be adjacent. Hence, at most one of the vertices L, M, N can be a graded simple ideal. Therefore, at least two of these vertices are non-graded-simple ideals, and they are adjacent.

Conversely, suppose that L and M are two adjacent non-graded-simple ideals. Then

$$L \cap M$$

is a nonzero graded simple ideal. Since L and M are not graded simple, the vertices

$$L, \quad M, \quad L \cap M$$

are distinct. Moreover,

$$L \leftrightarrow M, \qquad M \leftrightarrow L \cap M, \qquad L \cap M \leftrightarrow L,$$

and therefore they form a cycle of length 3. □

Theorem 21. If $GS_Q(R)$ contains a cycle, then its girth $g(GS_Q(R))$ is either 3 or an even number.

Proof. Assume that $GS_Q(R)$ contains a cycle, and let

$$J_1 \leftrightarrow J_2 \leftrightarrow \cdots \leftrightarrow J_n \leftrightarrow J_1$$

be a cycle of minimum length. Hence, $n = g(GS_Q(R))$.

If $n = 3$, then the conclusion is immediate. Assume that $n > 3$.

We first claim that two consecutive vertices in this cycle cannot both be non-graded-simple ideals. Indeed, suppose that J_i and J_{i+1} are both non-graded-simple ideals. Since they are adjacent, their intersection

$$J_i \cap J_{i+1}$$

is a nonzero graded simple ideal. By Theorem 20, the vertices

$$J_i, \quad J_{i+1}, \quad J_i \cap J_{i+1}$$

form a cycle of length 3, contradicting the minimality of the chosen cycle.

Therefore, in a cycle of length greater than 3, consecutive vertices must alternate between graded simple ideals and ideals that are not graded simple. Indeed, by Remark 19, two distinct graded simple ideals cannot be adjacent, and the above argument shows that two ideals that are not graded simple cannot be adjacent in such a cycle.

Now assume that n is odd. Without loss of generality, let J_1 be a graded simple ideal. Since the vertices alternate, we have

$$J_1, J_3, \ldots, J_n$$

graded simple and

$$J_2, J_4, \ldots, J_{n-1}$$

non-graded-simple. Because n is odd, J_n is graded simple. But J_n is adjacent to J_1, giving two distinct adjacent graded simple ideals, which contradicts Remark 19.

Hence, n cannot be odd. Therefore, every cycle of length greater than 3 has even length, and consequently

$$g(GS_Q(R)) = 3$$

or $g(GS_Q(R))$ is an even number. □

Corollary 22. Let R be a graded semisimple ring, say with decomposition $R = L_1 \oplus \cdots \oplus L_n$ exhausting all nonzero graded simple ideals of R (as in Remark 5). Then

$$g(GS_Q(R)) = \begin{cases} 3, & n \geq 3, \\ \infty, & n \in \{1, 2\}. \end{cases}$$

Proof. Case $n \geq 3$. Let L, M, N be three distinct components. The ideals $L \oplus M$ and $M \oplus N$ are not graded simple, since each contains a proper nonzero graded ideal, and $(L \oplus M) \cap (M \oplus N) = M$, a nonzero graded simple ideal. So $L \oplus M \leftrightarrow M \oplus N$. By Remark 19, both $L \oplus M$ and $M \oplus N$ are adjacent to M. Hence $L \oplus M \leftrightarrow M \leftrightarrow M \oplus N \leftrightarrow L \oplus M$ is a 3-cycle, so $g(GS_Q(R)) = 3$.

Case $n = 1$. Here $R = L_1$, so R is itself graded simple. The vertex set of $GS_Q(R)$ is $\{R\}$, a single vertex with no edges, so the graph is acyclic and $g(GS_Q(R)) = \infty$.

Case $n = 2$. Write $R = L \oplus M$. Let I be a nonzero graded ideal of R. By Lemma 3, I is graded semisimple. Since the only nonzero graded simple ideals of R are L and M, we have

$$I \in \{L, M, R\}.$$

Moreover, $L \cap M = 0$, so L and M are not adjacent. On the other hand, both L and M are adjacent to R by Remark 19. Hence $GS_Q(R)$ is the path

$$L \leftrightarrow R \leftrightarrow M,$$

which contains no cycle. Therefore, $g(GS_Q(R)) = \infty$. □

Theorem 23. The girth of $GS_Q(R)$ accepts one of the following values: ∞, 3, or 4.

Proof. If $GS_Q(R)$ is acyclic, then

$$g(GS_Q(R)) = \infty.$$

Otherwise, by Theorem 21, the girth is either 3 or an even integer. It remains to rule out the possibility that the girth is an even integer greater than or equal to 6.

Suppose, for contradiction, that the girth is exactly $n \geq 6$, and let

$$S_1 \leftrightarrow N_1 \leftrightarrow S_2 \leftrightarrow N_2 \leftrightarrow \cdots \leftrightarrow S_{n/2} \leftrightarrow N_{n/2} \leftrightarrow S_1$$

be a cycle of length n, alternating between graded simple vertices $S_1, \ldots, S_{n/2}$ and non-graded-simple vertices $N_1, \ldots, N_{n/2}$, as in the proof of Theorem 21. Here $n/2 \geq 3$.

Since the graph contains a cycle, R cannot be graded simple. Hence R is distinct from all graded simple vertices S_i. Moreover, since the vertices in a cycle are distinct, at most one of the vertices $N_1, \ldots, N_{n/2}$ can be equal to R. Because $n/2 \geq 3$, we may choose an index i such that

$$N_i \neq R.$$

By the cycle construction,

$$S_i \leftrightarrow N_i \leftrightarrow S_{i+1},$$

where indices are taken modulo $n/2$. By Remark 19, every nonzero graded simple ideal is adjacent to R. Hence

$$S_i \leftrightarrow R \leftrightarrow S_{i+1}.$$

Therefore,

$$S_i \leftrightarrow N_i \leftrightarrow S_{i+1} \leftrightarrow R \leftrightarrow S_i$$

is a cycle of length 4.

The four vertices are distinct: S_i and S_{i+1} are distinct vertices of the original cycle, R is not graded simple, and N_i is non-graded-simple with $N_i \neq R$. Hence this gives a cycle of length $4 < n$, contradicting the minimality of the assumed girth.

Thus no cycle of length at least 6 can occur. Consequently,

$$g(GS_Q(R)) \in \{\infty, 3, 4\}.$$

□

4 Paths, Connectedness, and Diameter in the Graph $GS_Q(R)$

This section studies the paths in $GS_Q(R)$ and presents criteria for when $GS_Q(R)$ is complete.

Lemma 24. In $GS_Q(R)$, the distance between any two distinct nonzero graded simple ideals is exactly 2.

Proof. Consider two nonzero graded simple ideals L and M in R. These ideals are not directly connected in $GS_Q(R)$, but each is adjacent to R. Thus, the path $L \leftrightarrow R \leftrightarrow M$ connects them and is the shortest such path. It follows that the distance between L and M is 2. □

Theorem 25. Let R be a graded ring. Then $GS_Q(R)$ is connected if and only if gr-Soc(R) is graded essential in R. Equivalently, every nonzero graded ideal of R contains a nonzero graded simple ideal.

Proof. If R is graded simple, all three conditions hold immediately: $GS_Q(R)$ consists of a single vertex and is therefore path-connected, gr-Soc$(R) = R$ is graded essential in itself, and every nonzero graded ideal (namely R itself) contains the nonzero graded simple ideal R. Assume henceforth that R is not graded simple.

($\Rightarrow$) Suppose $GS_Q(R)$ is path-connected, and let L be a nonzero graded ideal. There is a path from L to R. If $L \leftrightarrow R$, then L is graded simple (Remark 19), so $L \subseteq$ gr-Soc(R) and $L \cap$ gr-Soc$(R) = L \neq 0$. Otherwise, the path has a first edge $L \leftrightarrow M$ for some $M \neq R$, so $L \cap M$ is a nonzero graded simple ideal contained in L; being graded simple, $L \cap M \subseteq$ gr-Soc(R), so $L \cap$ gr-Soc$(R) \supseteq L \cap M \neq 0$. In either case $L \cap$ gr-Soc$(R) \neq 0$. As L was an arbitrary nonzero graded ideal, gr-Soc(R) is graded essential.

($\Leftarrow$) Conversely, suppose that gr-Soc(R) is graded essential. By Lemma 8(b), every nonzero graded ideal of R contains a nonzero graded simple ideal. Let L and M be vertices of $GS_Q(R)$. Choose graded simple ideals

$$N \subseteq L, \qquad O \subseteq M.$$

If $N = O$, then, whenever the vertices are distinct, Remark 19 gives

$$L \leftrightarrow N \leftrightarrow M.$$

Hence L and M are connected.

Suppose now that $N \neq O$. By Remark 19, distinct graded simple ideals have zero intersection. Thus

$$N \cap O = 0,$$

and so $N \oplus O$ is a graded ideal of R. Moreover,

$$N \cap (N \oplus O) = N, \qquad O \cap (N \oplus O) = O,$$

which implies

$$L \leftrightarrow N \leftrightarrow (N \oplus O) \leftrightarrow O \leftrightarrow M.$$

Therefore every two vertices are joined by a path, and $GS_Q(R)$ is path-connected. □

The proof of the following corollary has the same argument of the proof of Theorem 25.

Corollary 26. Let R be a graded ring with gr-Soc(R) essential, i.e., $GS_Q(R)$ is connected (in particular, this holds whenever R is graded Artinian, by Lemma 8). Then for any two nonzero graded ideals L, M of R,

$$\text{diam}(\text{GS}_\text{Q}(\text{R})) \leq 4.$$

Theorem 27. Every vertex with positive degree in $GS_Q(R)$ lies within the component containing R, while all other components consist of isolated vertices.

Proof. If R is graded simple, then $GS_Q(R)$ consists of a single vertex, so the statement is trivial. Assume that R is not graded simple, and let J be a vertex with $\deg(J) > 0$. Then there exists a vertex K such that

$$J \leftrightarrow K.$$

If J is graded simple, then $J \neq R$, and Remark 19 implies that

$$J \leftrightarrow R.$$

Hence J belongs to the connected component containing R.

Suppose now that J is not graded simple. Since

$$J \leftrightarrow K,$$

the intersection

$$I := J \cap K$$

is a nonzero graded simple ideal. Moreover,

$$I \subseteq J.$$

If $I = J$, then $J = J \cap K$ is graded simple, contradicting the assumption on J. Thus

$$I \subsetneq J.$$

By Remark 19,

$$I \leftrightarrow J$$

because $I \subsetneq J$, and

$$I \leftrightarrow R$$

because I is a nonzero graded simple ideal distinct from R. Therefore,

$$J \leftrightarrow I \leftrightarrow R$$

is a path, so J belongs to the connected component containing R.

Finally, every vertex of degree 0 is isolated and therefore forms a singleton connected component. Hence every vertex of positive degree belongs to the connected component containing R. □

Theorem 28. If the graph $GS_Q(R)$ is connected, then its diameter never equals 3.

Proof. Suppose, for contradiction, that $GS_Q(R)$ contains two nonzero graded ideals L, M with

$$d(L, M) = 3.$$

Let

$$L \leftrightarrow N \leftrightarrow P \leftrightarrow M$$

be a shortest path between them. By the minimality of this path, no pair of nonconsecutive vertices in this path can be adjacent. In particular,

$$L \not\leftrightarrow M, \qquad L \not\leftrightarrow P, \qquad N \not\leftrightarrow M.$$

Since $N \leftrightarrow P$, if both N and P were graded simple ideals, this would contradict Remark 19. Hence at least one of N and P is not graded simple. Because

$$L \leftrightarrow N \qquad \text{and} \qquad M \leftrightarrow P,$$

the definition of $GS_Q(R)$ gives that

$$L \cap N \qquad \text{and} \qquad M \cap P$$

are nonzero graded simple ideals of R.

We first claim that

$$L \cap M \cap P = 0.$$

Suppose otherwise. Since

$$L \cap M \cap P \subseteq M \cap P$$

and $M \cap P$ is a graded simple ideal, we obtain

$$L \cap M \cap P = M \cap P.$$

Hence

$$M \cap P \subseteq L.$$

Let

$$I = M \cap P.$$

Then I is a nonzero graded simple ideal contained in both L and M.

If $I \neq L$ and $I \neq M$, then by Remark 19,

$$L \leftrightarrow I \leftrightarrow M,$$

which gives a path of length 2, contradicting $d(L, M) = 3$.

If $I = M$, then $M \subseteq P$ and $M \subseteq L$. Hence

$$L \cap M = M,$$

which is a nonzero graded simple ideal, and therefore

$$L \leftrightarrow M,$$

again a contradiction.

If $I = L$, then $L = M \cap P \subseteq M$, and similarly

$$L \cap M = L$$

is a nonzero graded simple ideal, giving

$$L \leftrightarrow M,$$

which is again impossible.

Therefore,

$$L \cap M \cap P = 0.$$

By the same argument, using the fact that $L \cap N$ is a nonzero graded simple ideal, we also have

$$L \cap M \cap N = 0.$$

Now define

$$Z = (L \cap N) \oplus (M \cap P).$$

Since $L \cap N$ and $M \cap P$ are graded ideals and

$$(L \cap N) \cap (M \cap P) \subseteq L \cap M \cap P = 0,$$

the sum Z is a graded ideal of R.

Using the modular law and the fact that $L \cap N \subseteq L$, we obtain

$$\begin{aligned} L \cap Z &= (L \cap N) + \big(L \cap (M \cap P)\big) \\ &= (L \cap N) + (L \cap M \cap P) \\ &= L \cap N. \end{aligned}$$

Similarly,

$$\begin{aligned} M \cap Z &= (M \cap P) + \big(M \cap (L \cap N)\big) \\ &= (M \cap P) + (L \cap M \cap N) \\ &= M \cap P. \end{aligned}$$

Therefore,

$$L \cap Z = L \cap N$$

and

$$M \cap Z = M \cap P$$

are nonzero graded simple ideals. Hence, by the definition of the graph,

$$L \leftrightarrow Z \qquad \text{and} \qquad Z \leftrightarrow M.$$

If $Z \neq L$ and $Z \neq M$, then

$$L \leftrightarrow Z \leftrightarrow M$$

is a path of length 2, contradicting $d(L, M) = 3$.

If $Z = L$, then

$$L \cap M = M \cap Z = M \cap P$$

is a nonzero graded simple ideal, so

$$L \leftrightarrow M,$$

again a contradiction. The case $Z = M$ is symmetric.

Thus every possible case leads to a contradiction. Hence no two vertices of $GS_Q(R)$ can have distance 3, and therefore

$$\operatorname{diam}(GS_Q(R)) \neq 3.$$

□

As a direct consequence of the previous theorem we have the following corollary.

Corollary 29. If $GS_Q(R)$ is not connected, then the diameter of the component containing R never equals 3.

We give some sharp diameter results for some specific graded rings.

Proposition 30. The graph $GS_Q(R)$ is connected and $\operatorname{diam}(GS_Q(R)) = 1$ iff R has a unique nonzero proper graded ideal (That is, R is a graded local ring whose unique graded maximal ideal is graded simple either).

Proof. ($\Leftarrow$): Assume R has a unique proper graded ideal $\mathfrak{m}$. Then $\mathfrak{m}$ is graded simple and graded maximal simultaneously. Therefore, $GS_Q(R)$ is $R \leftrightarrow \mathfrak{m}$. This implies $GS_Q(R)$ is connected and $\operatorname{diam}(GS_Q(R)) = 1$.
($\Rightarrow$): Suppose that $GS_Q(R)$ is connected and $\operatorname{diam}(GS_Q(R)) = 1$. It follows from Lemma 24 that R has only one graded simple ideal, namely $\mathfrak{m}$. Next, assume, for contrary, that R has a nonzero proper graded ideal J which is not graded simple. If $\mathfrak{m} \subseteq J$, by Lemma 24 we have $R \not\leftrightarrow J$ but we have the path $R \leftrightarrow \mathfrak{m} \leftrightarrow J$, and hence $d(R, J) = 2 > \operatorname{diam}(GS_Q(R))$ which is a contradiction. If $\mathfrak{m} \not\subseteq J$, then $\mathfrak{m} \cap J = \{0\}$. This means that J is an isolated vertex in $GS_Q(R)$, which contradicts the connectedness of $GS_Q(R)$. As a conclusion, R has only one nonzero graded ideal $\mathfrak{m}$ and this ideal is graded simple. □

Corollary 31. Assume the graph $GS_Q(R)$ is connected. Then $\operatorname{diam}(GS_Q(R)) = 2$ or 4 iff R has more than one nonzero proper graded ideal.

Proof. Apply Theorem 28 and Proposition 30. □

Theorem 32. Suppose $GS_Q(R)$ is connected. Then $\operatorname{diam}(GS_Q(R)) = 2$ if and only if R has more than one nonzero proper graded ideals.

Proof. ($\Leftarrow$): Assume there exist at least two nonzero graded ideals. By Corollary 31, $\operatorname{diam}(GS_Q(R)) = 2$ or 4. We consider several cases.
Case 1: If I and J are distinct nonzero graded simple ideals. Then $I \not\leftrightarrow J$, however we have the path $I \leftrightarrow R \leftrightarrow J$. Thus $d(I, J) = 2$.
Case 2: If $I \neq \{0\}$ is graded simple, then $d(I, R) = 1$ and $d(I, J) = 1$ for every nonzero graded ideal J

that is not graded simple but containing I, because $R \leftrightarrow I$ and $J \leftrightarrow I$.

Case 3: If $I \neq \{0\}$ is graded but not graded simple, then $I \not\leftrightarrow R$. By Theorem 25, I contains a nonzero graded simple ideal K. Then we have the path $I \leftrightarrow K \leftrightarrow R$, and hence $d(I, R) = 2$.

Case 4: If $I \neq \{0\}$ is graded simple and J is a graded ideal which is not graded simple and not containing I. We have $I \not\leftrightarrow J$. By Theorem 25, J contains a nonzero graded simple ideal K. Then we have the path $I \leftrightarrow I \oplus K \leftrightarrow J$, and hence $d(I, J) = 2$.

Case 5: If I and J are distinct nonzero graded ideals which are not graded simple with $I \cap J$ is nonzero graded simple, then $I \leftrightarrow J$ and hence $d(I, J) = 1$.

Case 6: If I and J are distinct nonzero graded ideals which are not graded simple with $I \cap J$ is not nonzero graded simple. Then $I \not\leftrightarrow J$. If $I \cap J = \{0\}$, by Theorem 25, each of I and J contains a nonzero graded simple ideal, say I' and J', respectively, such that $I' \neq J'$. So, we have the path $I \leftrightarrow I' \oplus J' \leftrightarrow J$. Thus, we obtain that $d(I, J) = 2$. If $I \cap J \neq \{0\}$ is not graded simple, again, by Theorem 25, we get that $I \cap J$ contains a nonzero graded simple ideal K. So, we have the path $I \leftrightarrow K \leftrightarrow J$, and hence $d(I, J) = 2$. □

5 Bipartiteness of $GS_Q(R)$

This section examines the conditions under which $GS_Q(R)$ is bipartite. The null graph is clearly bipartite, hence our focus is on the non-null case, meaning when R has a proper nonzero graded simple ideal.

Proposition 33. If the girth of $GS_Q(R)$ is finite and greater than 3, then $GS_Q(R)$ is a bipartite graph.

Proof. Assume that $g(GS_Q(R))$ is finite and greater than 3. Then $GS_Q(R)$ contains no cycle of length 3. By the contrapositive of Theorem 20, no two non-graded-simple ideals of R can be adjacent. Moreover, by Remark 19, no two distinct nonzero graded simple ideals of R can be adjacent.

Let W_1 be the set of all nonzero graded simple ideals of R, and let W_2 be the set of all non-graded-simple ideals of R. Since no edge of $GS_Q(R)$ has both endpoints in W_1 and no edge has both endpoints in W_2, every edge joins a vertex in W_1 to a vertex in W_2. Hence (W_1, W_2) is a bipartition of $GS_Q(R)$, and therefore $GS_Q(R)$ is bipartite. □

Definition 34. Let J be a nonzero graded ideal of R. The local graded-simple-intersection graph of J, denoted by $GS_Q(R, J)$, is the subgraph of $GS_Q(R)$ induced by the vertex set

$$\{J\} \cup N(J),$$

where $N(J)$ is the set of all vertices adjacent to J in $GS_Q(R)$. Equivalently, $GS_Q(R, J)$ consists of J, all vertices adjacent to J, and all edges of $GS_Q(R)$ whose endpoints belong to this set.

Theorem 35. $GS_Q(R)$ is bipartite if and only if each nonzero graded simple ideal I in R has its local graph $GS_Q(R, I)$ structured as a star with I at the center.

Proof. ($\Rightarrow$) Suppose that $GS_Q(R)$ is bipartite, with bipartition (V, W). Let I be a nonzero graded simple ideal of R. Without loss of generality, assume that $I \in V$. Then every neighbor of I belongs to W. Since no two vertices of a bipartition class are adjacent, no two neighbors of I are adjacent. Therefore, the induced subgraph $GS_Q(R, I)$ is a star centered at I.

($\Leftarrow$) Suppose that for every nonzero graded simple ideal I, the local graph $GS_Q(R, I)$ is a star centered at I. Let V be the set of all nonzero graded simple ideals of R, and let W be the set of all non-graded-simple ideals.

By Remark 19, no two vertices of V are adjacent. Suppose, to the contrary, that two vertices $J, K \in W$ are adjacent. Then

$$J \cap K$$

is a nonzero graded simple ideal, say I. Hence both J and K are neighbors of I. Since J and K are adjacent, the local graph $GS_Q(R, I)$ contains an edge between two neighbors of its center, contradicting the assumption that it is a star. Therefore no two vertices of W are adjacent.

Thus every edge of $GS_Q(R)$ joins a vertex of V to a vertex of W, so (V, W) is a bipartition. Hence $GS_Q(R)$ is bipartite. □

We shall be careful about that a bipartite graph can have multiple bipartitions. However, for $GS_Q(R)$, having a bipartition (V, W) where V includes all nonzero graded simple ideals and W contains all non-graded simple ideals is both necessary and sufficient for it to be bipartite.

Corollary 36. The graph $GS_Q(R)$ is bipartite if and only if there exists a bipartition (V, W), where V is the set of all nonzero graded simple ideals of R and W is the set of all graded ideals of R that ar not graded simple..

Proof. Suppose $GS_Q(R)$ is bipartite. Since every bipartite graph contains no odd cycle, in particular $GS_Q(R)$ has no cycle of length 3. By Theorem 20, no two non-graded-simple ideals are adjacent. Hence the set

$$W = \{\text{all non-graded-simple ideals of } R\}$$

is an independent set.

On the other hand, by Remark 19, no two nonzero graded simple ideals are adjacent. Therefore the set

$$V = \{\text{all nonzero graded simple ideals of } R\}$$

is also an independent set.

Since every vertex of $GS_Q(R)$ is either graded simple or non-graded-simple, we have

$$V \cup W = V(GS_Q(R)) \quad \text{and} \quad V \cap W = \varnothing.$$

Moreover, every edge joins a vertex of V to a vertex of W. Hence (V, W) is a bipartition of $GS_Q(R)$.

Conversely, if (V, W) is a bipartition of $GS_Q(R)$, where V consists of all nonzero graded simple ideals and W consists of all non-graded-simple ideals of R, then by definition there are no edges joining two vertices of V or two vertices of W. Therefore $GS_Q(R)$ is bipartite. □

Example 37. Let

$$R = F_1 \oplus F_2 \oplus F_3,$$

where each F_i is a field equipped with the trivial grading, and let L_1, L_2, L_3 denote the corresponding graded simple ideals. Consider the three graded ideals

$$L_1 \oplus L_2, \qquad L_2, \qquad L_2 \oplus L_3.$$

Their pairwise intersections are

$$(L_1 \oplus L_2) \cap L_2 = L_2,$$

$$L_2 \cap (L_2 \oplus L_3) = L_2,$$

and

$$(L_1 \oplus L_2) \cap (L_2 \oplus L_3) = L_2.$$

Since $L_2 \neq 0$, these three vertices are pairwise adjacent in $GS_Q(R)$. Hence they form a triangle, and consequently

$$g(GS_Q(R)) = 3.$$

The corresponding subgraph is shown in Figure 1.

Example 38. Let $Q = \mathbb{Z}_2 = \{0, 1\}$ and consider the Q-graded ring

$$R = \mathbb{Z}_8 \oplus \mathbb{Z}_8 i = \mathbb{Z}_8[i]$$

where $R_0 = \mathbb{Z}_8$ and $R_1 = \mathbb{Z}_8 i$. We apply the theoretical results of the paper to $GS_{\mathbb{Z}_2}(R)$.

1. Vertices of $GS_{\mathbb{Z}_2}(R)$

The nonzero graded ideals of R are:

- $I_{(4)} = (4) \cdot R = \{0, 4, 4i, 4 + 4i\}$
- $I_{(2)} = (2) \cdot R = \{0, 2, 4, 6, 2i, 4i, 6i, \dots\}$
- $R = (1) \cdot R = \mathbb{Z}_8 \oplus \mathbb{Z}_8 i$
- **Graded Simple Ideal:** $I_{(4)}$ is the unique nonzero simple graded ideal, so gr-Soc$(R) = I_{(4)}$.
- **Non-Graded-Simple Ideals:** $I_{(2)}$ and R.

2. Graph Adjacencies and Structure

- $I_{(4)} \cap I_{(2)} = I_{(4)} \implies I_{(4)} \leftrightarrow I_{(2)}$
- $I_{(4)} \cap R = I_{(4)} \implies I_{(4)} \leftrightarrow R$
- $I_{(2)} \cap R = I_{(2)} \neq I_{(4)} \implies I_{(2)} \not\leftrightarrow R$

Hence, $GS_{\mathbb{Z}_2}(R)$ is the path graph P_3:
See Figure 2.

3. Theoretical Verification

1. **Connectedness & Essential Socle (Theorem 25):** gr-Soc$(R) = I_{(4)}$ intersects every nonzero graded ideal nontrivially, so gr-Soc(R) is graded essential in R. Consequently, $GS_{\mathbb{Z}_2}(R)$ is connected.
2. **Girth (Theorems 20 & 23):** The non-graded-simple ideals $I_{(2)}$ and R are not adjacent. Thus, no 3-cycle exists, and the graph is acyclic with girth $g(GS_{\mathbb{Z}_2}(R)) = \infty \in \{\infty, 3, 4\}$.
3. **Diameter (Theorem 28):** The maximum distance in $GS_{\mathbb{Z}_2}(R)$ is $d(I_{(2)}, R) = 2$. Thus, diam$(GS_{\mathbb{Z}_2}(R)) = 2 \neq 3$.
4. **Bipartiteness (Theorem 32 & Corollary 33):** The local graph $GS_{\mathbb{Z}_2}(R, I_{(4)})$ at the unique graded simple vertex $I_{(4)}$ is a star. The bipartition is (V, W) where $V = \{I_{(4)}\}$ and $W = \{I_{(2)}, R\}$.

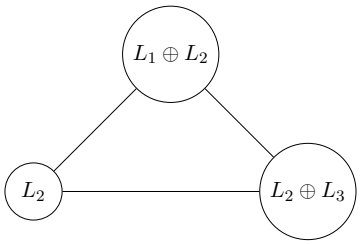


Fig. 1: A triangle in $GS_Q(F_1 \oplus F_2 \oplus F_3)$ showing that the girth is 3.
Source: created by the authors.

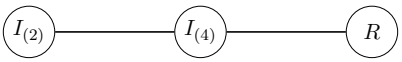


Fig. 2: The path graph $GS_{\mathbb{Z}_2}(\mathbb{Z}_8[i])$.
Source: created by the authors.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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